COLLOQUIUM MATHEMATICUM

VOL. 82 1999 NO. 1

FINITELY GENERATED GROUPS HAVING A FINITE SET OF CONJUGACY CLASSES MEETING ALL CYCLIC SUBGROUPS

BY

A. A. IVANOV (WROCŁAW)

Abstract. We study infinite finitely generated groups having a finite set of conjugacy classes meeting all cyclic subgroups. The results concern growth and the ascending chain condition for such groups.

0. Preliminaries. Throughout the paper we denote by G an infinite finitely generated group. We say that a group G has a *threading tuple* if there are non-trivial $w_1, \ldots, w_k \in G$ (the *threading tuple*) such that for every non-trivial $g \in G$ there exists a natural number n such that g^n is conjugate to some $w_i, 1 \leq i \leq k$. The paper is motivated by the following question: *Is there an infinite finitely presented group having a threading tuple?* The question has arisen in model theory. The positive answer would provide a finitely axiomatizable strongly minimal structure (see [9] for group-theoretic connections with questions of this kind).

S. V. Ivanov has constructed several examples of finitely generated groups with finitely many conjugacy classes. One of the examples satisfies the condition that any proper subgroup is cyclic of order p, where p is a fixed prime (see [13], p. 425). However the examples are not finitely presented and there is no hope to make them such. The reason is that Ivanov's construction uses the method of Olshanskiĭ, which is designed for producing infinitely presented groups.

Another way to obtain a finitely generated group having a threading tuple is to find a group G with a normal subgroup H such that:

(a) the group H has a threading tuple and is allowed to be infinitely generated (for example quasicyclic);

(b) G/H is a Burnside type group and for any non-trivial $g \in G$ there exists n such that $g^n \in H \setminus \{1\}$.

This idea is applied in Theorem 31.3 of [13] which provides a finitely generated group G with a normal finite subgroup H (contained in the center of G) such that G/H is a finitely generated group of exponent n and for any

1991 *Mathematics Subject Classification*: Primary 20F50.

non-trivial $g \in G$, $g^n \in H \setminus \{1\}$. It is clear that H forms a threading tuple of G.

In a sense the only known methods that give really unexpected finitely generated groups (at least in the class of periodic groups) are the ones of Olshanskiĭ and Grigorchuk. The latter can help to produce finitely presented groups with unusual properties [8]. Since *Grigorchuk type groups* ([1]) (realizable as Burnside groups of automorphisms of rooted trees) have subexponential growth, the following question looks quite important: *Is there an infinite finitely generated group of subexponential growth having a threading tuple?*

An example answering this question cannot be residually finite (see Corollary 0.2 below). Since Grigorchuk type groups are residually finite, the most natural way to obtain an example is to apply the idea described above: find a group G with a normal quasicyclic (or finite) subgroup H such that G/H is a Grigorchuk type group and for any non-trivial $g \in G$ there exists n such that $g^n \in H \setminus \{1\}$. Below we give some evidence that this is the only way that may give a required example, when additionally the FC-center is non-trivial.

On the other hand, it looks probable that demanding the ascending (descending) chain condition, a frequent property of Olshanskiĭ type groups, we can eliminate examples involving Grigorchuk type groups. This makes us think that most likely a finitely generated group having a threading tuple and satisfying the a.c.c. (d.c.c.) must be very similar to Ivanov's examples. We partially confirm this in Section 2.

The author is grateful to Krzysztof Krupiński for discussions.

Normal subgroups. The following observations show that the known Grigorchuk type groups do not have threading tuples.

LEMMA 0.1. *Let $w_1, \ldots, w_k$ be a threading tuple of G and let H be a non-trivial normal subgroup of G. Then*

(1) *$\{w_1, \ldots, w_k\} \cap H \neq \emptyset$.*

(2) *If $g_1, \ldots, g_m, \ldots$ are representatives of all cosets of H in G then for any element $g \in H$ there exist $n \in \omega$ and $w_j \in H$ such that g^n is conjugate in H to some $g_i^{-1} w_j g_i$. In particular, if H is of finite index in G then H has a threading tuple. If, moreover, the number of conjugacy classes of G is finite then the same holds for H.*

Proof. (1) is obvious. To see (2) let $\{w_1, \ldots, w_l\} = H \cap \{w_1, \ldots, w_k\}$. We claim that the elements $w_j^{g_i}$, $1 \leq i$, $1 \leq j \leq l$, form a threading tuple (possibly infinite) in H. Indeed, if $g \in H$ then there exist $n \in \omega$, $h \in H$,

[1] A better but longer name: Aleshin–Grigorchuk–Gupta–Sidki–Sushchanskiĭ groups.

