COLLOQUIUM MATHEMATICUM

VOL. 81 1999 NO. 2

POLYNOMIAL ALGEBRA OF CONSTANTS OF THE LOTKA–VOLTERRA SYSTEM

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Abstract. Let k be a field of characteristic zero. We describe the kernel of any quadratic homogeneous derivation $d : k[x,y,z] \to k[x,y,z]$ of the form $d = x(Cy+z)\frac{\partial}{\partial x} + y(Az+x)\frac{\partial}{\partial y} + z(Bx+y)\frac{\partial}{\partial z}$, called the Lotka–Volterra derivation, where $A, B, C \in k$.

1. Introduction. Let $k[x,y,z]$ be the algebra of polynomials in three variables x, y, z over a field k of characteristic zero. By a *derivation* of $k[x,y,z]$ we mean a k-linear mapping $d : k[x,y,z] \to k[x,y,z]$ of the form

$$d = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z},$$

where $f, g, h \in k[x,y,z]$. If the polynomials f, g, h are homogeneous of the same degree s, then we say that d is *homogeneous of degree* s.

For a given derivation d of $k[x,y,z]$ we denote by $k[x,y,z]^d$ the kernel of d, that is,

$$k[x,y,z]^d = \{w \in k[x,y,z] : d(w) = 0\}.$$

The set $k[x,y,z]^d$ is a k-subalgebra of $k[x,y,z]$ containing k, called *the k-algebra of constants of d*. The set $k[x,y,z]^d \setminus k$ coincides with the set of polynomial first integrals of the corresponding system

$$\dot{x} = f(x,y,z), \quad \dot{y} = g(x,y,z), \quad \dot{z} = h(x,y,z),$$

of ordinary differential equations in three variables (see [4], [5] or [6] for details).

It is well known ([7], [5]), that the algebra $k[x,y,z]^d$ is finitely generated over k. This means that either

$$k[x,y,z]^d = k$$

or there exist polynomials $f_1, \ldots, f_r \in k[x,y,z] \setminus k$ (where $r \ge 1$) such that

$$k[x,y,z]^d = k[f_1, \ldots, f_r],$$

1991 *Mathematics Subject Classification*: Primary 12H05; Secondary 13B25, 58Fxx.
Supported by KBN Grant 2 PO3A 017 16.

where $k[f_1, \ldots, f_r]$ means the smallest k-subalgebra of $k[x, y, z]$ containing k and $f_1, \ldots, f_r$.

The minimal number of generators of $k[x, y, z]^d$ is not bounded when d runs over the set of all derivations of $k[x, y, z]$, and even if d runs over the set of all homogeneous derivations of degree 1 (see [8]). If $k[x, y, z]^d = k$ then we say that the algebra of constants is *trivial*.

Assume now that A, B, C are elements of k. Following [1], by a *Lotka–Volterra derivation* defined by the triple (A, B, C) we mean the derivation $d : k[x, y, z] \to k[x, y, z]$ given by the formula

$$d = x(Cy + z)\frac{\partial}{\partial x} + y(Az + x)\frac{\partial}{\partial y} + z(Bx + y)\frac{\partial}{\partial z}. \tag{1.1}$$

Note that d is a quadratic homogeneous derivation such that

$$d(x) = x(Cy + z), \quad d(y) = y(Az + x), \quad d(z) = z(Bx + y).$$

The autonomous system of differential equations, corresponding to the polynomials $x(Cy + z)$, $y(Az + x)$, $z(Bx + y)$, is called the *Lotka–Volterra system*. This system has been studied for a long time; see for example [1], [2], [3], where many references on this subject can be found. We are interested in an algebraic description of the k-algebra $k[x, y, z]^d$.

In [1; pp. 687–689] a list of polynomials belonging to $k[x, y, z]^d$ is presented. In [2] the first named author characterizes all Lotka–Volterra derivations d such that $k[x, y, z]^d \neq k$, as follows.

THEOREM 1.2 ([3]). *Let $d : k[x, y, z] \to k[x, y, z]$ be the Lotka–Volterra derivation* (1.1) *with respect to* (A, B, C). *The algebra* $k[x, y, z]^d$, *of constants of d, is non-trivial if and only if one of the following cases holds*:

(1) $ABC = -1$.

