ON PEAKS IN CARRYING SIMPLICES

BY

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Dedicated to my teacher, Professor Andrzej Krzywicki,
on the occasion of his retirement

Abstract. A necessary and sufficient condition is given for the carrying simplex of
da dissipative totally competitive system of three ordinary differential equations to have a
peak singularity at an axial equilibrium. For systems of Lotka–Volterra type that result
translates into a simple condition on the coefficients.

1. Introduction. An $n$-dimensional system of $C^1$ ordinary differential
   equations (ODEs)
   \[(S_n) \dot{x}[i] = x[i] f[i](x),\]
   where $f = (f[1], \ldots, f[n]) : K \to \mathbb{R}^n$, $K := \{x = (x[1], \ldots, x[n]) \in \mathbb{R}^n : x[i] \geq 0 \text{ for } i = 1, \ldots, n\}$ is called totally competitive if
   \[
   \frac{\partial f[i]}{\partial x[j]}(x) < 0
   \]
   for all $x \in K$, $i, j = 1, \ldots, n$. We write $F = (F[1], \ldots, F[n])$ with $F[i](x) = x[i] f[i](x)$. The symbol $DF(x)$ denotes the Jacobian matrix of the vector
   field $F$ at $x \in K$, $DF(x) = [\frac{\partial F[i]}{\partial x[j]}(x)]_{i,j=1}^n$. Let $K^0$ stand for the
   interior of $K$ in $\mathbb{R}^n$, $K^0 = \{x \in K : x[i] > 0 \text{ for all } i = 1, \ldots, n\}$.
   
   We say that system $(S_n)$ is dissipative if there is a compact invariant set $\Gamma \subset K$ attracting all bounded subsets of $K$. A compact invariant set $A \subset K$ is a repeller if $\alpha(B) \subset A$ for some neighborhood $B$ of $A$ in $K$.
   (For the definitions of concepts from the theory of dynamical systems see
   Hale [5].)

   M. W. Hirsch proved in [6] the following result.

   Theorem 1.1. Assume that $(S_n)$ is a dissipative $n$-dimensional totally
   competitive system of ODEs having $\{0\}$ as a repeller. Then there exists a
   compact invariant set $\Sigma$ with the following properties:

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(i) $\Sigma$ is homeomorphic via radial projection to the standard $(n-1)$-dimensional probability simplex $\{ x \in K : \sum_{i=1}^{n} x[i] = 1 \}$.

(ii) For each $v \in V$ with positive components the restriction $P_v|_\Sigma$ of the orthogonal projection $P_v$ along $v$ is a Lipeomorphism onto its image.

(iii) For each $x \in K \setminus \{0\}$, $\omega(x) \subset \Sigma$.

We call $\Sigma$ the carrying simplex for $(S_n)$ (after M. L. Zeeman [13]).

In the aforementioned paper Hirsch asked about smoothness of $\Sigma$. The problem for $\Sigma \cap K^o$ was considered by P. Brunovský [3], I. Tereščák [12] and M. Benaïm [2]. The present author in [8], [9] gave criteria for the whole of $\Sigma$ to be a neatly embedded $C^1$ manifold-with-corners, whereas in [10] he presented (for $n = 3$) an example of a carrying simplex which is not of class $C^1$ on a part of its boundary. On the other hand, for $n = 2$ the carrying simplex is always a $C^1$ manifold-with-boundary, diffeomorphic to the real interval $[0,1]$ (Mierczyński [11]).

In this paper we investigate the situation (for the system $(S_3)$) when at some point the carrying simplex has a peak, which means that the tangent cone of $\Sigma$ is a halfline.

2. Preliminaries. We specialize to $n = 3$. For the remainder of the paper the standing assumption is:

$(S_3)$ is a dissipative three-dimensional totally competitive system of ODEs having $\{0\}$ as a repeller.