$i \in \omega$, $j \leq k$ such that $g^n = h^{-1} g_i^{-1} w_j g_i h$. Since H is a normal subgroup, $j \leq l$. If the index of H is finite then so is the number of elements of the form $w_j^{g_i}$. The same argument proves the last statement of the lemma. ∎

COROLLARY 0.2. *If $w_1, \ldots, w_k$ is a threading tuple of G then for any proper homomorphism $\phi : G \to G_1$ there exists w_i such that $\phi(w_i) = 1$. In particular, G has no infinite descending normal series with trivial intersection. If G has finitely many conjugacy classes then G has no infinite normal series.*

Proof. We only comment the last statement: any normal subgroup and its complement is closed under conjugation. ∎

Krzysztof Krupiński has pointed out that Corollary 0.2 implies that G in the corollary is not a finitely generated linear group: by a theorem of Maltsev (see [12], p. 408) a finitely generated linear group is residually finite.

Since all Grigorchuk type groups are residually finite, they do not have threading tuples. In the next section we show that if a finitely generated group G of subexponential growth has a threading tuple then G is periodic with $\pi(G)$ (the set of all primes dividing the orders of elements of G) finite.

Periodic groups. The finiteness of $\pi(G)$ of a periodic group having a threading tuple comes from the following observation.

LEMMA 0.3. *If a periodic group G has a threading tuple then $\pi(G)$ is finite. The group G has a finite number of conjugacy classes of elements of prime orders.*

Proof. Let $w_1, \ldots, w_m$ be a threading tuple. Let P be the set of all primes dividing the orders of w_i. Then $\pi(G) \subseteq P$. ∎

Let G be periodic and have a threading tuple. Since $\pi(G)$ is finite we can find a threading tuple that meets any conjugacy class of elements of prime order.

LEMMA 0.4. *Under the above circumstances if G is not a 2-group then there are at least two conjugacy classes of elements of prime orders.*

Proof. The lemma is obvious if $|\pi(G)| > 1$. Let $\pi(G) = \{p\}$. We now use an argument from [7]. Let $|g| = p$ and $n_0 \in \{2, \ldots, p-1\}$ be chosen so that g and g^{n_0} are conjugate in G. Consider h such that $h^{-1}gh = g^{n_0}$. Then for any i we have $h^{-i}gh^i = g^{n_0^i}$. Since G is a p-group, there exists $m = p^s$ such that $g = g^{n_0^m}$. This implies $n_0^m = 1 \pmod{p}$. Since n_0 is prime to p we see that $n_0^{(p-1)} = 1 \pmod{p}$ (apply Euler's Theorem). Choose the minimal $l > 1$ such that $n_0^l = 1 \pmod{p}$. Then $l \mid m$ and $l \mid p-1$. This is a contradiction. ∎

It is worth noting that Theorem 41.1 of [13] gives a finitely generated divisible p-group ($p \geq 3$) containing a quasicyclic H (isomorphic to C_{p^∞}) such that any element of G is conjugate to an element of H. It is clear that the elements of prime order in H form a threading tuple of G.

FC-*center.* The proof of the following lemma was suggested by the referee. It is more constructive than the original proof by the author.

