COLLOQUIUM MATHEMATICUM

VOL. 81 1999 NO. 2

WOLD DECOMPOSITION OF THE HARDY SPACE AND BLASCHKE PRODUCTS SIMILAR TO A CONTRACTION

BY

M. I. STESSIN (ALBANY, NY)

Abstract. The classical Wold decomposition theorem applied to the multiplication by an inner function leads to a special decomposition of the Hardy space. In this paper we obtain norm estimates for componentwise projections associated with this decomposition. An application to operators similar to a contraction is given.

1. Introduction. Let $V : X \to X$ be an isometry of a Hilbert space X. The well-known Wold decomposition theorem [9] states that

$$X = X_0 \oplus \bigoplus_{n=0}^{\infty} V^n X_1, \tag{1}$$

where $X_1 = X \ominus VX$ is a wandering subspace and $X_0 = \bigcap_{n=0}^{\infty} V^n X$. In the special case when $X = H^2$ and V is the operator of multiplication by an inner function g, we have $\bigcap_{n=0}^{\infty} g^n H^2 = \{0\}$, and (1) implies

$$H^2 = \bigoplus_{n=1}^{\infty} s_n H^2[g], \tag{2}$$

where $H^2[g]$ stands for the H^2-closure of the algebra of polynomials in g, $H^2[g] = \{f \circ g : f \in H^2\} = \mathrm{Closure}_{H^2}(\mathrm{span}\{g^k : k = 0, 1, \ldots\})$ and s_n, $n = 1, 2, \ldots$, is an orthonormal basis of the $*$-invariant subspace $H^2 \ominus gH^2$. We call (2) the *Wold decomposition* of H^2 *associated with* g.

The Wold decomposition of the Hardy space was investigated in [10] in connection with the description of the lattice of subspaces invariant under multiplication by g. In the case of a finite Blaschke product a similar question was considered by P. Lax [11].

In the case when the inner function g is a Blaschke product (in what follows we call it B) with zeros $\{a_k : k = 1, 2, \ldots\}$, the following special choice of the basis s_n, $n = 1, 2, \ldots$, was considered in [10]:

1991 *Mathematics Subject Classification*: 30D50, 30D55, 32A35.

$$(3)\qquad s_k(z) = \frac{\overline{a}_k}{|a_k|} \cdot \frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z} \prod_{m=1}^{k-1} \frac{\overline{a}_m}{|a_m|} \cdot \frac{a_m - z}{1-\overline{a}_m z}, \qquad k = 1, 2, \ldots$$

It follows from (2) that every H^2-function f is of the form

$$(4)\qquad f = \sum_{k=1}^{\infty} s_k f_k \circ B,$$

where $f_k \circ B \in H^2[B]$. This decomposition gives rise to the following component operators Q_k, $k = 1, 2, \ldots$:

$$(5)\qquad Q_k : H^2 \to H^2, \qquad Q_k(f) = f_k.$$

It was shown in [10] that these operators map bounded functions into bounded functions. Moreover, they can be extended from bounded functions to H^p, $1 \le p \le \infty$, as bounded operators acting on H^p. This fact plays an important role in establishing a Beurling type theorem for B-invariant subspaces.

In this paper we give norm estimates of Q_k as an operator acting on H^p, $p \ge 2$. In Section 3 we prove the following result.

THEOREM 1. *The following upper estimate holds for the norm of Q_k as an operator on H^p for $2 \le p \le \infty$*:

$$\|Q_k\|_{p\to p} \le \left(\frac{1+|B(0)|}{1-|B(0)|}\right)^{5/2}.$$

In the case when each level set of $B|_{\boldsymbol{T}}$ (where $\boldsymbol{T}$ stands for the unit circle) is either countable or its closure has Lebesgue measure zero we establish a lower bound for the norm of Q_k as an operator on H^∞. More precisely, let D be a subset of the unit circle where B has nontangential boundary limits. Then D is a subset of full Lebesgue measure. We denote this limit function on D by the same letter B. For $w \in \boldsymbol{T}$ write

$$E_w = \{z \in D : B(z) = w\}.$$

THEOREM 2. *If for almost all $w \in \boldsymbol{T}$ either $\overline{E}_w$ has Lebesgue measure zero, or E_w is countable, then there is a constant C independent of k and B such that the norm of Q_k as an operator on H^∞ satisfies*

$$\|Q_k\|_{\infty\to\infty} \ge C(1-|a_k|^2)^{1/2} \log \frac{1}{1-|a_k|^2}, \qquad k = 1, 2, \ldots$$

As a corollary we prove that if $B(0) = 0$ and almost all level sets E_w are countable, then there is a Blaschke product F such that

$$\|Q_k F\|_\infty = \|Q_k\|_{\infty\to\infty}, \qquad k = 1, 2, \ldots$$

We apply the above results to the following problem related to operators similar to a contraction.