The symbol $V := \{ v = (v[1], v[2], v[3]) \}$ stands for the vector space of all three-vectors, with the Euclidean norm $\| \cdot \|$. For $I \subset \{1,2,3\}$ we write $V_I := \{ v \in V : v[i] = 0 \text{ for } i \in I \}$, and $K_I := \{ x \in K : x[i] = 0 \text{ for } i \in I \}$. We also use a dual notation: $V_I$ means $V_{\{1,2,3\}\setminus I}$. Denote the $i$th vector of the standard basis of $V$ by $e_i$. Further, $\Sigma' := \Sigma \cap K' \setminus \{0\}$, $\Sigma_I := \Sigma \cap K_I$, $\Sigma^o := \Sigma \cap K^o$.

For a closed set $A \subset \mathbb{R}^3$ and $x \in A$, $C_x(A)$ denotes the tangent cone of $A$ at $x$, $C_x(A) := \{ \alpha v : \alpha \geq 0 \text{, there is a sequence } \{x_k\} \subset A \setminus \{x\}, x_k \to x \text{ as } k \to \infty \}$, such that $(x_k - x)/\|x_k - x\| \to v$. The cone $C_x(A)$ is closed, and if $x$ is not isolated in $A$ then $C_x(A) \neq \emptyset$.

For further reference we restate Hirsch’s result:

**Theorem 2.1.** There exists a compact invariant set $\Sigma$ with the following properties:

(a) $\Sigma$ is homeomorphic via radial projection to the standard two-dimensional probability simplex $\{ x \in K : \sum_{i=1}^{3} x[i] = 1 \}$.

(b) For each $v \in V$ with positive components the mapping $P_v|_\Sigma$ is a Lipeomorphism onto its image.
(c) Let \( v \in V^i \) with \( v[i] > 0 \) for both \( j \neq i \). Then the mapping \( P_v|_{\Sigma \cap K^i} \) is a Lipeomorphism onto its image.

(d) For each \( x \in K^i \setminus \{0\} \), \( \omega(x) \subset \Sigma \).

An equilibrium is called axial if only one of its coordinates is positive. By Theorem 2.1(a) there are precisely three axial equilibria \( y_i \in K_i \), and \( \Sigma_i = \{y_i\} \).

We say \( \Sigma \) has a peak singularity at \( y \in \Sigma \) if there is a nonzero vector \( p \in V \) such that \( C_y(\Sigma) = \{\alpha p : \alpha \geq 0\} \).

Proposition 2.2. If \( \Sigma \) has a peak singularity at \( y \) then \( y \) is an axial equilibrium.

Proof. Suppose first that \( y \in \Sigma^0 \), that is, all three coordinates of \( y \) are positive. Denote by \( P \) the orthogonal projection along \( v = (1,1,1) \) on \( S := \{x \in \mathbb{R}^3 : x[1] + x[2] + x[3] = 0\} \). Theorem 2.1(b) states that \( P|_\Sigma \) is a Lipeomorphism (hence a homeomorphism) onto its image. Put \( P \Sigma \) be a Lipschitz constant of the inverse \((P|_\Sigma)^{-1}\). The projection \( P \) takes the set \( \Sigma \cap K^0 \) onto the interior of \( P \Sigma \) in \( S \). Consequently, the tangent cone \( C_{Py}(P \Sigma) \) is the (two-dimensional) tangent space of \( S \) at \( Py \), that is, \( C_{Py}(P \Sigma) = \{v \in V : v[1] + v[2] + v[3] = 0\} \).

Take a unit vector \( r \in C_{Py}(P \Sigma) \). There is a sequence \( \{x_k\} \subset \Sigma^0 \setminus \{y\} \) such that \( \lim_{k \to \infty} x_k = y \) and \( \lim_{k \to \infty}(Px_k - Py)/\|Px_k - Py\| = r \). By choosing a subsequence if necessary, we can assume \( \lim_{k \to \infty}(x_k - y)/\|x_k - y\| = q \). As the derivative of \( P \) at \( y \) is equal to \( P \), one has \( Pq = \beta r \). We claim that \( \beta \neq 0 \). Indeed,

\[
\|Pq\| = \lim_{k \to \infty} \left\| P \frac{x_k - y}{\|x_k - y\|} \right\| = \lim_{k \to \infty} \frac{\|Px_k - Py\|}{\|x_k - y\|} \geq \frac{1}{L}.
\]

We have thus proved that \( P \) takes \( C_y(\Sigma) \) onto \( C_{Py}(P \Sigma) \). Therefore \( C_y(\Sigma) \) contains two noncollinear vectors, so \( \Sigma \) cannot have a peak singularity at \( y \).