LEMMA 0.5. *Let G have a threading tuple and let $g \in G$ be of infinite order. Then there is a natural number m such that for infinitely many n with $m \mid n$ the elements g^m and g^n are conjugate in G.*

Proof. For $h, h' \in G$ write $h \sim h'$ whenever h and h' are conjugate in G. Let $w_1, \ldots, w_k$ be a threading tuple in G. For g as in the statement, there exist integers l and i_1 such that $g^l \sim w_{i_1}$. Further, there exist exponents $d_i > 1$ such that

$$w_{i_1}^{d_1} \sim w_{i_2}, \quad w_{i_2}^{d_2} \sim w_{i_3}, \ \ldots, \ w_{i_k}^{d_k} \sim w_{i_{k+1}}, \ \ldots, \ w_{i_{k+r}}^{d_{k+r}} \sim w_{i_k}.$$

Then for $d = d_1 \cdot \ldots \cdot d_{k-1}$ and $d' = d_k \cdot \ldots \cdot d_{k+r}$ we have $g^{l \cdot d} \sim w_{i_k} \sim w_{i_k}^{d'}$. We set $m = l \cdot d$; then $g^n \sim g^m$ for any n of the form $n = m(d')^s$, $s \geq 0$. ∎

COROLLARY 0.6. *If a finitely generated group G has a threading tuple then its commutant G' is of finite index in G. In particular, G is not solvable.*

Proof. If $[G : G']$ is infinite then G/G' is an infinite finitely generated abelian group. So there exists $g \in G$ such that for all $i \in \omega \setminus \{0\}$ the elements g^i do not belong to G'. Choose $m < n$ as in Lemma 0.5. Then $(gG')^n, (gG')^m \in G/G'$ are distinct and conjugate. This is a contradiction. ∎

It is well known that the union of all finite conjugacy classes of a group G forms a characteristic subgroup $\mathrm{FC}(G)$. We call it the FC-*center* of G. It is also well known (see [6]) that for periodic G the FC-center is locally normal (any finite subset is contained in a finite normal subgroup) and its quotient by the center is residually finite. Moreover, if the center is infinite then it is a direct sum of finitely many quasicyclic groups.

PROPOSITION 0.7. *Let G be a finitely generated group having a threading tuple. Then*

(a) *G has infinite conjugacy classes.*

(b) *Let G be periodic and let H be an infinite normal subgroup of G. If H is a Chernikov group* (*i.e.*, *a finite extension of a direct sum of finitely many quasicyclic groups*), *then there is a normal subgroup $G_0 \leq G$ such that G_0 is of finite index in G and its center $Z(G_0)$ is a direct sum of finitely many quasicyclic groups.*

P r o o f. (a) If G is not periodic then apply Lemma 0.5. If G is periodic then $\mathrm{FC}(G)$ is locally finite and since G is finitely generated, $G/\mathrm{FC}(G)$ is infinite.

(b) Let N be the subgroup of H of finite index which does not have subgroups of finite index. Then N is normal in G and is a direct sum of finitely many quasicyclic groups (see 19.3.2 of [10]). Let G_0 be the centralizer of N in G. If G_0 is of infinite index in G then the natural homomorphic image of G in $\mathrm{Aut}(N)$ is infinite and periodic. This is impossible (see 19.3.4 of [10]). ∎

Theorem 31.3 of [13] provides a finitely generated group having a threading tuple with non-trivial (finite) FC-center.

1. Subexponential growth. As we noticed in the previous section, a finitely generated group having a threading tuple is not of polynomial growth (by Gromov's theorem finitely generated groups of polynomial growth are virtually nilpotent). Below we study what happens when the growth is subexponential. To a great extent the results of this section concern periodic finitely generated groups in general. The following proposition shows how such groups arise in our situation.

PROPOSITION 1.1. *Let G have a threading tuple. If G does not have free 2-generated subsemigroups* (*in particular*, *if G has subexponential growth*) *then G is periodic.*

P r o o f. Suppose G has an element g of infinite order. Choose m as in Lemma 0.5. Let $m \mid n$, $m \neq n$ and $h^{-1}g^m h = g^n$. We may assume that $m = 1$ and n is sufficiently large. It is clear that h is of infinite order. The group generated by all $h^i g h^{-i}$, $i \in \mathbb{Z}$, is locally cyclic but not cyclic. Thus it is not finitely generated. Lemma 1 of [11] states that if a group has no free subsemigroups then for any pair a, b of its elements the subgroup $\langle a^{\langle b \rangle} \rangle$ is finitely generated. So we get a contradiction. On the other hand, it is clear that a group having a free 2-generated subsemigroup is of exponential growth. ∎

The proposition (and Lemma 0.3) shows that in order to find a finitely generated group of subexponential growth and with a threading tuple we may consider only periodic groups with $\pi(G)$ finite.

