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ON THE ISOMORPHISM PROBLEM FOR MODULAR GROUP ALGEBRAS OF ELEMENTARY ABELIAN-BY-CYCLIC p-GROUPS

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Abstract. Let G be a finite p-group and let F be the field of p elements. It is shown that if G is elementary abelian-by-cyclic then the isomorphism type of G is determined by FG.

1. Introduction. The isomorphism problem for modular group algebras is whether the isomorphism of the group algebras FG and FH implies the isomorphism of the groups G and H, where F is a field of characteristic p, $p > 0$, and G is a finite p-group. The problem, though studied for more than fifty years, is solved only for some special classes of finite p-groups. For a survey of existing results see e.g. [5], [6], [7] and [10].

In this paper, developing some ideas from [2] and [5], we solve the problem for elementary abelian-by-cyclic p-groups. This extends essentially the results of [2]. In fact, we show something more: if the centralizer N of the commutator subgroup G_2 modulo its Frattini subgroup $\Phi(G_2)$ is elementary abelian and G/N is cyclic then the isomorphism class of $G/\Phi(G_2)$ is determined by the group algebra FG, where F is the field of p elements. Similarly to [2] in the proof we use only the information provided by the factor algebra $FG/I(G_2)^2FG$. If G_2 is elementary abelian then the inclusion $G \subset FG$ induces a monomorphism of G into the factor algebra $FG/I(G_2)I(G)$. This factor algebra is called the *small group algebra*. In [5] the structure of this algebra was used in solving the problem for groups of nilpotency class two with elementary abelian commutator subgroup. We show that when studying the small group algebra it is not possible to solve the problem even for p-groups of maximal class with elementary abelian commutator subgroup.

Throughout, F denotes the field of p elements, where p is a fixed prime, and G is a finite p-group with FG its modular group algebra. $I(G)$ denotes

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the augmentation ideal of FG and $V = 1 + I(G)$ the group of normalized units of FG. The terms of the lower central series of G are denoted by G_i, in particular G_2 is the commutator subgroup of G. We will denote by $(g,h) = g^{-1}h^{-1}gh$ the group commutator of two elements g, h of a group and by $[a,b] = ab - ba$ the Lie commutator of two elements of an associative algebra.

We also often use well-known standard identities, such as:

$$(xy-1) = (x-1) + (y-1) + (x-1)(y-1), \tag{1}$$

$$\begin{aligned}[x-1, y-1] &= ((y,x)-1) + (x-1)((y,x)-1) \\ &\quad + (y-1)((y,x)-1) + (y-1)(x-1)((y,x)-1),\end{aligned} \tag{2}$$

$$[x, y^t] = \sum_{k=1}^{t} y^{t-k}[x, \underbrace{y, \ldots, y}_{k}]. \tag{3}$$

A subgroup H of V is called a *base subgroup* if $|H| = |G|$ and all elements of H are linearly independent over F.

2. Preliminary results. We begin with some general combinatorial observations. Let K be an arbitrary field and let A be a K-algebra. For arbitrary elements $x, y \in A$ and for $i > j > 0$ we define elements $c_{ij} = c_{ij}(x,y)$ of A in the following way:

$$c_{21} = [y,x], \quad c_{31} = [x,y,y] = [-c_{21}, y], \quad c_{32} = [c_{21}, x],$$

and inductively

$$c_{n1} = [-c_{n-1,1}, y], \quad c_{ni} = [c_{n-1,i-1}, x] \quad \text{for } 2 \le i \le n-1.$$

The following formula can be easily derived from (3):

$$\begin{aligned}[y^m, x] &= \sum_{j=1}^{m} \binom{m}{j} (-1)^{j-1} [c_{21}, \underbrace{y, \ldots, y}_{j-1}] y^{m-j} \\ &= \sum_{j=2}^{m+1} \binom{m}{j-1} c_{j1} y^{m-j+1}.\end{aligned} \tag{4}$$

Let

$$c_n = \sum_{i=1}^{n-1} c_{ni}.$$

It is seen from the definition that $c_{n+1} = [c_n, x] + [-c_{n1}, y]$.