Suppose now that only one of the coordinates of \( y \) is zero, say \( y \in \Sigma^3 \setminus (\Sigma_1 \cup \Sigma_2) \). Thm. 1 in Mierczyński [11] yields that \( \Sigma^3 \) is a \( C^1 \) one-dimensional manifold-with-boundary containing \( y \) in its (manifold) interior \( \Sigma^3 \setminus (\Sigma_1 \cup \Sigma_2) \). Hence \( C_y(\Sigma^3) \subset C_y(\Sigma^3) \) is a one-dimensional vector space. Therefore \( \Sigma \) does not have a peak singularity at \( y \).

For an axial equilibrium \( y_i \), the Jacobian matrix \( DF(y_i) \) leaves both the two-dimensional vector subspaces \( V^j \), \( j \neq i \), as well as their one-dimensional intersection \( V_i \), invariant. As \( V_i = \text{span } e_i \), \( e_i \) is an eigenvector of \( DF(y_i) \). Adapting the terminology from Mierczyński [9] we will call the eigenvalue of \( DF(y_i) \) corresponding to \( e_i \) the internal eigenvalue at \( y_i \). By the external eigenvalue at \( y_i \) in \( K^j \), \( j \neq i \), we mean the (unique) eigenvalue of the quotient linear mapping \( (DF(y_i)|_{V^j})/V_i \). An eigenvector for \( DF(y_i) \) belonging to an
external eigenvalue is called an external eigenvector (such an eigenvector need not exist, see Lemma 3.1(iii)).

We are now ready to formulate our principal result.

**Main Theorem.** The carrying simplex $\Sigma$ has a peak singularity at an axial equilibrium $y_i$ if and only if the internal eigenvalue at $y_i$ is larger than or equal to the maximum external eigenvalue at $y_i$. In that case, $C_{y_i}(\Sigma) = \{-\alpha e_i : \alpha \geq 0\}$. 

3. **Proof of the Main Theorem.** To streamline the argument and limit the number of indices we assume in the present section that the axial equilibrium under consideration is $y = y_1$. Similarly, we write $e = e_1$.

**Lemma 3.1.** (i) If the internal eigenvalue at $y$ is larger than the external eigenvalue at $y$ in $K^3$ [resp. in $K^2$] then there is an external eigenvector in $V^3$ [resp. in $V^2$] of the form $(1, a_2, 0)$ with $a_2 > 0$ [resp. of the form $(1, 0, a_3)$ with $a_3 > 0$].

(ii) If the internal eigenvalue at $y$ is smaller than the external eigenvalue at $y$ in $K^3$ [resp. in $K^2$] then there is an external eigenvector in $V^3$ [resp. in $V^2$] of the form $(1, -b_2, 0)$ with $b_2 > 0$ [resp. of the form $(1, 0, -b_3)$ with $b_3 > 0$].

(iii) If the internal eigenvalue at $y$ is equal to the external eigenvalue at $y$ in $K^3$ [resp. in $K^2$] then there are no external eigenvectors in $V^3$ [resp. in $V^2$].

**Proof.** It suffices to observe that the matrix of the restriction of $DF(y)$ to $K^j$, $j = 2, 3$, has the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $a < 0$ and $b < 0$, and compute eigenvectors (compare Lemma 2 in [11]).