We now describe some general property which is inconsistent with subexponential growth. This motivates our further results.

DEFINITION 1.2. Let G be finitely generated with a fixed generating set. A word w in the generators of G is *geodesic* if the corresponding element of G cannot be written as a shorter word. The *length* $|g|$ of an element $g \in G$ is the length of the corresponding geodesic word. A geodesic word w is called *wide* if there are infinitely many geodesic words which have prefix w.

The *trace* of G (denoted by $\mathrm{Tr}(G)$) is the set of all words which are geodesic and wide with respect to the generators of G.

It is clear that the trace of an infinite finitely generated group is infinite.

Let C_g be the conjugacy class of $g \in G$. We say that a tuple $c_1, \ldots, c_l \in C_g$ is *orthogonal* if for distinct $X, Y \subset \{1, \ldots, l\}$, $\prod_{i\in X} c_i \neq \prod_{i\in Y} c_i$.

Let $\mathrm{Prod}(C_g)$ be the set of all $g' \cdot g''$ where g' and $(g'')^{-1}$ are of the form $c_1 \cdot \ldots \cdot c_k$ (not necessarily with the same k) with $c_1, \ldots, c_k \in C_g$ orthogonal.

LEMMA 1.3. *Let G be a finitely generated group, $g \in G$ and let C_g be the conjugacy class of g. Assume that there are natural numbers s and s_0 such that there exist infinitely many k with the property that the trace* $\mathrm{Tr}(G)$ *contains a geodesic word w satisfying the following conditions*:

(1) *Any subword of w having empty intersection with the prefix of length s_0 is not conjugate to any element of* $\mathrm{Prod}(C_g)$.

(2) *There exists a sequence of prefixes $\emptyset \subset w_0 \subset w_1 \subset \ldots w_k \subseteq w_{k+1} = w$ such that $|w_0| \leq s_0$ and for any $j \leq k$, $||w_{j+1}| - |w_j|| \leq s$ and the sequence $w_j^{-1} w_{i-1} g w_{i-1}^{-1} w_j$, $1 \leq i \leq j$, is orthogonal.*

Then the growth of G is exponential.

Proof. For G as above and a tuple $\overline{h} \in G$ we say that an element $c \in G$ is *$\overline{h}$-paradoxical* if for some word w_0 the trace of G contains infinitely many elements w with the following property: w can be presented as an $\overline{h}$-word extending w_0 and geodesic with respect to the generators of G and if $w_0 \subset w_1 \subset \ldots \subset w$ is the sequence of its $\overline{h}$-prefixes extending w_0 then all elements $w_k \cdot \prod_{i=1}^{k} (w_k^{-1} w_{i-1} c w_{i-1}^{-1} w_k)^{e_i}$, $1 \leq k$, $e_i \in \{0,1\}$, are pairwise distinct.

We prove that the element g from the statement of the lemma is paradoxical with respect to the sequence $\overline{h}$ of all words of length not greater than s. Take an infinite geodesic word α such that any subword $w' \subseteq \alpha$ satisfies (1) and (2) of the lemma. The existence of such an α follows from König's Lemma.

König's Lemma also implies the existence of an infinite sequence of prefixes $w_0, w_1, \ldots, w_k, \ldots$ such that for any k the elements $w_k^{-1} w_{i-1} g w_{i-1}^{-1} w_k$, $i \leq k+1$, form an orthogonal tuple and $||w_{k+1}| - |w_k|| \leq s$.

To see that g is $\overline{h}$-paradoxical suppose that for some $k \leq l$ and $e_i, d_i \in \{0,1\}$, the elements

$$w_k \cdot \prod_{i=1}^{k} (w_k^{-1} w_{i-1} g w_{i-1}^{-1} w_k)^{e_i} \quad \text{and} \quad w_l \cdot \prod_{i=1}^{l} (w_l^{-1} w_{i-1} g w_{i-1}^{-1} w_l)^{d_i}$$

are the same. Since $w_k^{-1}w_l$ is a subword of α, condition (1) implies that the case $k \neq l$ is impossible. Then the equalities $e_i = d_i$, $i \leq k$, follow from the orthogonality.