LEMMA 1. *Let L be the ideal of A generated by all elements of the form $[a,b][c,d]$, where $a,b,c,d \in A$. For all $x,y \in A$ and all natural $n \geq 2$,*

$$\text{(a)} \qquad (x+y)^n \equiv \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j + \sum_{i=2}^{n} \binom{n}{i} \sum_{k=0}^{n-i} \binom{n-i}{k} x^{n-i-k} c_i y^k \pmod{L},$$

$$\text{(b)} \qquad (xy)^n \equiv x^n y^n + \sum_{j=2}^{n} \binom{n}{j} \sum_{i=1}^{j-1} x^{n-i} c_{ji} y^{n-j+i} \pmod{L}.$$

Proof. (a) The case $n=2$ is obvious:

$$(x+y)^2 = x^2 + 2xy + y^2 + c_2.$$

For the induction step we have

$$\begin{aligned}(5) \qquad (x+y)^n &= (x+y)^{n-1}(x+y) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j x \\ &\quad + \sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} c_i y^k x \\ &\quad + \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^{j+1} \\ &\quad + \sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} c_i y^{k+1}.\end{aligned}$$

Now using (4) and the equality $\binom{l}{j}\binom{j}{i} = \binom{l}{i}\binom{l-i}{l-j}$, which is true for all $i \leq j \leq l$, we obtain

$$\begin{aligned}(6) \qquad &\sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j x \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-j} y^j \\ &\quad + \sum_{j=1}^{n-1} \binom{n-1}{j} x^{n-1-j} \sum_{i=1}^{j} \binom{j}{i} (-1)^{i-1} [c_2, \underbrace{y, \ldots, y}_{i-1}] y^{j-i}\end{aligned}$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-j} y^j$$

$$+ \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{i-1} \sum_{j=i}^{n-1} \binom{n-1-i}{n-1-j} x^{n-1-j} [c_2, y, \ldots, y] y^{j-i}.$$

So taking $j - i = k$ we get

$$\sum_{j=1}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j x \tag{7}$$

$$= \sum_{j=1}^{n-1} \binom{n-1}{j} x^{n-j} y^j$$

$$+ \sum_{i=1}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} c_{i+1,1} y^k.$$

We also have

$$\sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} c_i y^k x \tag{8}$$

$$\equiv \sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-i-k} c_i y^k$$

$$+ \sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} [c_i, x] y^k.$$

Let α be the first right hand double sum of (8) and β the second one. Then by (6), (7) and the definition of c_m,

$$\sum_{j=1}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j x + \beta \tag{9}$$

$$= \sum_{j=1}^{n-1} x^{n-j} y^j + \binom{n-1}{1} \sum_{k=0}^{n-2} \binom{n-2}{k} x^{n-2-k} c_2 y^k$$

$$+ \sum_{i=2}^{n-1} \binom{n-1}{i} \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} x^{n-1-i-k} c_{i+1} y^k$$

$$= \sum_{j=1}^{n-1} \binom{n-1}{j} x^{n-j} y^j + \sum_{i=2}^{n-1} \binom{n-1}{i-1} \sum_{k=0}^{n-i} \binom{n-i}{k} x^{n-i-k} c_i y^k + c_n$$

and similarly

$$\alpha+\sum_{i=2}^{n-1}\binom{n-1}{i}\sum_{k=0}^{n-1-i}\binom{n-1-i}{k}x^{n-1-i-k}c_iy^{k+1} = \sum_{i=2}^{n-1}\binom{n-1}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}x^{n-i-k}c_iy^k. \tag{10}$$

Using the last two equalities for extending (5) one obtains part (a) of Lemma.

For the induction step in the proof of part (b) one only needs to use the equality $\binom{n-1}{j}+\binom{n-1}{j-1}=\binom{n}{j}$. This part is much easier so we leave it to the reader. ■

COROLLARY 2. *If F is a field of characteristic $p>0$, then for all $x,y\in A$ and $n\ge 1$,*

(a) $$(x+y)^{p^n}\equiv x^{p^n}+y^{p^n}+c_{p^n} \pmod{L},$$

(b) $$(xy)^{p^n}\equiv x^{p^n}y^{p^n}+\sum_{i=1}^{p^n-1}x^{p^n-i}c_{p^n,i}y^i \pmod{L},$$

where L is the ideal of A generated by all elements of the form $[a,b][c,d]$ with arbitrary $a,b,c,d\in A$. ■

Now let N be a normal subgroup of G such that G/N is cyclic of order p^n. Let g be an element of G with $G=\langle g,N\rangle$ and let ϱ denote the automorphism of N induced by conjugation by g. If u is an element of N let μ_u denote conjugation in N by u. The following lemma is an easy strengthening of Lemma 1 from [9] and can be proved in the same way.