Let X be a Hilbert space. Recall that an operator $A : X \to X$ is called a *contraction* if $\|A\| \le 1$, and it is *similar to a contraction* if there is a linear isomorphism $C : X \to X$ such that $\|C^{-1}AC\| \le 1$.

Let B be a Blaschke product such that the closure of the set of poles of B lies off the spectrum of A. If A is similar to a contraction, then also $B(A)$ is similar to a contraction. This follows directly from the celebrated inequality of von Neumann [13].

The question is when the converse statement is true. That is, given that $B(A)$ is similar to a contraction, does this imply that A is similar to a contraction? If $B(z) = z^n$ for some n, the affirmative answer to this question was obtained by Halmos and others (see [8]). In [12] V. Mascioni proved that the result holds for any finite Blaschke product (in fact, [12] deals with the more general case of operators on Banach spaces). An alternative proof of the Hilbert version of Mascioni's result was given in [10] by using estimates related to the Wold decomposition of the Hardy space H^2. Neither of these proofs worked for the case of an infinite Blaschke product. It was R. G. Douglas who suggested that the Wold decomposition should be used in the above problem and made a conjecture that the answer is affirmative for all or, at least, some wide class of infinite Blaschke products.

The following result partially confirms Douglas's conjecture.

THEOREM 3. *Let B be a Blaschke product whose zeros satisfy the condition*

$$\sum_{k=1}^{\infty}(1-|a_k|^2)^{1/2} < \infty. \tag{6}$$

If $B(A)$ is similar to a contraction, then also A is similar to a contraction.

The paper is organized as follows. Section 2 contains necessary background results about singular measures generated by inner functions. It was proved by Clark [6] that these measures are spectral measures of one-dimensional perturbations of the shift operator on $*$-invariant subspaces of H^2. They were further investigated by Aleksandrov [2, 3] and Poltoratski [15, 16]. We also state the theorem of Arveson [4] which generalizes the inequality of von Neumann and Paulsen's criteria for an operator to be similar to a contraction [14]. Section 3 is devoted to norm estimates of the component operators Q_k as operators on H^p. We also show how Theorem 3 above follows from these estimates. Section 4 contains some concluding remarks.

Acknowledgements. The author would like to thank R. G. Douglas and J. A. Cima for useful discussions, and the referee for very helpful suggestions about both the content of the paper and the presentation style. In particular, the referee's observation sharpened the estimate in Theorem 1.

2. Background results

2.1. *Measures σ_w.* If $w \in \boldsymbol{T}$, then the harmonic function

$$\varphi(z) = \mathrm{Re}\frac{w + B(z)}{w - B(z)}$$

is positive in the unit disk. By Herglotz's theorem there is a nonnegative measure σ_w on $\boldsymbol{T}$ such that

$$\mathrm{Re}\frac{w + B(z)}{w - B(z)} = \int_{\boldsymbol{T}} \mathrm{Re}\frac{\zeta + z}{\zeta - z}\, d\sigma_w(\zeta).$$

The measures σ_w, $w \in \boldsymbol{T}$, were introduced by Clark [6] in connection with his investigation of one-dimensional pertubations of the shift operator on $*$-invariant subspaces. They are singular measures supported on the level set of B, i.e. E_w. That is,

$$\sigma_w(\boldsymbol{T}\backslash E_w) = 0 \tag{7}$$

(cf. [2]). The following result is due to Aleksandrov.

Theorem A (Aleksandrov [2]). *Any function $f \in L^1(\boldsymbol{T}, dm)$ belongs to $L^1(\boldsymbol{T}, d\sigma_w)$ for almost all $w \in \boldsymbol{T}$ and*

$$\int_{\boldsymbol{T}} f(z)\, dm(z) = \int_{\boldsymbol{T}} \Big(\int_{\boldsymbol{T}} f(z)\, d\sigma_w(z)\Big) dm(w). \tag{8}$$

2.2. *Completely polynomially bounded operators.* Let $D(z)$ be a holomorphic polynomial $(n \times n)$-matrix function in the unit disk Δ. Its ∞-norm is defined by

$$\|D\|_\infty = \sup_{|z|<1} \big(\sup_{|\zeta|\le 1} |D(z)\zeta|\big) \tag{9}$$

where $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and, as usual, $|(b_1, \ldots, b_n)| = (\sum |b_i|^2)^{1/2}$.

If $A : X \to X$ is a bounded operator on a Hilbert space X, then for any polynomial matrix $D(z)$ the operator $D(A) : X^n \to X^n$ is bounded (X^n is equipped with the standard norm $\|(x_1, \ldots, x_n)\|^2 = \sum \|x_i\|^2$). The following generalization of von Neumann's inequality is due to Arveson.

Theorem B (Arveson [4]). *If A is a contraction, then $\|D(A)\| \le \|D\|_\infty$ for any polynomial matrix D.*

An operator A is called *completely polynomially bounded* if there is a constant C such that for any polynomial matrix D,

$$\|D(A)\| \le C\|D\|_\infty.$$

It immediately follows from Arveson's result that if A is similar to a contraction, then A is completely polynomially bounded. The following result by Paulsen states that the converse is also true.

