COLLOQUIUM MATHEMATICUM

VOL. 81 1999 NO. 2

REAL REPRESENTATIONS OF QUIVERS

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Abstract. The Dynkin and the extended Dynkin graphs are characterized by representations over the real numbers.

The aim of this note is to put together some old results on representations of quivers without oriented cycles with some simple considerations on representations of quivers which contain oriented cycles, to obtain a characterization of the Dynkin and the extended Dynkin quivers in terms of representations over the real numbers. For a quiver Q let $\mathrm{fd}(Q,\mathbb{R})$ denote the category of finite-dimensional representations of Q over $\mathbb{R}$, the real numbers. For a representation (V,f) in $\mathrm{fd}(Q,\mathbb{R})$ we denote by $\overline{\mathrm{End}}(V,f)$ the ring $\mathrm{End}(V,f)/\mathrm{rad}\,\mathrm{End}(V,f)$, the endomorphism ring of (V,f) modulo its radical. With this notation the result reads as follows:

THEOREM. *Let Q be a connected quiver. Then*:

(I) *Q is a Dynkin quiver if and only if $\overline{\mathrm{End}}(V,f)\simeq\mathbb{R}$ for all indecomposable objects (V,f) in $\mathrm{fd}(Q,\mathbb{R})$.*

(II) *Q is an extended Dynkin quiver if and only if $\overline{\mathrm{End}}(V,f)\simeq\mathbb{R}$ or $\mathbb{C}$ for all indecomposable objects in $\mathrm{fd}(Q,\mathbb{R})$, and both cases do occur.*

(III) *Q is neither Dynkin nor extended Dynkin if and only if there is an indecomposable object (V,f) in $\mathrm{fd}(Q,\mathbb{R})$ such that $\overline{\mathrm{End}}(V,f)\simeq\mathbb{H}$.*

Proof. First we take care of the situation where Q does not contain any oriented cycles. The proof is then obtained by putting together old results of Brenner, Dlab and Ringel. If Q has no oriented cycles, then $\mathrm{fd}(Q,\mathbb{R})$ is equivalent to the category of finitely generated modules over the path algebra $\mathbb{R}Q$ of Q over $\mathbb{R}$. We then have the following:

(1) If N is an indecomposable preprojective module, then $N\simeq(\mathrm{Tr}\,D)^nP$ for some $n\in\mathbb{N}_0$ and some indecomposable projective module P, and $\mathrm{End}\,N \simeq \mathrm{End}\,P$ by [ARS, VIII, 1.5], hence $\mathrm{End}\,N\simeq\mathbb{R}$.

(2) If N is preinjective, then $\mathrm{End}\,N\simeq\mathbb{R}$ by duality.

1991 *Mathematics Subject Classification*: 16E10, 16G10, 16G20,16P10.

(3) If N is regular, then there is a chain of irreducible monomorphisms $N_1 \subset \ldots \subset N_{n-1} \subset N_n = N$ such that N_1 is quasi-simple [ARS, VIII, 4.15], [R2], and since the valuation in the Auslander–Reiten quiver is $(1,1)$ we have $\overline{\mathrm{End}}\, N \simeq \overline{\mathrm{End}}\, N_{n-1} \simeq \ldots \simeq \overline{\mathrm{End}}\, N_1$. So, it suffices to consider quasi-simple regular representations N.

(4) If Q is an extended Dynkin quiver, then we can use the classification of the regular modules given in [DR]. By [DR, 5.1] there is a bimodule ${}_F M_G$ of type $\widetilde{A}_{11}$ or $\widetilde{A}_{12}$ with a full exact embedding T of the category of all homogeneous representations of ${}_F M_G$ into the category of all regular representations of Q, and for every (simple) homogeneous representation (S,f) of Q there is a (simple) homogeneous representation (Y,g) of ${}_F M_G$ such that $(S,f) = T(Y,g)$ and thus $\mathrm{End}(S,f) \simeq \mathrm{End}(Y,g)$. Now, Q is one of $\widetilde{A}_{12}, \widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7$ and $\widetilde{E}_8$. Looking at the tables in [DR, §6], we see that in all these cases $F = G = \mathbb{R}$ and ${}_F M_G$ is of type $\widetilde{A}_{12}$. Then by [DR, Addendum (case (2) with $F = \mathbb{R}$)] we know that $\mathrm{End}\,(Y,g) \simeq \mathbb{R}$ or $\mathbb{C}$, and both cases do occur.