3.1. **Necessity.** Suppose by way of contradiction that the internal eigenvalue at $y$ is smaller than the external eigenvalue at $y$ in, say, $K^3$. By Mierczyński [11], $\Sigma = \Sigma \cap K^3$ is a $C^1$ one-dimensional manifold-with-boundary. By the theory of invariant manifolds (see e.g. Hirsch, Pugh and Shub [7]) any locally invariant $C^1$ one-dimensional submanifold passing through $y$ is tangent either to $e$ or to another eigenvector of $DF(y)|_{V^3}$ (not collinear with $e$); moreover, in the former case the submanifold is locally unique. As $K_1$ is an invariant one-dimensional submanifold tangent at $y$ to $e$, $\Sigma^3$ cannot be locally equal to it, since otherwise the radial projection of $\Sigma$ would not be injective (Theorem 2.1(a)). Consequently, $\Sigma^3$ is tangent at $y$ to the vector $(1, -b_2, 0)$ with nonzero second component (by Lemma 3.1(ii)). Hence $C_{y}(\Sigma) \supset C_{y}(\Sigma^3)$ contains $(1, -b_2, 0)$. On the other hand, any vector in $C_{y}(\Sigma^3) \subset C_{y}(\Sigma)$ has zero second component (and $C_{y}(\Sigma^3) \neq \{0\}$). Therefore $C_{y}(\Sigma)$ contains two noncollinear vectors, so $\Sigma$ does not have a peak singularity at $y$. 
3.2. Sufficiency. Put $C := C_y(\Sigma)$. We write $A$ for the linear operator $DF(y)$. In the standard basis, $A$ has the matrix

$$
\begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{12} & d_{22} & 0 \\
d_{13} & 0 & d_{33}
\end{bmatrix}
$$

with

$$
d_{11} = y^{[1]} \frac{\partial f^{[1]}}{\partial x^{[1]}}(y) < 0, \quad d_{12} = y^{[1]} \frac{\partial f^{[1]}}{\partial x^{[2]}}(y) < 0, \quad d_{13} = y^{[1]} \frac{\partial f^{[1]}}{\partial x^{[3]}}(y) < 0, \\
d_{22} = f^{[2]}(y), \quad d_{33} = f^{[3]}(y).
$$

We will prove sufficiency by carefully analyzing the action of the group $\{e^{tA}\}_{t \in \mathbb{R}}$ on the tangent cone $C$.

As $y \in \Sigma$ is an equilibrium and $\Sigma$ is invariant, each of the linear operators $e^{tA}$ leaves $C$ invariant.

Put $C^N := C \cap S$, where $S := \{v \in V : \|v\| = 1\}$ is the unit sphere in $V$.

For $t \in \mathbb{R}$ define the mapping $\psi_t : S \to S$ as

$$
\psi_t v := \frac{e^{tA} v}{\|e^{tA} v\|}.
$$

The family $\psi = \{\psi_t\}_{t \in \mathbb{R}}$ is the solution flow of the system of ODEs

$$
\dot{v} = Av - \langle Av, v \rangle v,
$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^3$. The flow $\psi$ leaves $C^N$ invariant.

**Lemma 3.2.** For any nonzero $v \in C$ we have $v^{[1]} < 0$, $v^{[2]} \geq 0$, $v^{[3]} \geq 0$.

**Proof.** The last two inequalities follow by the definition of $C$ and the fact that $(x - y)^{[2]} \geq 0$ and $(x - y)^{[3]} \geq 0$ for any $x \in K$. Suppose first that there is $v \in C$ with $v^{[1]} > 0$. As a consequence of the definition of $C$ there is a point $x \in \Sigma$ such that $x^{[1]} > y^{[1]}$, $x^{[2]} \geq y^{[2]}$ and $x^{[3]} \geq y^{[3]}$. If $x^{[2]} > y^{[2]}$ and $x^{[3]} > y^{[3]}$ then the restriction $P_{x-y}|\Sigma$ of the orthogonal projection along $x - y \in \Sigma$ is not injective, which contradicts Theorem 2.1(b). If $x^{[2]} > y^{[2]}$ and $x^{[3]} = y^{[3]}$ then $x \in \Sigma^3 \subset K^3$ and $P_{x-y}|\Sigma^3$ is not injective, which is in contradiction with Theorem 2.1(c). The case $x^{[2]} = y^{[2]}$ and $x^{[3]} > y^{[3]}$ is treated in an analogous way. If $x^{[2]} = y^{[2]}$ and $x^{[3]} = y^{[3]}$ then $y \in \Sigma_1$ and the radial projection of $\Sigma$ is not injective, contrary to Theorem 2.1(a).