THEOREM C (Paulsen [14]). *If A is completely polynomially bounded, then A is similar to a contraction.*

3. Norm estimates for operators Q_k. Given a Blaschke product B we introduce $L^2[B]$ in a similar way as $H^2[B]$ was introduced in the previous section:

$$L^2[B] = \mathrm{Closure}_{L^2(\boldsymbol{T})}(\mathrm{span}\{B^k|_{\boldsymbol{T}} : k = 0, \pm 1, \ldots\}).$$

Let $P_{L^2[B]}$ and $P_{H^2[B]}$ stand for the orthogonal projections

$$P_{L^2[B]} : L^2(\boldsymbol{T}) \to L^2[B], \qquad P_{H^2[B]} : L^2(\boldsymbol{T}) \to H^2[B].$$

If $\mathcal{B}$ denotes the σ-subalgebra generated by B in the σ-algebra of Lebesgue measurable subsets of $\boldsymbol{T}$, then it is easily seen that $P_{L^2[B]}$ coincides with the conditional expectation operator associated with $\mathcal{B}$ (cf. [7, p. 183]). It was proved in [10] that this leads to the following result: *if $f \in H^p$, $p \ge 2$, then $f_k \in H^p$ for all $k \ge 1$.* Here we give a norm estimate for this inclusion.

Obviously, the restrictions of $P_{L^2[B]}$ and $P_{H^2[B]}$ to H^2 coincide. Since $P_{L^2[B]}$ is a conditional expectation, this implies (cf. [7, p. 184]) that for any $1 \le p \le \infty$ the restriction of $P_{H^2[B]}$ to H^∞ can be extended to a norm one operator $P_{H^p[B]}$ that maps H^p onto $H^p[B] = \mathrm{Closure}_{H^p}(\mathrm{span}\{B^k : k = 0, 1, \ldots\})$.

THEOREM 1. *Let $p \ge 2$. The following estimate for the norm of Q_k as an operator on H^p holds*:

$$\|Q_k\|_{p\to p} \le \left(\frac{1+|B(0)|}{1-|B(0)|}\right)^{5/2}, \qquad k = 1, 2, \ldots \tag{10}$$

To prove this theorem we will need some auxiliary results.

Along with the functions s_k, $k = 1, 2, \ldots$, given by (3) we introduce the following functions t_k:

$$t_k(z) = \frac{\overline{a}_k}{|a_k|} \cdot \frac{z\sqrt{1-|a_k|^2}}{1-\overline{a}_k z} \prod_{j=k+1}^{\infty} \frac{\overline{a}_j}{|a_j|} \cdot \frac{a_j - z}{1-\overline{a}_j z}.$$

Note that both s_k and t_k belong to the $*$-invariant subspace $H^2 \ominus BH^2$ and their restrictions to the unit circle $\boldsymbol{T}$ satisfy the relation

$$t_k|_{\boldsymbol{T}} = -B\overline{s}_k|_{\boldsymbol{T}}. \tag{11}$$

LEMMA 1. *Let $f \in L^2(\boldsymbol{T})$ and $f_0 \circ B = P_{L^2[B]} f$. Then for almost all $w \in \boldsymbol{T}$ we have*

$$\frac{1-|B(0)|^2}{|1-\overline{B(0)}w|^2} f_0(w) = \int_{\boldsymbol{T}} f(z)\, d\sigma_w(z).$$

Proof. This follows from the fact that $P_{L^2[B]}$ coincides with conditional expectation. The direct proof of the statement above is as follows. Let $\varphi \in$

$L^2(\boldsymbol{T}) \ominus L^2[B]$. Write

$$\psi(w) = \int_{\boldsymbol{T}} \varphi(z)\, d\sigma_w(z), \qquad w \in \boldsymbol{T}.$$

By the result of Aleksandrov (Theorem A in this paper) $\psi \in L^1(\boldsymbol{T})$ and for all $k = 0, \pm 1, \pm 2, \ldots$ we have

$$0 = \int_{\boldsymbol{T}} \varphi(z) B(z)^k\, dm(z) = \int_{\boldsymbol{T}} w^k \Big(\int_{\boldsymbol{T}} \varphi(z)\, d\sigma_w(z) \Big) dm(w) = \int_{\boldsymbol{T}} w^k \psi(w)\, dm(w).$$

By the uniqueness theorem this implies $\psi = 0$. If $f = f_0 \circ B + f_1$ where $f_1 \in L^2(\boldsymbol{T}) \ominus L^2[B]$, this and (7) yield

$$\begin{aligned}\int_{\boldsymbol{T}} f(z)\, d\sigma_w(z) &= \int_{\boldsymbol{T}} (f_0 \circ B)(z)\, d\sigma_w(z) \\ &= f_0(w) \int_{\boldsymbol{T}} d\sigma_w(z) = \frac{1 - |B(0)|^2}{|1 - \overline{B(0)} w|^2} f_0(w). \ \blacksquare\end{aligned}$$

Since the restrictions of $P_{L^2[B]}$ and $P_{H^2[B]}$ to H^2 coincide, we obtain the following result.