Assume now (S,f) is quasi-simple non-homogeneous. Then by [DR, 3.5] we have $(S,f) \simeq C^{+j}\, E(t)$ for some $j \in \mathbb{N}$ and $1 \le t \le m$, where C^+ denotes the Coxeter functor and $E(1), \ldots, E(m)$ is a generating set in the sense of [DR, §3]. But then $\mathrm{End}(S,f) \simeq \mathrm{End}\, E(t) \simeq \mathbb{R}$ by [DR, 2.5 and 3.4].

(5) If Q is neither Dynkin nor extended Dynkin, then the quadratic form q is indefinite [DR, 1.2], and we know by [R1, Theorem 2] that there is a full exact embedding $T : w(\mathbb{R}) \to \mathrm{fd}(Q, \mathbb{R})$, where $w(\mathbb{R})$ denotes the category of all finite-dimensional modules over the wild algebra $\mathbb{R}\langle x, y\rangle$.

By [B] we know that every finite-dimensional $\mathbb{R}$-algebra occurs as the endomorphism ring of some object in $w(\mathbb{R})$, thus there is some $Y \in w(\mathbb{R})$ such that $\mathbb{H} \simeq \mathrm{End}\, Y \simeq \mathrm{End}\, T(Y)$.

Now the proof of the Theorem in case there are no oriented cycles follows readily:

(A) If Q is Dynkin, then $\mathrm{fd}(Q, \mathbb{R})$ consists of the preprojectives, and (1) yields $\overline{\mathrm{End}}(V,f) \simeq \mathbb{R}$ for all indecomposable representations (V,f) in $\mathrm{fd}(Q, \mathbb{R})$.

(B) If Q is extended Dynkin, then (4) yields $\overline{\mathrm{End}}(V,f) \simeq \mathbb{R}$ or $\mathbb{C}$ for all indecomposable representations (V,f) in $\mathrm{fd}(Q, \mathbb{R})$, and both cases do occur.

(C) If Q is neither Dynkin nor extended Dynkin, then by (5) there is an indecomposable representation $(V,f) \in \mathrm{fd}(Q, \mathbb{R})$ such that $\overline{\mathrm{End}}(V,f) \simeq \mathbb{H}$.

Now we turn to the situation where Q contains an oriented cycle.

Since no Dynkin quiver contains cycles, one implication in (I) is proven.

We consider the quiver $\widetilde{A}_n$ with cyclic orientation. Then the category $\mathrm{fd}(Q, \mathbb{R})$ is a coproduct of uniserial finite length hereditary abelian categories

$\mathcal{C}_g$, indexed by the monic irreducible polynomials g in $\mathbb{R}[X]$, where for each monic irreducible polynomial g except X there is one simple representation in $\mathcal{C}_g$ and for X there are $n+1$ simple representations in $\mathcal{C}_X$ (see [S]). Now $\overline{\mathrm{End}}(V,f)$ is isomorphic to $\mathbb{R}[X]/(g)$ for each indecomposable (V,f) in $\mathcal{C}_g$ and thus for $\widetilde{A}_n$ with cyclic orientation, both $\mathbb{R}$ and $\mathbb{C}$ occur this way. That completes the proof of (I) and gives one implication in (II).

To prove the rest, it is enough to produce a representation in $\mathrm{fd}(Q,\mathbb{R})$ with $\mathbb{H}$ as endomorphism ring if Q contains $\widetilde{A}_n$ with cyclic orientation as a proper subquiver. To do this, fix a copy of $\widetilde{A}_n$ in Q with cyclic orientation. Take two simple representations (S,g) and (T,h) of this subquiver which are not isomorphic and with endomorphism ring $\mathbb{C}$. This can be done by taking the fixed two-dimensional $\mathbb{R}$-space $\mathbb{C}$ at each vertex of $\widetilde{A}_n$, with the identity map for all arrows in $\widetilde{A}_n$ except one which is an $\mathbb{R}$-map with characteristic polynomial equal to g and h respectively, where g and h are different irreducible monic polynomials of degree two in $\mathbb{R}[X]$. Since $\widetilde{A}_n$ is a proper subquiver of the connected quiver Q, there is at least one additional arrow α starting or ending at one of the vertices of the fixed subquiver $\widetilde{A}_n$.