Let g be paradoxical. It is easy to see that there is an infinite $\overline{h}$-word γ, geodesic with respect to the generators of G, such that any finite sequence of its prefixes has the property of the above definition for g.

Then the rest of the lemma follows from the argument of Subsection 52 of [4]: express γ in the form $w_0h_1h_2\ldots$, where $w_i = w_0h_1\ldots h_i$; to a sequence $e_1\ldots e_l$, $e_i \in \{0,1\}$, assign the element $w_0x_1^{e_1}\cdot\ldots\cdot x_l^{e_l}$, where $x_i^e = h_i(h_i^{-1}gh_i)^e$ is considered as a color of an edge. Since g is $\overline{h}$-paradoxical, this creates a 2-tree (called paradoxical in [4]). ■

We now return to finitely generated periodic groups. The theorem below is applicable if G has infinite *normal* locally normal subgroups (for example, if $\mathrm{FC}(G)$ is infinite). In a sense it states that then case (b) of Proposition 0.7 is most probable for G of subexponential growth (compare with Lemma 1.3).

THEOREM 1.4. *Let G be a finitely generated periodic group and let $H < G$ be infinite, locally normal and normal in G. For $g \in H$ of prime order let W_g be the set of all geodesic words w such that any subword of w and its inverse is not conjugate to any element of* $\mathrm{Prod}(C_g)$, *where C_g is the conjugacy class of g. Assume that H is not Chernikov. Then there exists $g \in H$ of prime order such that*

(∗) *there exists an infinite sequence $w_0, w_1, \ldots, w_k, \ldots \in W_g$ such that for any k the sequence $w_j^{-1}gw_j$, $0 \leq j \leq k$, is orthogonal.*

P r o o f. If for any $p \in \pi(H)$ the conjugacy class of elements of order p is finite then the set of all elements of prime order generates a finite subgroup that meets every subgroup of H. In [3] V. Belyaev describes locally finite groups P for which there exists a finite subgroup $F_0 < P$ such that for any non-trivial finite F_0-invariant subgroup $F_1 < P$, $F_0 \cap F_1 \neq \{1\}$. In particular, Theorem 2.5 of that paper asserts that such a group P has a normal subgroup N which is Chernikov and P/N does not have non-trivial finite normal subgroups. By local normality of H this implies that H is Chernikov, a contradiction.

Applying Theorem 2.5 of [3] again we deduce that for any finite subgroup $F_0 < H$ there is a non-trivial finite F_0-invariant subgroup $F_1 < H$ with $F_0 \cap F_1 = \{1\}$. Let $g \in H$ be of prime order with an infinite conjugacy class C_g such that for any finite F_0 generated by elements of C_g an appropriate F_1 meets C_g. As $\pi(H)$ is finite, such a g exists. We show that g satisfies (∗).

Since H is localy finite, it is of infinite index in G. Then G/H under the generators corresponding to the ones of G has an infinite geodesic word.

Its natural preimage γ in G is geodesic as well. All subwords of γ form an infinite subset of W_g.

Since H is normal in G, any element of G is a product hw, where $h \in H$ and $w \in W_g$. By local normality of H we see that C_g consists of finitely many sets $g_i^{W_g}$ where g_i is conjugate to g in H. By the choice of g we obtain an infinite orthogonal sequence in one of the sets $g_i^{W_g}$. ∎

REMARKS. 1. If in the condition $(*)$ we can additionally claim that there exists a sequence of prefixes $w_{-1} \subset w_{-2} \subset \ldots \in W_g$ such that the elements $g^{w_{-i}^{-1}}$ form an orthogonal sequence and all $||w_{-i}| - |w_{-i+1}||$ are bounded by some number s, then for any natural number k the word w_{-k-1}^{-1} satisfies conditions (1) and (2) of Lemma 1.3: take $w_i := w_{-k-1}^{-1} w_{i-k-1}$ for $i \leq k$. Then by Lemma 1.3 the growth of G is exponential.