Corollary. *If* $f \in H^2$ *and* $f_0 \circ B = P_{H^2[B]} f$, *then*

$$\frac{1 - |B(0)|^2}{|1 - \overline{B(0)} w|^2} f_0(w) = \int_{\boldsymbol{T}} f(z)\, d\sigma_w(z) \qquad a.e. \text{ on } \boldsymbol{T}.$$

For $a \in \Delta$ denote by $R_a : H^1 \to H^1$ the following operator:

$$R_a f(z) = (1 - |a|^2) \frac{\dfrac{f(z)}{1 - \overline{a} z} - \dfrac{f(a)}{1 - |a|^2}}{z - a}.$$

It is obvious that R_a maps H^p into H^p for all $1 \le p \le \infty$. If $f(0) = 0$, then the following estimate holds:

$$\|R_a f\|_p \le \frac{(1 + |a|)^{2 - 1/p}}{(1 - |a|)^{1 + 1/p}} \|f\|_p. \tag{12}$$

Indeed, write $h(z) = f(z)/(1 - \overline{a} z)$. Since $h(0) = 0$, the standard H^p-point evaluation implies

$$|h(a)| \le \frac{|a|}{(1 - |a|^2)^{1/p}} \|h\|_p.$$

Now we have

$$\begin{aligned}\|R_a f\|_p &\le (1 - |a|^2) \frac{\|h(\cdot) - h(a)\|_p}{1 - |a|} \le (1 + |a|) \Big(1 + \frac{|a|}{(1 - |a|^2)^{1/p}} \Big) \|h\|_p \\ &\le \frac{(1 + |a|)^{2 - 1/p}}{(1 - |a|)^{1/p}} \|h\|_p \le \frac{(1 + |a|)^{2 - 1/p}}{(1 - |a|)^{1 + 1/p}} \|f\|_p.\end{aligned}$$

LEMMA 2. *Let* $f \in H^2$, $f = \sum_{k=1}^{\infty} s_k f_k \circ B$. *Write* $\widehat{f}_k \circ B = P_{H^2[B]}(f t_k)$. *Then*

$$f_k = -R_{B(0)} \widehat{f}_k.$$

Proof. Write $B(0) = \beta$. It follows from (2) that for all $w \in \Delta$,

$$f_k(w) = \sum_{j=1}^{\infty} \Big(\int_{\boldsymbol{T}} f(z) \overline{s_k(z) B(z)}^j \, dm(z) \Big) w^j = \int_{\boldsymbol{T}} \frac{f(z)\overline{s_k(z)}}{1 - w\overline{B(z)}} \, dm(z).$$

Now, if $w \neq \beta$, then (11), the theorem of Aleksandrov and the Corollary to Lemma 1 yield

$$\begin{aligned} f_k(w) &= -\int_{\boldsymbol{T}} \frac{\overline{\tau}}{1 - w\overline{\tau}} \Big(\int_{\boldsymbol{T}} f(z) t_k(z) \, d\sigma_\tau(z) \Big) \, dm(\tau) \\ &= -(1 - |\beta|^2) \int_{\boldsymbol{T}} \frac{\overline{\tau} \widehat{f}_k(\tau)}{(1 - w\overline{\tau})|1 - \beta\overline{\tau}|^2} \, dm(\tau) \\ &= -\frac{1 - |\beta|^2}{w - \beta} \int_{\boldsymbol{T}} \Big(\frac{1}{1 - w\overline{\tau}} - \frac{1}{1 - \beta\overline{\tau}} \Big) \frac{\widehat{f}_k(\tau)}{1 - \overline{\beta}\tau} \, dm(\tau) = -R_\beta \widehat{f}_k(w). \end{aligned}$$

The case $w = \beta$ follows by continuity. ∎

Proof of Theorem 1. We use the same notation as in Lemma 2. Let $f \in H^p$ and $|w| = 1$. If $2 \le p < \infty$, we have by the Corollary to Lemma 1,