(2) $C = -1 - 1/A$, $A = -1 - 1/B$ *and* $B = -1 - 1/C$.

(3) $C = -k_2 - 1/A$, $A = -k_3 - 1/B$, $B = -k_1 - 1/C$, *where*, *up to a permutation*, (k_1, k_2, k_3) *is one of the triples*: $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$.

The polynomials $x - Cy + ACz$ *and* $A^2B^2x^2 + y^2 + A^2z^2 - 2ABxy - 2A^2Bxz - 2Ayz$ *belong to* $k[x, y, z]^d$ *in the cases* (1) *and* (2), *respectively. In each of the cases of* (3), *there exists a homogeneous polynomial in* $k[x, y, z]^d$ *of degree* 3, 4 *or* 6 *respectively.* ∎

The main result of the present paper is the following theorem, giving a complete description of the algebra $k[x, y, z]^d$ of constants in each of the cases (1), (2), (3) in Theorem 1.2.

THEOREM 1.3. *Let $k[x, y, z]$ be the algebra of polynomials in three variables over a field k of characteristic zero. Let $d : k[x, y, z] \to k[x, y, z]$ be a Lotka–Volterra derivation* (1.1) *such that* $k[x, y, z]^d \neq k$.

(1) *Assume that* $ABC = -1$ *and let* $\mathbb{Q}_- \subseteq k$ *be the set of negative rational numbers.*

(1a) *If* $A, B, C \in \mathbb{Q}_-$ *then there exist positive integers* p, q, r *such that* $\gcd(p, q, r) = 1$, $A = -\frac{p}{q}$, $B = -\frac{q}{r}$, $C = -\frac{r}{p}$, *and* $k[x, y, z]^d = k[t, w]$, *where*

$$\begin{cases} t = pqx + rqy + rpz, \\ w = x^p y^q z^r. \end{cases}$$

(1b) *If some of the scalars* A, B, C *belongs to* $k \setminus \mathbb{Q}_-$ *then* $k[x, y, z]^d = k[x - Cy + ACz]$.

(2) *If* $C = -1 - 1/A$, $A = -1 - 1/B$ *and* $B = -1 - 1/C$, *then* $k[x, y, z]^d = k[g]$, *where* $g = A^2B^2x^2 + y^2 + A^2z^2 - 2ABxy - 2A^2Bxz - 2Ayz$.

(3) *Let* $C = -k_2 - 1/A$, $A = -k_3 - 1/B$, $B = -k_1 - 1/C$, *where, up to a permutation,* (k_1, k_2, k_3) *is one of the triples*: $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$. *In every case there exists a homogeneous irreducible polynomial* g *in* $k[x, y, z]$ (*of degree* 3, 4 *or* 6, *respectively*) *such that* $k[x, y, z]^d = k[g]$.

The proof of Theorem 1.3 is presented in Section 5 and is based on a sequence of preparatory results given in Sections 2–4.

2. Darboux polynomials and strict polynomial constants. Assume that $d : k[x, y, z] \to k[x, y, z]$ is the Lotka–Volterra derivation with respect to (A, B, C).

We say that a nonzero polynomial $f \in k[x, y, z]$ is a *Darboux polynomial* of d if $d(f) = hf$ for some $h \in k[x, y, z]$. In this case the polynomial h is unique and it is called the *eigenvalue* of f.

It is easy to show that the product of Darboux polynomials is a Darboux polynomial. Moreover, if $f \in k[x, y, z]$ is a Darboux polynomial then so is each factor of f. Nonzero polynomials which belong to $k[x, y, z]^d$ are simply Darboux polynomials with the zero eigenvalue.

The variables x, y, z are Darboux polynomials with the eigenvalues $Cy + z$, $Az + x$, $Bx + y$, respectively. Every monomial $x^\alpha y^\beta z^\gamma$ is a Darboux polynomial with the eigenvalue equal to

$$\alpha(Cy + z) + \beta(Az + x) + \gamma(Bx + y).$$

We say that a polynomial $g \in k[x, y, z]$ is *strict* if g is nonzero, homogeneous and not divisible by x, y or z. Every nonzero homogeneous polynomial $f \in k[x, y, z]$ has a unique representation

$$f = x^\alpha y^\beta z^\gamma g,$$

where α, β, γ are nonnegative integers and $g \in k[x, y, z]$ is strict.