LEMMA 3. *Let α be an automorphism of N. Then there is a bijection between*

$$A_\alpha=\{\varphi\in \mathrm{Aut}(G)\mid \varphi|_N=\alpha,\ \varphi|_{G/N}=\mathrm{id}_{G/N}\}$$

and

$$N_\alpha=\{u\in N\mid (\varrho,\alpha)=\mu_u,\ (g^{p^n})^\alpha=(gu)^{p^n}\}$$

given by $\varphi\mapsto g^{-1}g^\varphi$. In particular, for a fixed $y\in N$ the function

$$g\mapsto gy,\qquad x\mapsto x\qquad \text{for all } x\in N$$

can be extended to an automorphism of G if and only if $(gy)^{p^n}=g^{p^n}$. ■

The following lemma is a special case of the main result of [4].

LEMMA 4. *If G is a cyclic group of order p^n generated by an element g and F is a field of p elements then the group V of normalized units of the group algebra FG is the direct product of the cyclic groups generated by all elements of the form $1+(g-1)^\delta$, where $0<\delta<p^n$ and $p\nmid\delta$. In particular,*

the subgroup $U = \langle 1 + (g-1)^\delta \mid 1 < \delta < p^n,\ p \nmid \delta \rangle$ *has index* p^n *in* V *and* $V = \langle g \rangle \times U$. ∎

3. The structure of the algebra $FG/I(G_2)^2FG$**.** Let N and H, $H \le N$, be normal subgroups of G and let $|G:N| = p^n$, $|N:H| = p^m$. Let $g_1, \ldots, g_n$ be elements of G such that the elements

$$(g_1-1)^{i_1} \ldots (g_n-1)^{i_n}, \quad 0 \le i_1, \ldots, i_n \le p,\ 0 < \sum_j i_j < p^n, \tag{11}$$

form a linear basis of $I(G)$ modulo $I(N)FG$. In particular, if G/N is a cyclic group generated by gN, where $g \in G \setminus N$ is a fixed element, we can take $g_1 = g,\ g_{i+1} = g_i^p = g^{p^i},\ i = 1, \ldots, n-1$, and then the basis (11) has the form

$$(g-1)^{i_1}(g^p-1)^{i_2} \ldots (g^{p^{n-1}}-1)^{i_n} = (g-1)^{i_1+i_2p+\ldots+i_np^{n-1}}.$$

Let now similarly $x_1, \ldots, x_m \in N$ be such that

$$(x_1-1)^{i_1} \ldots (x_m-1)^{i_m}, \quad 0 \le i_1, \ldots, i_m < p,\ 0 < \sum_j i_j < p^m, \tag{12}$$

form a basis of $I(N)$ modulo $I(H)FN$, which is of course a linear basis of $I(N) + I(H)FG$ modulo $I(H)FG$.

Let $h_1, \ldots, h_k$ be a minimal set of generators of H. It is well known that the set $\{h_1-1, \ldots, h_k-1\}$ is a basis of the space $I(H)$ modulo $I(H)^2$ ([8], Prop. III.1.15(i)). It is also a basis of $I(H)FG$ modulo $I(H)I(G)$ ([8], Prop. III.1.15(ii)). Notice that we can treat the space $I(H)FG/I(H)I(G)$ as a module over the ring $FG/I(H)FG$, where the action is induced by conjugation by elements of G.

Now assume that $H = G_2$ and $N = C_G(G_2/\Phi(G_2))$. Let also G/N be cyclic of order p^n generated by gN, where $g \in G \setminus N$ is a fixed element. Since the annihilator of the $FG/I(G_2)FG$-module $M = I(G_2)FG/I(G_2)I(G)$ contains $I(N)FG/I(G_2)FG$ we can view M as an $FG/I(N)FG$-module. But $FG/I(N)FG \simeq F(G/N)$ is a group ring of a cyclic p-group, so M splits into a direct sum of cyclic submodules. Let $z_{11}, \ldots, z_{1s}$ be elements of H such that the images of the elements $z_{11}-1, \ldots, z_{1s}-1$ in M are generators of all different direct summands of M. Let M_i be the submodule of M generated by $z_{1i}-1$ and let $\dim_F M_i = k_i$. Moreover, assume that $k_1 \ge \ldots \ge k_s$. It is clear that for fixed i, $1 \le i \le s$, the submodule M_i is spanned over F by the images of the elements $z_{1i}-1, \ldots, z_{k_ii}-1$, where $z_{j+1,i} = (z_{ji}, g)$. Now observe that the elements