Suppose now that there is a nonzero $v \in C$ with $v^{[1]} = 0$. Then at least one of the remaining components of $v$ is positive. We have

$$
\left. \frac{d}{dt} (e^{tA} v)^{[1]} \right|_{t=0} = (Av)^{[1]} = d_{12} v^{[2]} + d_{13} v^{[3]} < 0,
$$

from which it follows that $(e^{-tA} v)^{[1]} > 0$ for $t > 0$ sufficiently close to 0. As $e^{-tA} v \in C$ for all $t \in \mathbb{R}$, this is in contradiction to the above paragraph. $\blacksquare$
Denote by $\lambda_1$ the internal eigenvalue at $v$, and by $\lambda_2$ [resp. $\lambda_3$] the external eigenvalue at $v$ in $K^2$ [resp. in $K^3$]. The symbol $w_j$, $j = 2, 3$, stands for the unit external eigenvector corresponding to $\lambda_j$ (provided it exists) having positive first component.

Suppose that $u \in C^N \setminus \text{span } e$. The idea of the proof is to find a vector in $C$ with first component positive, contradicting Lemma 3.2.

We consider four cases (up to relabeling).

**Case I:** $\lambda_1 > \lambda_2 > \lambda_3$. For the flow $\psi$ the set $\{w_3, -w_3\}$ is a repeller, its dual attractor being $V^2 \cap S$. The flow $\psi$ restricted to $V^2 \cap S$ has repeller $\{w_2, -w_2\}$ with $\{e, -e\}$ as its dual attractor (for those concepts see Conley [4] or Akin [1]).

Therefore, if $u \notin V^2$ (notice that in such a case $u^{[2]} > 0$) then $\psi_{-t} u$ converges, as $t \to \infty$, to either $w_3$ or $-w_3$. The latter case is impossible, since as $u^{[2]} > 0$ and $(-w_3)^{[2]} < 0$ (Lemma 3.1(i)), the image of the mapping $\mathbb{R} \ni t \mapsto \psi_t u$ would meet $V^2 \cap S = \{v \in S : v^{[2]} = 0\}$, which is invariant under $\psi$. By the closedness of the tangent cone we have $w_3 \in C^N$, which contradicts Lemma 3.2.

Similarly, if $u \in V^2$ (notice that in such a case $u^{[3]} > 0$) then $\psi_{-t} u$ converges, as $t \to \infty$, to either $w_2$ or $-w_2$. The latter case is impossible, since as $u^{[3]} > 0$ and $(-w_2)^{[3]} < 0$ (Lemma 3.1(i)), the image of the mapping $\mathbb{R} \ni t \mapsto \psi_t u$ would meet $V^3 \cap S = \{v \in S : v^{[3]} = 0\}$, which is invariant under $\psi$. By the closedness of the tangent cone, $w_2 \in C^N$, contradicting Lemma 3.2.

**Case II:** $\lambda_1 = \lambda_2 > \lambda_3$. The set $\{w_3, -w_3\}$ is a repeller, with dual attractor $V^2 \cap S$. If $u \notin V^2$ the proof goes along the lines of Case I.

On $V^2 \cap S$, $\{e, -e\}$ is the set of equilibria, and for $u \in V^2$ we have $\psi_t u \to e$ or $\psi_{-t} u \to e$ as $t \to \infty$ (compare the proof of Lemma 2 in [11]). By the closedness of the tangent cone, $e \in C$, which is impossible.