Let us recall the following result.

PROPOSITION 2.1 ([2], [3]). *If g is a strict Darboux polynomial of d then its eigenvalue is a linear form*

$$\lambda x + \mu y + \nu z,$$

where λ, μ, ν are nonnegative integers. ∎

Using Proposition 2.1 we get an important consequence.

PROPOSITION 2.2. *Let $g \in k[x, y, z]$ be a strict polynomial and let $g = g_1 g_2$, for some $g_1, g_2 \in k[x, y, z]$. If $d(g) = 0$ then $d(g_1) = d(g_2) = 0$.*

Proof. Let $d(g) = 0$. Then g_1, g_2 are strict Darboux polynomials of d, and hence (by Proposition 2.1) $d(g_1) = h_1 g_1$, $d(g_2) = h_2 g_2$, where $h_1 = \lambda_1 x + \mu_1 y + \nu_1 z$, $h_2 = \lambda_2 x + \mu_2 y + \nu_2 z$, for some nonnegative integers λ_1, μ_1, ν_1, λ_2, μ_2 and ν_2. The equalities $0 = d(g) = d(g_1 g_2) = (h_1 + h_2) g$ imply that $h_1 + h_2 = 0$, and hence $\lambda_1 + \lambda_2 = 0$, $\mu_1 + \mu_2 = 0$ and $\nu_1 + \nu_2 = 0$, that is, $\lambda_1 = \mu_1 = \nu_1 z = \lambda_2 = \mu_2 = \nu_2 = 0$. Therefore $d(g_1) = 0 g_1 = 0$, $d(g_2) = 0 g_2 = 0$. ∎

COROLLARY 2.3. *If the set $k[x, y, z]^d \setminus k$ contains a strict polynomial then it contains a strict irreducible polynomial.* ∎

Now we recall some facts from [2].

PROPOSITION 2.4 ([2]). *If $k[x, y, z]^d \neq k$, then $A \neq 0$, $B \neq 0$ and $C \neq 0$.* ∎

PROPOSITION 2.5 ([2]). *If g is a strict polynomial of degree m, belonging to $k[x, y, z]^d$, then*

$$g(0, y, z) = a(y - Az)^m, \quad g(x, 0, z) = b(z - Bx)^m, \quad g(x, y, 0) = c(x - Cy)^m,$$

for some nonzero elements $a, b, c \in k$. Moreover, $a = c(-C)^m$, $b = a(-A)^m$ and $c = b(-B)^m$. ∎

PROPOSITION 2.6 ([2]). *The ring $k[x, y, z]^d$ contains a nonzero homogeneous polynomial of degree 1 if and only if $ABC = -1$.* ∎

3. Monomial constants. In this section we characterize all the Lotka–Volterra derivations d such that the algebra $k[x, y, z]^d$ contains a nontrivial monomial.

Assume again that $d : k[x, y, z] \to k[x, y, z]$ is the Lotka–Volterra derivation with respect to (A, B, C).

PROPOSITION 3.1. *The following two conditions are equivalent*:

(1) *The set $k[x, y, z]^d \setminus k$ contains a monomial.*

(2) *The parameters A, B, C are negative rational numbers and $ABC = -1$.*

Proof. (1)⇒(2). Let $d(x^\alpha y^\beta z^\gamma) = 0$, where α, β, γ are nonnegative integers with $\alpha+\beta+\gamma > 0$. Then $\alpha(Cy+z)+\beta(Az+x)+\gamma(Bx+y) = 0$ and so

$$\alpha C = -\gamma, \quad \beta A = -\alpha, \quad \gamma B = -\beta.$$

If $\alpha = 0$, then $\gamma = -0C = 0$, $\beta = -0B = 0$, and we have a contradiction because $\alpha+\beta+\gamma > 0$. Hence $\alpha > 0$ and analogously $\beta > 0$, $\gamma > 0$. This implies that $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$ are negative rational numbers and $ABC = (-\alpha/\beta)(-\beta/\gamma)(-\gamma/\alpha) = -1$.

(2)⇒(1). If A, B, C are negative rational numbers and $ABC = -1$, then there exist integers $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$. Then $d(x^\alpha y^\beta z^\gamma) = 0$. ∎

Let us note the following corollary from the above proof.