$$(g-1)^t(x_1-1)^{i_1} \ldots (x_m-1)^{i_m} \tag{13}$$

form a basis of FG modulo $I(G_2)FG$ and the elements

$$(g-1)^t(x_1-1)^{i_1} \ldots (x_m-1)^{i_m}(z_{ji}-1) \tag{14}$$

form a basis of $I(G_2)FG$ modulo $I(G_2)^2FG$, where $0 \le t < p^n$, $0 \le i_1, \ldots, i_m < p$, $0 < \sum_r i_r < p^m$, $1 \le j \le k_i$. Now put $z = z_0 = z_{11}$ and for $1 \le i < k_1$ let $z_i = z_{i+1,1}$.

LEMMA 5. *The function*

$$\alpha \mapsto [\alpha, z] \tag{15}$$

induces a monomorphism from the space spanned by elements of the form (13) *with* $1 \le t < p^n$ *into* $I(G_2)FG/I(G_2)^2FG$.

Proof. Let A be the space spanned by all elements of the form (13) and let A_t be its subspace spanned by such elements for fixed t, $1 \le t < p^n$. It is clear that $A = A_1 \oplus \ldots \oplus A_{p^n-1}$. The image of an arbitrary element α is contained in the subspace of $I(G_2)FG/I(G_2)^2FG$ spanned modulo $I(G_2)^2FG$ by all elements of the form

$$(g-1)^t(x_1-1)^{i_1}\ldots(x_k-1)^{i_k}(z_j-1).$$

Now observe that by (2),

$$[g, z_i] \equiv (z_{i+1}-1) + (g-1)(z_{i+1}-1) \equiv g(z_{i+1}-1) \pmod{I(H)^2FG}$$

and then for $1 \le l < k_1$,

$$[g, z, \underbrace{g, \ldots, g}_{l-1}] \equiv (-1)^{l-1}g^l(z_l-1) \pmod{I(G_2)^2FG}.$$

Hence by a slight modification of (4) for all t, $1 \le t < p^n$, we have

$$\begin{aligned}[(g-1)^t, z] &\equiv \sum_{l=1}^{t}\binom{t}{l}(g-1)^{t-l}[g, z, \underbrace{g, \ldots, g}_{l-1}]\\ &\equiv \sum_{l=1}^{t}\binom{t}{l}(g-1)^{t-l}(-1)^{l-1}g^l(z_l-1) \pmod{I(G_2)^2FG}.\end{aligned}$$

Therefore

$$\begin{aligned}&\Big[\sum_{t=1}^{p^n-1}a_t(g-1)^t, z\Big]\\ &\equiv \sum_{t=1}^{p^n-1}a_t\sum_{l=1}^{t}\binom{t}{l}(g-1)^{t-l}(-1)^{l-1}g^l(z_l-1)\\ &\equiv \sum_{l=1}^{k_1}\Big(\sum_{t=l}^{p^n-1}a_t\binom{t}{l}(g-1)^{t-l}\Big)(-1)^{l-1}g^l(z_l-1) \pmod{I(G_2)^2FG}\end{aligned}$$

as $p^{n-1} < k_1 < p^n$ and for $l \ge k_1$ we have $z_l - 1 \equiv 0 \pmod{I(G_2)^2FG}$.

Now, if

$$\Big[\sum_{t=1}^{p^n-1} a_t(g-1)^t, z\Big] \equiv 0 \pmod{I(G_2)^2FG},$$

then for $t = l < k_1$ we get $a_t g^t(z_t - 1) \equiv 0$, which implies $a_t = 0$. For $k_1 \le t \le p^n - 1$ we obtain $a_t = 0$ upon taking $l = 1$ when $t \not\equiv 0 \pmod p$, and $l = p^j$ when $t = ip^j$, $i \not\equiv 0 \pmod p$. The lemma now follows from the fact that

$$[(g-1)^t(x_1-1)^{i_1}\dots(x_m-1)^{i_m}, z] \equiv [(g-1)^t, z](x_1-1)^{i_1}\dots(x_m-1)^{i_m}. \ \blacksquare$$

The following result is a more general version of Proposition 1.4 of [2].