Consider the case where α ends at a vertex q of $\widetilde{A}_n$, and let p be the start of α. Suppose first p is not a vertex of $\widetilde{A}_n$. Take the real representation of Q obtained by taking two copies of each of the simple representations (S,g) and (T,h) on the subquiver $\widetilde{A}_n$, the vector space $\mathbb{R}^4$ at the vertex p, and the zero space at all other vertices. Then the space at q is $\mathbb{C}^2 \amalg \mathbb{C}^2$ where the first two $\mathbb{C}$, come from the direct sum of the two copies (S,g) and the last two $\mathbb{C}$'s come from the two copies of (T,h). Let the maps for the arrows in $\widetilde{A}_n$ be as those in the direct sum of the four simple representations. Further, let the map corresponding to α be given by the 8×4 real matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

relative to the standard basis of $\mathbb{R}^4$ and the ordered real basis $(1\ 0\ 0\ 0)$, $(i\ 0\ 0\ 0)$, $(0\ 1\ 0\ 0)$, $(0\ i\ 0\ 0)$, $(0\ 0\ 1\ 0)$, $(0\ 0\ i\ 0)$, $(0\ 0\ 0\ 1)$, $(0\ 0\ 0\ i)$ in $\mathbb{C}^2 \amalg \mathbb{C}^2$. Then since the endomorphism rings of $(S,g)^2$ and $(T,h)^2$ are both the 2×2 matrix ring over the complex numbers, and there are no non-zero maps between (S,g) and (T,h), the endomorphism ring of the constructed representation consists of pairs $\left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right), C\right)$ where

$$A = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} i & j & k & l \\ -j & i & -l & k \\ m & n & s & t \\ -n & m & -t & s \end{pmatrix}$$

are 2×2 "complex" matrices and C is a 4×4 real matrix such that $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)M = MC$. This is the same as the set of triples (A, B, C) of matrices such that $AI = IC$ and $BJ = JC$, where I (the 4×4 identity matrix) is the first 4 rows of M and

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the last 4 rows of M. Solving these equations, one ends up with

$$A = C = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & -c & b & d \\ c & a & -d & b \\ -b & d & a & c \\ -d & -b & -c & a \end{pmatrix},$$

and the set of these matrices is one representation of $\mathbb{H}$ as a ring of matrices.

Now consider the case where p is one of the vertices in $\widetilde{A}_n$. Then we add 4 copies of the simple representation (U, p) which is given by a one-dimensional space at the vertex p and with all maps being zero. We consider the representation of $\widetilde{A}_n$ consisting of the direct sum $(S, g)^2 \amalg (T, h)^2 \amalg (U, p)^4$. If $p \neq q$, let the maps corresponding to the arrows of $\widetilde{A}_n$ be given by the maps of the eight simple representations. Let the map corresponding to the arrow α be given by the 8×12 matrix $\left(\begin{smallmatrix} 0 & 0 & I \\ 0 & 0 & J \end{smallmatrix}\right)$ relative to the decomposition of the space at p as $\mathbb{C}^2 \amalg \mathbb{C}^2 \amalg \mathbb{R}^4$ and at q as $\mathbb{C}^2 \amalg \mathbb{C}^2$ where I and J are as before. The endomorphism ring of this representation consists of pairs

$$\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \right)$$

where A and B are "complex" matrices and C is a real matrix such that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & J \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & J \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Then one finds that the relations between the matrices A, B and C have to be as before, giving again $\mathbb{H}$ as endomorphism ring.

If p and q are the same vertex, let the maps corresponding to the arrows of $\widetilde{A}_n$ be given by the maps as before. Write the space at p as $\mathbb{C}^2 \amalg \mathbb{C}^2 \amalg \mathbb{R}^4$

and let the matrix associated with the arrow α be the 12×12 matrix

$$\begin{pmatrix} 0 & 0 & I \\ 0 & 0 & J \\ 0 & 0 & 0 \end{pmatrix}$$

where I and J are as before. Then the same result is obtained.

We now leave it to the reader to consider the case where the additional arrow α starts at a vertex of $\widetilde{A}_n$.

This shows that if the quiver Q contains $\widetilde{A}_n$ with cyclic orientation as a proper subquiver then there exists an indecomposable representation in $\mathrm{fd}(Q, \mathbb{R})$ with endomorphism ring $\mathbb{H}$. This finishes the proof of the Theorem.

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Received 24 February 1999;
revised 12 March 1999