COROLLARY 3.2. *Let α, β, γ be nonnegative integers with $\alpha+\beta+\gamma > 0$. If $d(x^\alpha y^\beta z^\gamma) = 0$, then $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $A = -\alpha/\beta$, $B = -\beta/\gamma$, $C = -\gamma/\alpha$.* ∎

We say that a monomial $x^p y^q z^r$ is *primitive* if $p > 0$, $q > 0$, $r > 0$ and $\gcd(p, q, r) = 1$. As a consequence of the above facts we obtain

COROLLARY 3.3. *Assume that the set $k[x, y, z]^d \setminus k$ contains a monomial. Then there exists a unique primitive monomial w belonging to $k[x, y, z]^d$. Every monomial belonging to $k[x, y, z]^d$ is, up to a nonzero coefficient, a power of w.* ∎

Let us also note a fact from [2].

PROPOSITION 3.4 ([2]). *Let $f = x^\alpha y^\beta z^\gamma g$, where α, β, γ are nonnegative integers and $g \in k[x, y, z]$ is strict. If $d(f) = 0$, then $d(x^\alpha y^\beta z^\gamma) = 0$ and $d(g) = 0$.* ∎

4. The algebra of constants. The following theorem describes the algebra $k[x, y, z]^d$ in the case when a monomial belongs to $k[x, y, z]^d \setminus k$. This proves the statement (1a) of Theorem 1.3.

THEOREM 4.1. *Let d be a Lotka–Volterra derivation with respect to $(-p/q, -q/r, -r/p)$, where p, q, r are positive integers and $\gcd(p, q, r) = 1$. Then $k[x, y, z]^d = k[t, w]$, where*

$$t = pqx + rqy + rpz, \quad w = x^p y^q z^r.$$

Proof. It is clear that $k[t, w] \subseteq k[x, y, z]^d$. Since d is homogeneous, it is sufficient to prove that if $f \in k[x, y, z]$ is a homogeneous polynomial such that $d(f) = 0$, then $f \in k[t, w]$. Assume therefore that $0 \neq f \in k[x, y, z]^d$ and f is homogeneous.

Let $f = x^\alpha y^\beta z^\gamma g$, where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$ and $g \in k[x,y,z]$ is strict. Then $d(x^\alpha y^\beta z^\gamma) = 0$ and $d(g) = 0$ (see Proposition 3.4).

The equality $d(x^\alpha y^\beta z^\gamma) = 0$ implies (by Corollary 3.3) that $x^\alpha y^\beta z^\gamma$ is, up to a nonzero coefficient, a power of w (because w is a unique primitive monomial belonging to $k[x,y,z]^d$). This means that $x^\alpha y^\beta z^\gamma$ belongs to $k[t,w]$.

Therefore it is sufficient to prove that if g is a strict polynomial belonging to $k[x,y,z]^d$, then $g \in k[t,w]$. We will prove it by induction on the degree of g. If $\deg g = 1$ then it is obvious. Assume now that $\deg g = m > 1$.

Since g is strict, there exists (by Proposition 2.5) a nonzero element $c \in k$ such that

$$g(x,y,0) = c\left(x + \frac{r}{p}y\right)^m.$$

Consider now the polynomial

$$h = g - \frac{c}{p^m q^m}t^m.$$

It is a homogeneous polynomial belonging to $k[x,y,z]^d$. Observe that

$$h(x,y,0) = c\left(x + \frac{r}{p}y\right)^m - \frac{c}{p^m q^m}(pqx + rqy)^m = 0.$$

This implies that h is divisible by z.