PROPOSITION 6. *Let G be a finite p-group, and assume that the subgroup*

$$N = C_G(G_2 : \Phi(G_2))$$

is such that G/N is cyclic. Then

$$I(N) + I(G_2)FG = C_{I(G)}(I(G_2)FG : I(G_2)^2FG).$$

In particular,

(a) *the subring $I(N)+I(G_2)FG$ is determined by FG, i.e. it is canonical in the sense of Passman, and*

(b) *the algebra $(I(N) + I(G_2)FG)/(I(G_2)^2FG)$ is commutative iff the group $N/\Phi(G_2)$ is commutative.*

Proof. Let

$$S = I(N) + I(G_2)FG \quad \text{and} \quad U = C_{I(G_2)}(I(G_2)FG : I(G_2)^2FG).$$

The inclusion $S \subset U$ follows from the assumption $N = C_G(G_2 : \Phi(G_2))$ and one can prove it in the same way as in the proof of Prop. 1.4 of [2]. The proof of the reverse inclusion is also similar to that in [2] and is an immediate consequence of Lemma 5. In fact, if α is an element of $I(G)$ then it can be uniquely expressed in the form

$$\alpha = \sum a_{ti_1\dots i_m}(g-1)^t(x_1-1)^{i_1}\dots(x_m-1)^{i_m},$$

where $a_{ti_1\dots i_m} \in F$, $0 < t$, $0 \le i_1, \dots, i_m < p$. So if $[\alpha, \beta] \in I(G_2)^2FG$ for all $\beta \in I(G_2)FG$ then in particular $[\alpha, z-1] \in I(G_2)^2FG$ and by Lemma 5 for $t < p^n$ we must have $a_{ti_1\dots i_m} = 0$, that is, $\alpha \in U$. $\blacksquare$

COROLLARY 7. *Let G be a finite p-group, and assume that the subgroup $N = C_G(G_2 : \Phi(G_2))$ is such that G/N is cyclic. Then $N/\Phi(G_2)$ is determined by the structure of the algebra FG.*

Proof. Observe first that by standard considerations one can easily obtain

$$(I(N) + I(G_2)FG)/I(G_2)I(G) \simeq I(N)/I(G_2)I(N) \simeq I(\overline{N})/I(\overline{G}_2)I(N),$$

where $\overline{X}$ means the image of a subset $X \subset N$ under the natural epimorphism $N \to N/\Phi(G_2)$. But by definition of N, $N/\Phi(G_2)$ is of nilpotency class two with elementary abelian commutator subgroup. So the result follows from the main result of [6]. ∎

LEMMA 8. *Let G and N be as in the assumptions of Corollary 7. Then for all $n \geq 1$:*

(a) $(x+y)^{p^n} \equiv x^{p^n} + y^{p^n} \pmod{I(N)I(G)}$ *if* $x, y \in I(G)^2$,

(b) $(xy)^{p^n} \equiv x^{p^n} y^{p^n} \pmod{I(N)I(G)}$ *if* $y \in I(G)^2$.

Proof. Let $J_2 = I(G_2)FG$ and for $k > 2$,

$$J_k = J_{k-1}I(G) + I(G)J_{k-1} = \sum_{i=2}^{k-1} I(G_i)I(G)^{k-i}$$

where $I(G)^0 = FG$ (see [2]). An easy induction shows that for all $m > 1$,

$$[\underbrace{I(G)^2, I(G)^2, \ldots, I(G)^2}_{m}] \subset J_{2k}.$$

Therefore for $x, y \in I(G)^2$ in Corollary 2 the element c_{p^n} must belong to J_{2p^n}. Since by definition of N, $G_{2p^n} \leq \Phi(G_2)$, it follows that $c_{p^n} \equiv 0 \pmod{I(G_2)I(G)}$ and (a) follows. By Corollary 2(b) part (b) is obvious. ∎

THEOREM 9. *Let G be a finite p-group such that for the subgroup $N = C_G(G_2 : \Phi(G_2))$ the factor group G/N is cyclic. Then the isomorphism class of $G/\Phi(N)$ is determined by the structure of FG.*