$$\begin{aligned} |\widehat{f}_k(w)|^p &= \Big(\frac{|1 - \overline{\beta} w|^2}{1 - |\beta|^2} \Big)^p \Big| \int_{\boldsymbol{T}} f(z) t_k(z) \, d\sigma_w(z) \Big|^p \\ &\le \Big(\frac{|1 - \overline{\beta} w|^2}{1 - |\beta|^2} \Big)^p \Big(\int_{\boldsymbol{T}} |f(z)|^p \, d\sigma_w(z) \Big) \Big(\int_{\boldsymbol{T}} |t_k(z)|^{p'} \, d\sigma_w(z) \Big)^{p/p'} \\ &\le \Big(\frac{|1 - \overline{\beta} w|^2}{1 - |\beta|^2} \Big)^p \Big(\int_{\boldsymbol{T}} |f(z)|^p \, d\sigma_w(z) \Big) \\ &\quad \times \Big(\int_{\boldsymbol{T}} |t_k(z)|^2 \, d\sigma_w(z) \Big)^{p/2} \Big(\int_{\boldsymbol{T}} d\sigma_w(z) \Big)^{(p-2)/2} \\ &= \Big(\frac{|1 - \overline{\beta} w|^2}{1 - |\beta|^2} \Big)^{1+p/2} \int_{\boldsymbol{T}} |f(z)|^p \, d\sigma_w(z) \\ &\le \Big(\frac{1 + |\beta|}{1 - |\beta|} \Big)^{1+p/2} \int_{\boldsymbol{T}} |f(z)|^p \, d\sigma_w(z). \end{aligned}$$

In the last inequality we used the facts that

$$\int_{\boldsymbol{T}} d\sigma_w(z) = \mathrm{Re} \frac{w + \beta}{w - \beta} = \frac{1 - |\beta|^2}{|1 - \overline{\beta} w|^2} \le \frac{1 + |\beta|}{1 - |\beta|}$$

and

$$\int_{\boldsymbol{T}} |t_k(z)|^2 \, d\sigma_w(z) = \mathrm{Re} \frac{w + B(a_k)}{w - B(a_k)} = 1.$$

After integration with respect to $dm(w)$ over the unit circle, application of the theorem of Aleksandrov and raising to the power $1/p$ we obtain

$$\|\widehat{f}_k\|_p \le \left(\frac{1 + |\beta|}{1 - |\beta|} \right)^{1/2+1/p} \|f\|_p.$$

Since $t_k(0) = 0$, we obviously have $\widehat{f}(0) = 0$. Now, since $p \ge 2$, by (12) and Lemma 2 we get

$$\|Q_k f\|_p = \|f_k\|_p = \|R_\beta \widehat{f}_k\|_p \le \frac{(1 + |\beta|)^{5/2}}{(1 - |\beta|)^{3/2+2/p}} \|f\|_p \le \left(\frac{1 + |\beta|}{1 - |\beta|} \right)^{5/2} \|f\|_p.$$

Since for any bounded analytic function its H^∞-norm is the supremum of H^p-norms, the case $p = \infty$ follows from the above by passing to the limit as $p \to \infty$. ∎

REMARK. In fact, we proved that

$$\|Q_k\|_{p \to p} \le \frac{(1 + |\beta|)^{3-1/p}}{(1 - |\beta|)^{2+1/p}} \sup_{w \in \boldsymbol{T}} \|t_k\|_{L^{p'}(\boldsymbol{T}, d\sigma_w)}, \tag{13}$$

where, as usual, $1/p + 1/p' = 1$.

COROLLARY. *If f is a disk-algebra function, then $Q_k f$ are disk-algebra functions for all $k = 1, 2, \ldots$*

Proof. First we note that if q is a polynomial and

$$q = \sum_{k=1}^{\infty} s_k q_k \circ B$$

is the Wold decomposition (4), then for all $k \ge 1$, q_k are functions analytic in the closed unit disk (if $B(0) = 0$, all q_k are polynomials of the same degree as q or lower). Indeed, it is enough to prove this for monomials. Let $q(z) = z^n$. For $|w| < 1$ we have

$$\begin{aligned}(z^n)_k(w) &= \sum_{l=0}^{\infty} \Big(\int_{\boldsymbol{T}} z^n \overline{s_k(z) B(z)}^{\,l} \, dm(z) \Big) w^l = \int_{\boldsymbol{T}} z^n \frac{\overline{s_k(z)}}{1 - w\overline{B(z)}} \, dm(z) \\ &= \overline{\frac{1}{n!} \cdot \frac{d^n}{dz^n} \cdot \frac{s_k(z)}{1 - \overline{w} B(z)} \bigg|_{z=0}} = \frac{\gamma(w)}{(1 - \overline{B(0)} w)^{n+1}},\end{aligned}$$

where $\gamma(w)$ is a polynomial of degree n in w. Obviously, this function is analytic in the disk $\{|w| < 1/|B(0)|\}$.