2. The property that W_g in Theorem 1.4 is infinite is a consequence of the fact that H is normal in G of infinite index. Notice that this condition entails that the words from W_g not having subwords of finite conjugacy classes form an infinite set. To see this observe that $H \cdot \mathrm{FC}(G)$ is a normal subgroup of infinite index. Indeed, if the index is finite, then $H \cdot \mathrm{FC}(G)$ is a finitely generated group. We may assume that $G = H \cdot \mathrm{FC}(G)$. Then G/H is a homomorphic image of the locally finite group $\mathrm{FC}(G)$. So G/H cannot be infinite and finitely generated. Now apply the argument of Theorem 1.4 to the group $G/(H \cdot \mathrm{FC}(G))$.

2. The ascending chain condition. In this section we study what happens when G does not have an infinite ascending chain of subgroups.

PROPOSITION 2.1. *Let G have a threading tuple. If G satisfies the ascending chain condition for subgroups then G is periodic.*

Proof. Suppose G has an element g of infinite order. Choose m as in Lemma 0.5. Let $m \mid n, m \neq n$ and $h^{-1} g^n h = g^m$. Then $G_0 = \langle g^m \rangle$ is a subgroup of the group G_1 generated by $g_1 = h^{-1} g^m h$ (notice that $g_1^{n/m} = g^m$). Take $g_2 = h^{-1} g_1 h$ and note that it generates a group G_2 containing G_1. By the ascending chain condition we have $G_0 = G_1$, a contradiction to the choice of m and n. ∎

As we have already noted there is an example of a group having a threading tuple and satisfying the a.c.c. for subgroups (S. V. Ivanov). All elements of this group have the same prime order greater than 2. Our next result together with Proposition 2.1 shows that this example is not casual.

THEOREM 2.2. *Let G be periodic, have a threading tuple and satisfy the a.c.c. for subgroups. Then there is an infinite $G_1 < G$ having a threading tuple such that the set of involutions of G_1 is a finite subset of the center.*

P r o o f. Most of the arguments that we use are standard (for example, see [1]).

Assume that G has involutions: $2 \in \pi(G)$. For $p \in \pi(G)$ we define a p-Sylow subgroup to be a *maximal locally finite* p-subgroup. By the a.c.c. we deduce that any p-Sylow subgroup is finite.

We now prove that all 2-Sylow subgroups are conjugate. Suppose $\mathbf{S}$ is a conjugacy class of 2-Sylow subgroups and $S \in \mathbf{S}$ and T are non-conjugate 2-Sylow subgroups with $|S \cap T|$ maximal. First observe that $|S \cap T| \neq 1$. Suppose $|S \cap T| = 1$. Since S and T are nilpotent there are two involutions $s \in Z(S)$ and $t \in Z(T)$. If $s^h = t$ then $t \in S^h \cap T$ contradicting the maximality of $|S \cap T|$. Since $v = st$ is of finite order and s and t are not conjugate, the order of v is even and the corresponding involution $u \in \langle v \rangle$ commutes with both s and t. Let S_1 be a 2-Sylow subgroup containing $\langle u, s \rangle$ and let T_1 be a 2-Sylow subgroup containing $\langle u, t \rangle$. By maximality, S and S_1 (and then S_1 and T_1) are conjugate. So, T_1 and T are conjugate and S and T are conjugate, a contradiction.

Let $I = S \cap T$. By nilpotency, I is self-normalizing neither in S nor in T. Let $i \in S \cap N_G(I)$ and $j \in T \cap N_G(I)$ be involutory modulo I. Since the order of ij is finite in $N_G(I)/I$ we can conjugate T by an element of $N_G(I)$ such that i and j generate a 2-group in $N_G(I)$. Then a 2-Sylow subgroup containing I, i, j is conjugate to S. By the choice of S and T that 2-Sylow subgroup is conjugate to T, a contradiction.