Proof. Let, as previously, $G = \langle g, N\rangle$ and $N = \langle x_1, \ldots, x_k, \Phi(G_2)\rangle$. Since

$$(I(N) + I(G_2)FG)FG = I(N)FG,$$

by Proposition 6 the ideal $I(N)FG$, and then also the ideal $I(N)I(G)$, is determined by FG. Let H be an arbitrary base subgroup of FG. Then again by Proposition 6, H contains a subgroup M such that

$$I(M) + I(H_2)FH = I(N) + I(G_2)FG.$$

We will use the bar convention to denote images in the quotient algebra $FG/I(N)I(G)$. It is clear that we may assume that $\overline{M} = \overline{N}$. Let $h \in H \setminus M$ be such that $\overline{H} = \langle \overline{h}, \overline{N}\rangle$. Then h generates FG modulo $I(N)FG$ and by Lemma 4, $\overline{h} = \overline{g}^i\, \overline{u}\, \overline{t}$, where

$$u \in \langle 1 + (g-1)^\delta \mid 1 < \delta < p^n,\ p \nmid \delta\rangle \quad \text{and} \quad t \in N.$$

Since $t - 1 \in I(G)^2$ the element $x = g^i t$ must generate G modulo N. By Lemma 8(b) for $y = t^{-1}ut$ we have $(xy)^{p^n} \equiv x^{p^n} y^{p^n} \pmod{I(G_2)I(G)}$. Moreover, since $y-1 \in I(G)^2$, by Lemma 8(a) we have $y^{p^n} \equiv 0 \pmod{I(G_2)I(G)}$.

Hence by the inclusion $G_2 \le N$ we obtain $(xy)^{p^n} \equiv x^{p^n} \pmod{I(N)I(G)}$. It is clear that $G = \langle x, N\rangle$. Let

$$\overline{W} = \langle \overline{N},\ 1 + (\overline{g} - 1)^{\delta} \mid 1 < \delta < p^n,\ p \nmid \delta\rangle$$

be a subgroup of the group of normalized units $\overline{V} = 1 + \overline{I(G)}$. By Lemma 4, $|\overline{V} : \overline{W}| = p^n$ and $\overline{V} = \langle \overline{x}, \overline{W}\rangle$. Therefore by the above and by Lemma 3 the function

$$\overline{x} \mapsto \overline{xy}, \quad \overline{z} \mapsto \overline{z}, \quad z \in W,$$

can be extended to an automorphism of $\overline{V}$. This automorphism maps $\overline{G}$ onto $\overline{H}$. ∎

Corollary 10. *If G is an elementary abelian-by-cyclic p-group and $FG \simeq FH$ then $G \simeq H$.* ∎

4. The small group algebra. In the proof of Proposition 6 we used the information provided only by the factor algebra $FG/I(G_2)^2FG$. Having this proposition we proved Theorem 9 using even a smaller factor algebra, namely $FG/I(G_2)I(G)$. Notice that the inclusion $G \subset FG$ determines a monomorphism of $G/\Phi(G_2)$ into $FG/I(G_2)I(G)$. The following example shows that in this last factor algebra there is not enough information about G to determine its isomorphism type.

Example 1. Let G be a p-group of maximal class with elementary abelian maximal subgroup N. By III.14 of [3], $|G| \le p^{p+1}$ and $G = \langle g, N\rangle = \langle g, h\rangle$, with $h \in N$. Assume that $|G| \ge p^5$. Consider the subgroup H of the group of normalized units generated by the elements g and $h + (g-1)^2$. It is clear that H generates the algebra FG and its image $\overline{H}$ in $FG/I(G_2)I(G)$ is a p-group of maximal class and of order equal to $|G|$, in particular $FG/I(G_2)I(G) \simeq F\overline{H}/I(\overline{H}_2)I(\overline{H})$. Moreover, it can be easily proved that $\overline{H}$ does not contain a maximal abelian subgroup, that is, $FG \not\simeq F\overline{H}$.

The situation described in the example is a special case of a more general property of the factor algebra $FG/I(G_2)I(G)$.