Since any disk-algebra function f is the uniform limit of a sequence of polynomials, Theorem 1 and the above computation show that $Q_k f$ is the limit of a sequence of rational functions analytic in the disk of radius $1/|B(0)|$ which converges uniformly in the closed unit disk. ■

THEOREM 2. *If for almost all* $w \in \boldsymbol{T}$ *the set* E_w *is either countable, or satisfies the condition* $\operatorname{mes}\{\overline{E}_w\} = 0$, *then there is a constant* C *independent of* k *and of the Blaschke product* B *such that the norm of* Q_k *as an operator on* H^∞ *satisfies the following lower estimate*:

$$\|Q_k\|_{\infty\to\infty} \ge C\sqrt{1-|a_k|^2}\log\frac{1}{1-|a_k|^2}.$$

Proof. Write

$$(14)\qquad {}_k\varphi(z) = \overline{z}\sqrt{\frac{1-\overline{a}_k z}{1-a\overline{z}}}\prod_{j=1}^{k}\frac{\overline{a}_j}{|a_j|}\cdot\frac{a_j - z}{1-\overline{a}_j z}.$$

Note that the restriction of ${}_k\varphi$ to the unit circle is continuous and unimodular. Let $w \in \boldsymbol{T}$. If $\operatorname{mes}\{\overline{E}_w\} = 0$, then by the theorem of Rudin [17] there is a disk-algebra function ${}_k\psi(z)$ of ∞-norm one whose restriction to $\overline{E}_w$ is equal to ${}_k\varphi|_{\overline{E}_w}$.

Now we have

$$|(\widehat{{}_k\psi})_k(w)| = \frac{|1-\overline{\beta}w|^2}{1-|\beta|^2}\Big|\int_{\boldsymbol{T}} t_k(z)\,{}_k\psi(z)\,d\sigma_w(z)\Big|$$
$$= \frac{|1-\overline{\beta}w|^2}{1-|\beta|^2}\int_{\boldsymbol{T}}\frac{\sqrt{1-|a_k|^2}}{|1-\overline{a}_k z|}\,d\sigma_w(z) = \frac{|1-\overline{\beta}w|^2}{1-|\beta|^2}\|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}.$$

Let $h_n = {}_k\psi B^n$. Then

$$(\widehat{h}_n)_k \circ B = P_{H^2[B]}({}_k\psi B^n t_k) = B^n(\widehat{{}_k\psi})_k \circ B.$$

This implies

$$(\widehat{h}_n)_k(\tau) = \tau^n(\widehat{{}_k\psi})_k(\tau), \qquad \tau \in \Delta.$$

Therefore,

$$(\widehat{h}_n)_k(\beta) \to 0 \qquad \text{as } n \to \infty.$$

Since $\|h_n\|_\infty = 1$ for all $n = 1, 2, \ldots$, we obtain

$$\|Q_k\|_{\infty\to\infty} \ge \sup_n\{\|Q_k h_n\|_\infty\} \ge \lim_{n\to\infty}|Q_k h_n(w)|$$
$$= \frac{|1-\overline{\beta}w|^2}{1-|\beta|^2}(1-|\beta|^2)\frac{\|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}}{|w-\beta|\cdot|1-\overline{\beta}w|}$$
$$= \left|\frac{1-\overline{\beta}w}{w-\beta}\right|\|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)} = \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}.$$

Thus,

$$(15)\qquad \|Q_k\|_{\infty\to\infty} \ge \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}.$$

If E_w is countable, but its closure has positive Lebesgue measure, instead of Rudin's theorem we use the theorem of Belna, Colwell and Piranian [5]. Applied to this case it claims the existence of a Blaschke product which has nontangential limits at every point in E_w and such that these limits coincide with the restriction of φ_k to E_w. Then the above argument shows that (15) holds in this case as well.

Since almost all E_w are either countable or their closures have Lebesgue measure zero, we conclude

$$\|Q_k\|_{\infty\to\infty} \ge \sup_{w\in\boldsymbol{T}} \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}.$$

It is well known that $\|1/(1-\overline{a}z)\|_{L^1(\boldsymbol{T},dm)}$ has the order of $\log(1/(1-|a|^2))$ for a in the unit disk (cf. [18], p. 18). Therefore, for some constant C we have

$$\begin{aligned} C\sqrt{1-|a_k|^2}\log\frac{1}{1-|a_k|^2} &\le \|t_k\|_{L^1(\boldsymbol{T},dm)} = \int_{\boldsymbol{T}} |t_k(z)|\,dm(z) \\ &= \int_{\boldsymbol{T}}\Big(\int_{\boldsymbol{T}} |t_k(z)|\,d\sigma_w(z)\Big)dm(w) \\ &= \int_{\boldsymbol{T}} \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}\,dm(w) \\ &\le \sup_{w\in\boldsymbol{T}} \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}. \ \blacksquare \end{aligned}$$

Remark. It immediately follows from (13) and Theorem 2 that if $B(0) = 0$, then

$$\|Q_k\|_{\infty\to\infty} = \sup_{w\in\boldsymbol{T}} \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}. \tag{16}$$

In this case Lemma 2 claims that for almost all $w \in \boldsymbol{T}$,

$$Q_k f(w) = f_k(w) = -\overline{w}\int_{\boldsymbol{T}} f(z)t_k(z)\,d\sigma_w(z). \tag{17}$$

Corollary 1. *If a Blaschke product B is such that $B(0) = 0$ and almost all level sets E_w, $w \in \boldsymbol{T}$, are countable, then there is an H^∞-function φ, $\|\varphi\| = 1$, such that*

$$\|Q_k\varphi\|_\infty = \|Q_k\|_{\infty\to\infty} \quad \textit{for all } k = 1, 2, \ldots$$