If $h = 0$, then

$$g = \frac{c}{p^m q^m}t^m \in k[t,w].$$

Suppose now that $h \neq 0$. Then $h = x^\alpha y^\beta z^\gamma g_1$, where $g_1 \in k[x,y,z]$ is strict and $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma \geq 1$. The equality $d(h) = 0$ implies (by Proposition 3.4) that $d\left(x^\alpha y^\beta z^\gamma\right) = 0$ and $d(g_1) = 0$. But $\deg g_1 < \deg g$ so, by induction, $g_1 \in k[t,w]$. Moreover, the monomial $x^\alpha y^\beta z^\gamma$ also belongs to $k[t,w]$, because it is (by Corollary 3.3), up to a nonzero coefficient, a power of w. Therefore $g \in k[t,w]$. ∎

EXAMPLE 4.2. Let d be the derivation of $k[x,y,z]$ such that

$$d(x) = x(z-y), \quad d(y) = y(x-z), \quad d(z) = z(y-x).$$

Then (by Theorem 4.1) $k[x,y,z]^d = k[x+y+z, xyz]$. It is easy to check that d coincides with the jacobian derivation $\mathrm{Jac}(xyz, x+y+z, \_)$. ∎

The next theorem decribes the algebra of constants in the case when the set $k[x,y,z]^d \setminus k$ has no monomials.

THEOREM 4.3. *Let d be a Lotka–Volterra derivation. Assume that $k[x,y,z]^d \neq k$ and the set $k[x,y,z]^d \setminus k$ has no monomials. Then there exists an irreducible homogeneous polynomial $g \in k[x,y,z]$ such that $k[x,y,z]^d = k[g]$.*

Proof. The idea of the proof is similar to that in Theorem 4.1. It follows from the assumptions and Proposition 3.4 that there exists a strict polynomial g belonging to $k[x,y,z]^d$. We may assume (by Corollary 2.3) that g is irreducible. Let $m = \deg g$.

It is sufficient to prove that every nonzero homogeneous polynomial belonging to $k[x,y,z]^d$ is, up to a nonzero coefficient, a power of g.

Assume that f is a nonzero homogeneous polynomial, of degree $n \geq 1$, belonging to $k[x,y,z]^d$. Then f is strict (by the assumptions and Proposition 3.4) and hence, by Proposition 2.5,

$$f(0,y,z) = p(y - Az)^n,$$

for some $0 \neq p \in k$. Moreover, also by Proposition 2.5,

$$g(0,y,z) = a(y - Az)^m,$$

for some $0 \neq a \in k$. Consider now the polynomial

$$h = a^n f^m - p^m g^n.$$

It is a homogeneous polynomial belonging to $k[x,y,z]^d$. Observe that

$$h(0,y,z) = a^n p^m (y - Az)^{nm} - a^n p^m (y - Az)^{nm} = 0.$$

This implies that h is divisible by x.

Suppose that $h \neq 0$. Then $h = x^\alpha y^\beta z^\gamma h_1$, where $h_1 \in k[x,y,z]$ is strict and $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma \geq 1$. Since $d(h) = 0$, Proposition 3.4 implies that $d(x^\alpha y^\beta z^\gamma) = 0$, which is a contradiction with our assumptions.

Therefore $h = 0$, that is, $a^n f^m = p^m g^n$ and we see that f is, up to a nonzero coefficient, a power of g (since g is irreducible). ∎

5. Conclusion. Now it is easy to prove our main result.

Proof of Theorem 1.3. The statement (1a) is a consequence of Theorem 4.1.

Let $ABC = -1$ and assume that some of the scalars A, B, C belong to $k \setminus \mathbb{Q}_-$. Then $k[x,y,z]^d \neq k$ (by Theorem 1.2) and the set $k[x,y,z]^d \setminus k$ has no monomials (Proposition 3.1). Hence, by Theorem 4.3, there exists an irreducible homogeneous polynomial $g \in k[x,y,z]$ such that $k[x,y,z]^d = k[g]$. Since $ABC = -1$, Proposition 2.6 implies that $\deg g = 1$. It is easy to check that $g = x - Cy + ACz$. This completes the proof of (1b).

The statements (2) and (3) are simple consequences of Theorems 1.2, 4.3 and Proposition 3.1. ∎

COROLLARY 5.1. *Let d be a Lotka–Volterra derivation. If the ring of constants of d is nontrivial, then it is a polynomial ring in one or two variables.*

Acknowledgments. We are very grateful to Jean-Marie Strelcyn for many helpful discussions concerning the Lotka–Volterra systems. The final version of this paper was written when the second author was in the GAGE Laboratory of the École Polytechnique (Palaiseau, France) headed by Marc Giusti. The Laboratory's hospitality and excellent working conditions are gratefully acknowledged.

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Received 12 November 1998;
revised 24 February 1999