Let G be a finite p-group and assume that the nilpotency class of $G/\Phi(G_2)$ is equal to c. Let $\{t_{21}, \ldots, t_{2k_2}, t_{31}, \ldots, t_{3n_3}, \ldots, t_{c1}, \ldots, t_{cn_c}\}$ be a minimal set of generators of G_2 such that $\{t_{i1}, \ldots, t_{in_i}\}$ is a minimal set of generators of $\Phi(G_2)G_i$ modulo $\Phi(G_2)G_{i+1}$. For $k \ge 2$ we define ideals $I_k = I(G_2)I(G) + I(G_k)FG$. It is clear that each element of I_k has the form $\sum_{i\ge k} a_{ij}(t_{ij} - 1) + \alpha$, where $a_{ij} \in F$ and $\alpha \in I(G_2)I(G)$. The following two lemmas can be proved by easy induction.

Lemma 11. *For every $k \ge 2$, $G_2 \cap (1 + I_k) = \Phi(G_2)G_k$. In particular, if $k \ge c$ then $G_2 \cap (1 + I_k) = \Phi(G_2)$.* ∎

LEMMA 12. *Let* $\alpha_1, \ldots, \alpha_n \in V$, $n \geq 2$. *Then* $[\alpha_1, \ldots, \alpha_n] \in I_n$ *and* $(\alpha_1, \ldots, \alpha_n) - 1 \equiv [\alpha_1, \ldots, \alpha_n] \pmod{I_{n+1}}$. *Moreover*, *if* $\alpha_i = x_i + \beta_i$, $i = 1, \ldots, n$, *where* $x_i \in G$ *and* $\beta_i \in I(G)^2$, *then* $(\alpha_1, \ldots, \alpha_n) \equiv (x_1, \ldots, x_n) \pmod{I_{n+1}}$. ■

PROPOSITION 13. *Let* G *be a* p-*group of maximal class with elementary abelian commutator subgroup* G_2. *Then there exists a* p-*group* H *of maximal class having maximal abelian subgroup such that*

$$FG/I(H) \simeq FH/I(H_2)I(H).$$

Proof. Let $G_1 = C_G(G_2/G_4)$. It is well known (e.g. [3], III.14) that $|G : G_1| = p$. Let $y \in G \setminus G_1$, $x_1 \in G_1 \setminus G_2$ and for $i \geq 1$ let $x_{i+1} = (x_i, y)$. If G_1 is abelian there is nothing to prove. So assume that G_1 is non-abelian and let $(G_1, G_1) = G_i$. Since all normal subgroups of G containing x_2 contain also G_2, the subgroup $C = C_{G_2}(G_1/G_{i+1})$ as a normal subgroup of G does not contain x_2. Otherwise $(G_1, G_2) \leq G_{i+1}$. Hence $(x_2, x_1) \equiv x_1^k \pmod{G_{i+1}}$ for some k, $1 \leq k < p$. Let H be the subgroup of the unit group of $FG/I(G_2)I(G)$ generated by the elements $\overline{y} = y + I(G_2)I(G)$ and $\overline{x}_1 = x_1 - k(y-1)^{i-1} + I(G_2)I(G)$. Then H is of course of maximal class, has the same order as G and since

$$\begin{aligned}(\overline{x}_2, \overline{x}_1) - 1 \equiv [\overline{x}_2, \overline{x}_1] &\equiv [x_2, x_1] - k[x_2, (y-1)^{i-1}] \\ &\equiv ((x_2, x_1) - 1) - k[x_2, y, \ldots, y] \\ &\equiv (x_i^k - 1) - k(x_2, y, \ldots, y) \equiv 0 \pmod{J_{i+1}}\end{aligned}$$

we have $(H_1, H_2) \leq H_{i+1}$. Observe now that the natural embedding of H into the small group algebra $FG/I(G_2)I(G)$ can be extended to a homomorphism of FH onto $FG/I(G_2)I(G)$ and the kernel of this homomorphism is equal to $I(H_2)I(H)$. This means that

$$FH/I(H_2)I(H) \simeq FG/I(G_2)I(G).$$

Repeating this construction for H we again get a group K with small group algebra isomorphic to $FH/I(H_2)I(H)$ such that $(K_1, K_2) \leq K_{i+2}$. So the proposition follows by easy induction. ■

It is clear by [7] that $FG/I(G_2)I(G)$ determines $G/G_2^pG_3$. Finally, notice that a slight modification of the proof from [7] also implies that $G/G_2^pG_4$ is determined by this factor algebra if G is two-generated.

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