Proof. Fix $\{w_{k,n}\}_{k,n=1}^\infty$, $|w_{k,n}| = 1$, $k, n = 1, 2, \ldots$, such that $E_{w_{k,n}}$ is countable for all k, n and

$$\sup_n \|t_k\|_{L^1(\boldsymbol{T},d\sigma_{w_{k,n}})} = \sup_{w\in\boldsymbol{T}} \|t_k\|_{L^1(\boldsymbol{T},d\sigma_w)}.$$

Define a function φ on $E = \bigcup_{k,n=1}^\infty E_{w_{k,n}}$ by

$$\varphi|_{E_{w_{k,n}}} = ({}_k\varphi)|_{E_{w_{k,n}}}$$

where ${}_k\varphi$ are the functions (14). Since E is countable, the result of Belna, Colwell and Piranian [5] mentioned above implies that there is a Blaschke product F whose nontangential limits coincide with φ everywhere on E. Now, the argument of Theorem 2, (16) and (17) yield the claimed equality. ∎

COROLLARY 2. *If* $B(0)=0$ *and either the zeros of the Blaschke product* B *satisfy the condition*

$$\sum_{k=1}^{\infty}(1-|a_k|)^{1/2}<\infty, \tag{18}$$

or the set of accumulation points of zeros of B *has linear Lebesgue measure* 0 *in the unit circle*, *then there is a Blaschke product* F *such that for all* $k=1,2,\ldots,$

$$\|Q_kF\|_\infty=\|Q_k\|_{\infty\to\infty}\|F\|_\infty\ge C\sqrt{1-|a_k|^2}\log\frac{1}{1-|a_k|^2}. \tag{19}$$

Proof. In both cases the level sets E_w are countable. If (18) holds, this follows from the result of Ahern and Clark [1]. In the second case, B has angular derivatives at every point of an open set complement to the points of accumulation of zeros of B, which has full measure. By the result of Aleksandrov [2, Proposition 4] this implies that all level sets of B are countable. Now the required statement follows from Corollary 1. ∎

Finally, we notice that the following vector form of Theorem 1 holds.

Let $D(z)$ be a polynomial matrix function,

$$D(z)=\|d_{ij}(z)\|_{i,j=1}^n.$$

Any polynomial d_{ij} has a representation (4),

$$d_{ij}=\sum_{k=1}^{\infty}s_kd_{ijk}\circ B.$$

This leads to the following decomposition, which we call the *Wold decomposition of* D *associated with* B:

$$D=\sum_{k=1}^{\infty}s_kD_k\circ B \tag{20}$$

where

$$D_k\circ B=\|d_{ijk}\circ B\|_{i,j=1}^n.$$

If we denote by $\mathcal{M}_n$ the space of $(n\times n)$-matrix functions in the unit disk whose entries are in H^∞ equipped with the ∞-norm given by (9), then we can introduce operators

$$Q_{n,k}:\mathcal{M}_n\to\mathcal{M}_n,\qquad Q_{n,k}(D)=D_k.$$

The following result is the extension of Theorem 1 to the vector case. Its proof goes along the same lines as the proof of Theorem 1 and, thus, we omit it.

THEOREM 1′. *For all* $n = 1, 2, \ldots$ *and* $k = 1, 2, \ldots$ *the norm of* $Q_{n,k}$ *satisfies the estimate*

$$\|Q_{n,k}\|_{\mathcal{M}_n \to \mathcal{M}_n} \le \left(\frac{1 + |B(0)|}{1 - |B(0)|}\right)^{5/2}.$$

As a direct corollary to this theorem we obtain the following result about operators similar to a contraction.

Let $A : X \to X$ be a bounded operator in a Hilbert space X. Following the standard notation we let $\mathrm{Sp}(A)$ stand for the spectrum of A. Let B be a Blaschke product,

$$B(z) = z^l \prod_{k=1}^{\infty} \frac{\overline{a}_k}{|a_k|} \cdot \frac{a_k - z}{1 - \overline{a}_k z},$$

and

$$\overline{\{1/\overline{a}_k\}_{k=1}^{\infty}} \cap \mathrm{Sp}(A) = \emptyset. \tag{21}$$

It is easy to show that in this case the following infinite product converges and defines a bounded operator on X:

$$B(A) = A^l \prod_{k=1}^{\infty} \frac{\overline{a}_k}{|a_k|} (a_k - A)(1 - \overline{a}_k A)^{-1}. \tag{22}$$

THEOREM 3. *Let* B *be a Blaschke product whose zeros satisfy the condition*

$$\sum_{k=1}^{\infty} (1 - |a_k|^2)^{1/2} < \infty$$

and $A : X \to X$ *be a bounded operator in a Hilbert space* X *such that*

$$\overline{(\{1/\overline{a}_k\}_{k=1}^{\infty})} \cap \mathrm{Sp}(A) = \emptyset.$$

If $B(A)$ *is similar to a contraction, then also* A *is similar to a contraction.*

Proof. Let $B(A) = C^{-1} S C$ where $C : X \to X$ is a linear isomorphism of X and $S : X \to X$ is a contraction. We will show that A is completely polynomially bounded.