**Case III:** $\lambda_1 > \lambda_2 = \lambda_3$. The flow $\psi$ has a repeller, $\text{span}\{w_2, w_3\} \cap S$, consisting of fixed points. Its dual attractor is $\{e, -e\}$.

We write $u$ as $\alpha e + \beta \tilde{u}$, where $\tilde{u}$ is a unit vector in $\text{span}\{w_2, w_3\}$ such that $\tilde{u}^{[2]} \geq 0$ and $\tilde{u}^{[3]} \geq 0$ (at least one of these components must be positive). Such a $\tilde{u}$ is unique. Writing $\tilde{u} = \gamma w_2 + \delta w_3$ and observing that $w_2$ has sign pattern $(+, 0, +)$ and $w_3$ has sign pattern $(+, +, 0)$ (Lemma 3.1(i)) we have $\tilde{u}^{[1]} > 0$. The vector subspace $U := \text{span}\{e, \tilde{u}\}$ is invariant under $A$, hence $U \cap S$ is invariant under $\psi$. The flow $\psi$ restricted to $U \cap S$ has repeller $\{\tilde{u}, -\tilde{u}\}$ with dual attractor $\{e, -e\}$. Consequently, $\psi_{-t} u \to \tilde{u}$ or $\psi_{-t} u \to -\tilde{u}$ as $t \to \infty$ (in fact, the former is the case, but we do not need it here). By Lemma 3.2 neither $\tilde{u}$ nor $-\tilde{u}$ can belong to $C$, a contradiction.
CASE IV: $\lambda_1 = \lambda_2 = \lambda_3$. In this case we will investigate the action of $e^{tA}$ on $V$ rather than the action of $\psi$ on $S$. The matrix of $A$ can be written as

$$
\begin{bmatrix}
a & b & c \\
0 & a & 0 \\
0 & 0 & a
\end{bmatrix}
$$

with $a < 0$, $b < 0$ and $c < 0$. Consequently,

$$
e^{tA}u = e^{at} \begin{bmatrix} 1 & bt & ct \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ u[3] \end{bmatrix} = e^{at} \begin{bmatrix} u[1] + btu[2] + ctu[3] \\ u[2] \\ u[3] \end{bmatrix}.
$$

As, by Lemma 3.2, $u[2] \geq 0$ and $u[3] \geq 0$, and by hypothesis, one of these components is positive (recall that $u \not\in \text{span } e$), one has $(e^{-tA}u)[1] > 0$ for $t$ sufficiently large, which contradicts Lemma 3.2.

It would be perhaps interesting to look at the action of $\psi$ in the last case. There is a two-dimensional vector subspace $W = \text{span}\{e, \bar{w}\}$, $\bar{w} = (0, 1, -b/c)$, such that $W \cap S$ consists of the fixed points for the flow $\psi$. For any $v \in S \setminus W$ one finds that $\psi_tv$ converges to $e$ (or $-e$) as $t \to \infty$ (and similarly as $t \to -\infty$, with changed sign).

4. Lotka–Volterra systems. Now we apply our Main Theorem to three-dimensional systems $(S_3)$ of Lotka–Volterra type, that is, to systems

$$(4.1) \quad \dot{x}[i] = b_ix[i]\left(1 - \sum_{j=1}^{3} a_{ij}x[j]\right)$$

where $a_{ij} > 0$ and $b_i > 0$.

It is straightforward that for system (4.1),

$$y_1 = (1/a_{11}, 0, 0), \quad y_1 = (0, 1/a_{22}, 0), \quad y_1 = (0, 0, 1/a_{33}).$$

At $y_i$ the internal eigenvalue equals $-b_i$, whereas the external eigenvalue in $V^j$ is equal to $b_k(1 - a_{ki}/a_{ii})$, with $k \neq i$, $k \neq j$. As a consequence of the Main Theorem we obtain the following.

**Theorem 4.1.** For system (4.1) the carrying simplex $\Sigma$ has a peak singularity at $y_i$ if and only if

$$a_{ii}(b_i + b_j) \leq b_j a_{ji}$$

for both $j \neq i$. 
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