Note that the argument above uses only the a.c.c. and thus it works in any subgroup of G. Let S be a 2-Sylow subgroup, $Z = Z_G(S)$, $N = N_G(S)$. We want to show that Z works as G_1 in the statement of the theorem (it is clear that all involutions of Z belong to S). If for $x, y \in Z$ we have $x^g = y$ then $S, S^g < Z_G(y)$ and there is $g_1 \in Z_G(y)$ such that $S = S^{gg_1}$. This implies that x and y are conjugate in N (by gg_1). So Z has a threading tuple with respect to the action of N. Since S is finite, Z is of finite index in N and each N-orbit splits into a finite number of conjugacy classes in Z. As a result Z has a threading tuple.

It remains to show that Z is infinite. First note that a centralizer of any involution in G is infinite. Otherwise by a well-known theorem of Shunkov [15] the group G contains an infinite locally finite subgroup. Let A be a maximal subgroup of S with $Z_G(A)$ infinite. If $A \neq S$ then there is $B < N_G(A)$ such that $[B : A] = 2$ and B/A defines an involution with finite centralizer in $N_G(A)/A$. Then by Shunkov's theorem $N_G(A)/A$ (and then $N_G(A)$) contains an infinite locally finite subgroup, contradicting the a.c.c. ∎

3. Remarks. *The ascending chain condition.* 1. Let G be an infinite group having a threading tuple, satisfying the a.c.c. and having a minimal number of infinite conjugacy classes of elements of prime order. Then by

Proposition 2.1 and Theorem 2.2, G is periodic with finite $\mathrm{FC}(G)$ and finitely many involutions. Let $p \in \pi(G)$ witness an infinite conjugacy class. Then $p \neq 2$. Moreover, the argument of Theorem 2.2 shows that if a p-Sylow subgroup S has infinite centralizer Z, then there is $y \in Z$ such that $Z_G(y)$ has p-Sylow subgroups non-conjugate in $Z_G(y)$.

2. The ascending chain condition implies the descending chain condition for centralizers of tuples. This shows that G has an infinite subgroup each of whose elements has finite centralizer. This implies the existence of $g \in G$ of prime order such that g together with some conjugate generates an infinite subgroup (and the number of such conjugates is infinite, [16]).

BN-pairs. We start this remark with the definition of Λ-trees (see [2]). Let Λ be a totally ordered abelian group. A Λ-metric space (X, d) is called a *Λ-tree* if:

(1) (X, d) is *geodesically linear*: for any $x, y \in X$ there exists a unique metric morphism $\alpha : [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$ (then $[x, y] := \alpha([0, d(x, y)])$).

(2) $\forall x, y, z \exists! w([x, y] \cap [x, z] = [x, w])$.

(3) $\forall x, y, z \in X([x, y] \cap [y, z] = \{y\} \to [x, y] \cup [y, z] = [x, z])$.

An isometry s of X is called an *inversion* if s^2 has a fixed point but s does not. If s is not an inversion, the *characteristic set* A_s is defined to be $\{p \in X : [p, sp] \cap [p, s^{-1}p] = \{p\}\}$. Then one can define the *hyperbolic length*: $l(s) = \min\{d(x, sx) : x \in X\}$. In this case $A_s = \{p \in X : d(p, sp) = l(s)\}$.

PROPOSITION 3.1. *Let G have a threading tuple. Then any action of G on a Λ-tree is trivial*: *if $g \in G$ is not an inversion*, *then $0 = l(g)$. If G is finitely generated*, *then G has property* (FA): *whenever G acts without inversions on a $\mathbb{Z}$-tree*, *then there exists a vertex fixed by all elements of G.*

Proof. Let $w_1, \ldots, w_k$ be a threading tuple. Suppose $0 < l(g)$ for some $g \in G$. By Corollary 6.13 of [2], $l(g^2) = 2l(g)$; therefore we may assume that for each w_i if $l(w_i)$ is defined then $l(w_i) < l(g)$. Let $m \in \omega$ and $h \in G$ satisfy $h^{-1} g^m h = w_i$. Then again by [2], Corollary 6.13, we have $|m| l(g) = l(g^m) = l(w_i)$, a contradiction.