Let D be a polynomial matrix. The decomposition (20) yields

$$D(A) = \sum_{k=1}^{\infty} s_k(A) D_k(B(A)). \tag{23}$$

Since the spectrum of A is off the poles of B, (3) implies that there is a constant M such that

(24) $$\|s_k(A)\| \le M\sqrt{1-|a_k|^2}, \qquad k=1,2,\ldots$$

The Corollary to Theorem 1 implies that Arveson's theorem (Theorem B) can be applied to $D_k(S)$, $k=1,2,\ldots$ Now, Theorem 1′, (24), (25) and Arveson's theorem yield

$$\begin{aligned}\|D(A)\| &\le \sum_{k=1}^{\infty} \|s_k(A)\| \cdot \|C\| \cdot \|C^{-1}\| \cdot \|D_k(S)\| \\ &\le M \cdot \|C\| \cdot \|C^{-1}\| \Big(\frac{1+|B(0)|}{1-|B(0)|}\Big)^{5/2} \Big(\sum_{k=1}^{\infty}(1-|a_k|^2)^{1/2}\Big)\|D\|_\infty.\end{aligned}$$

Thus, A is completely polynomially bounded and, therefore, by Paulsen's theorem (Theorem C) it is similar to a contraction. ■

4. Concluding remarks. It would be very interesting to find sharp upper and lower estimates of norms of component operators Q_k since these estimates are crucial for the description of the lattice of multiplication invariant subspaces in Hardy spaces (cf. [10]). It was already mentioned in the Remark after the proof of Theorem 1 that the upper estimate in that theorem was not the best possible: for $p > 2$ the relation (13) gives a better one. Still, (13) is based on the estimate (12) which is rather rough.

In the case when the Blaschke product vanishes at the origin and has countable multiplicity on the boundary, and $p = \infty$, the norm of Q_k is given by (16). It would be interesting to know if the same result holds without the assumption of countable boundary multiplicity.

Further, our method of getting lower estimates (Theorem 2) requires countable boundary multiplicity since it is based on the application of the result by Belna, Colwell and Piranian [5]. The author is unaware of a result similar to [5] for uncountable sets of zero Lebesgue measure.

Finally, lower estimates in the case $p < \infty$ would be of considerable interest as well. The difficulty here obviously lies in the problem of interpolation (or rather interpolation and approximation) on a set of positive Lebesgue measure.

REFERENCES

[1] P. R. Ahern and D. N. Clark, *On inner functions with B^p derivatives*, Michigan Math. J. 23 (1976), 393–396.

[2] A. B. Aleksandrov, *Multiplicity of boundary values of inner functions*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 22 (1987), 490–503 (in Russian).

[3] A. B. Aleksandrov, *Inner functions and related spaces of pseudocontinuable functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 170 (1989), 7–33 (in Russian); English transl.: J. Soviet Math. 63 (2) (1993).
[4] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), 141–224.
[5] C. L. Belna, P. Colwell and G. Piranian, *The radial limits of Blaschke products*, Proc. Amer. Math. Soc. 93 (1985), 267–271.
[6] D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Anal. Math. 25 (1972), 169–191.
[7] J. L. Doob, *Measure Theory*, Springer, New York, 1994.
[8] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. 76 (1970), 887–933.
[9] —, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961), 102–112.
[10] T. L. Lance and M. I. Stessin, *Multiplication invariant subspaces of Hardy spaces*, Canad. J. Math. 49 (1997), 100–118.
[11] P. Lax, *Translation invariant subspaces*, Acta Math. 101 (1959), 163–178.
[12] V. Mascioni, *Ideals of the disk algebra, operators related to Hilbert space contractions and complete boundedness*, Houston J. Math. 20 (1994), 299–311.
[13] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. 4 (1950/51), 258–281.
[14] V. I. Paulsen, *Every completely polynomially bounded operator is similar to a contraction*, J. Funct. Anal. 55 (1984), 1–17.
[15] A. G. Poltoratski, *The boundary behavior of pseudocontinuable functions*, St. Petersburg Math. J. 5 (1994), 389–406.
[16] —, *On the distributions of the boundary values of Cauchy integrals*, Proc. Amer. Math. Soc. 124 (1996), 2455–2463.
[17] W. Rudin, *Boundary values of continuous analytic functions*, ibid. 7 (1956), 808–811.
[18] —, *Function Theory in the Unit Ball of C^n*, Springer, New York, 1980.

Department of Mathematics and Statistics
University at Albany
Albany, NY 12222, U.S.A.
E-mail: stessin@math.albany.edu

Received 29 October 1998;
revised 2 March 1999