To see the second statement of the lemma apply Theorem 15 of [14]: G is not a non-trivial free product with amalgamation (by the previous paragraph) and $\mathbb{Z}$ is not a homomorphic image of G (by Corollary 0.6). ■

COROLLARY 3.2. *If a BN-pair (G, B, N, S) has a threading tuple then its rank* (*that is*, $|S|$) *equals* 1: *G is doubly transitive.*

Proof. A theorem of Tits (Theorem II.1.8 in [14]) states that G is the sum of N and the standard parabolic subgroups G_s, $s \in S$, amalgamated along their intersections (recall that for $H = B \cap N$ the set S is a set of involutions generating $W = N/H$ and the map $X \to G_X = B\langle X\rangle B$, where $X \subset S$, is a bijection onto the set of all subgroups containing B).

Suppose $|S| > 1$. Notice that for any $s_i \in S$ the set $G_{s_i} \setminus (G_{S\setminus\{s_i\}} \cup N)$ is not empty. Indeed, since $s_i \subseteq N$ and $B \not\subseteq N$, there exists $b \in B$ such that $bs_i \cap N = \emptyset$. It is clear that $bs_i \not\subset G_{S\setminus\{s_i\}}$.

We now see (by [14]) that G has a non-trivial action on a $\mathbb{Z}$-tree, contradicting Proposition 3.1. ∎

The following question looks very interesting: *Does there exist a finitely generated doubly transitive permutation group having a threading tuple?*

REFERENCES

[1] I. Aguzarov, R. E. Farey and J. B. Goode, *An infinite superstable group has infinitely many conjugacy classes*, J. Symbolic Logic 56 (1991), 618–623.

[2] C. Alperin and H. Bass, *Length functions of groups actions on Λ-trees*, in: Combinatorial Group Theory and Topology, S. M. Gersten and J. R. Stallings (eds.), Ann. of Math. Stud. 111, Princeton Univ. Press, 1987, 265–378.

[3] V. V. Belyaev, *Locally finite groups with a finite non-separable subgroup*, Sibirsk. Mat. Zh. 34 (1993), 23–41 (in Russian).

[4] T. Ceccherini-Silberstein, R. Grigorchuk and P. de la Harpe, *Amenability and paradoxes for pseudogroups and for metric spaces*, preprint, Geneve 1997, 33 pp.

[5] H. Furstenberg, *Poincaré recurrence and number theory*, Bull. Amer. Math. Soc. 5 (1981), 211–234.

[6] Yu. Gorchakov, *Groups with Finite Conjugacy Classes*, Nauka, Moscow, 1978 (in Russian).

[7] Yu. Gorchinskiĭ, *Periodic groups with a finite number of conjugacy classes*, Mat. Sb. 31 (1952), 209–216 (in Russian).

[8] R. Grigorchuk, *An example of a finitely presented amenable group not belonging to the class EG*, ibid. 189 (1998), 79–100 (in Russian).

[9] A. Ivanov, *The problem of finite axiomatizability for strongly minimal theories of graphs*, Algebra and Logic 28 (1989), 183–194 (English translation from Algebra i Logika 28 (1989)).

[10] M. Kargapolov and Yu. Merzlyakov, *Basic Group Theory*, Nauka, Moscow, 1977 (in Russian).

[11] P. Longobardi, M. Maj and A. H. Rhemtulla, *Groups with no free subsemigroups*, Proc. Amer. Math. Soc., to appear.

[12] Yu. I. Merzlyakov, *Rational Groups*, Nauka, Moscow, 1980 (in Russian).

[13] A. Yu. Olshanskiĭ, *Geometry of Defining Relations in Groups*, Nauka, Moscow, 1989 (in Russian).

[14] J.-P. Serre, *Trees*, Springer, New York, 1980.

[15] V. P. Shunkov, *On periodic groups with almost regular involutions*, Algebra i Logika 11 (1972), 470–493 (in Russian).

[16] A. I. Sozutov, *On groups with Frobenius pairs*, ibid. 16 (1977), 204–212 (in Russian).

[17] A. I. Sozutov and V. P. Shunkov, *On infinite groups with Frobenius subgroups*, ibid. 16 (1977), 711–735 (in Russian).

Institute of Mathematics
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: ivanov@math.uni.wroc.pl

Received 3 December 1998;
revised 10 March 1999