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TAME THREE-PARTITE SUBAMALGAMS OF TILED ORDERS OF POLYNOMIAL GROWTH

BY

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Abstract. Assume that K is an algebraically closed field. Let D be a complete discrete valuation domain with a unique maximal ideal $\mathfrak{p}$ and residue field $D/\mathfrak{p} \cong K$. We also assume that D is an algebra over the field K. We study subamalgam D-suborders $\Lambda^{\bullet}$ (1.2) of tiled D-orders Λ (1.1). A simple criterion for a tame lattice type subamalgam D-order $\Lambda^{\bullet}$ to be of polynomial growth is given in Theorem 1.5. Tame lattice type subamalgam D-orders $\Lambda^{\bullet}$ of non-polynomial growth are completely described in Theorem 6.2 and Corollary 6.3.

1. Introduction. Throughout this paper K is an algebraically closed field and D is a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D. We denote by $F = D_0$ the field of fractions of D.

We recall that a D-*order* Λ in a finite-dimensional semisimple F-algebra C is a D-subalgebra Λ of C which is a finitely generated free D-submodule of C and Λ contains an F-basis of C [**?**]. We denote by $\operatorname{latt}(\Lambda)$ the category of right Λ-lattices, that is, finitely generated right Λ-modules which are free as D-modules. It is well known that any D-order is a semiperfect ring and the category $\operatorname{latt}(\Lambda)$ has the finite unique decomposition property [**?**, Section 1.1].

A D-order Λ is said to be of *finite lattice type* if the category $\operatorname{latt}(\Lambda)$ has finitely many isomorphism classes of indecomposable modules. A D-order Λ is said to be of *tame lattice type* if the indecomposable Λ-lattices of any fixed D-rank form a finite set of at most one-parameter families. The orders of tame lattice type are divided into two classes: the orders of polynomial and of non-polynomial growth (see [**?**], [**?**, Section 3], [**?**, Section 7]). The definitions are presented at the end of this section.

In the present paper we continue our study of tame three-partite subamalgams of tiled D-orders discussed in [27]–[29]. We use the terminology and notation introduced there. We denote by $\mathbb{M}_m(D)$ the full $m \times m$-matrix ring with coefficients in D. We suppose that $n, n_1, n_2 > 0$ and $n_3 \geq 0$ are

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natural numbers and Λ is a tiled D-suborder of $\mathbb{M}_n(D)$ of the form

$$\Lambda = \left.\begin{pmatrix} D & {}_1D_2 & \dots & {}_1D_n \\ \mathfrak{p} & D & \dots & {}_2D_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & {}_{n-1}D_n \\ \mathfrak{p} & \mathfrak{p} & \dots & D \end{pmatrix}\right\} n \tag{1.1}$$

such that

(a) ${}_iD_j$ is either D or $\mathfrak{p}$,
(b) Λ admits a three-partition

$$\Lambda = \left(\begin{array}{c|c|c} \Lambda_1 & \mathcal{X} & \mathbb{M}_{n_1}(D) \\ \hline \mathbb{M}_{n_3\times n_1}(\mathfrak{p}) & \Lambda_3 & \mathcal{Y} \\ \hline \mathbb{M}_{n_1}(\mathfrak{p}) & \mathbb{M}_{n_1\times n_3}(\mathfrak{p}) & \Lambda_2 \end{array}\right) \begin{array}{l} \}n_1 \\ \}n_3 \\ \}n_2 \end{array} \tag{1.2}$$

where $\Lambda_2 = \Lambda_1$, $n_1 = n_2$, $n_1 + n_2 + n_3 = n$ and Λ_3 is a hereditary $n_3 \times n_3$ -matrix D-order

$$\Lambda_3 = \left.\begin{pmatrix} D & D & \dots & D & D \\ \mathfrak{p} & D & \dots & D & D \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \dots & D & D \\ \mathfrak{p} & \mathfrak{p} & \dots & \mathfrak{p} & D \end{pmatrix}\right\} n_3 \tag{1.3}$$

In particular ${}_iD_j = D$ for $1 \le i \le n_1$ and $n_1 + n_3 < j \le n$.

Note that $1 = \varepsilon_1 + \varepsilon_3 + \varepsilon_2$, where ε_1, ε_3 and ε_2 are the matrix idempotents of Λ corresponding to the identity elements of Λ_1, Λ_3 and Λ_2, respectively. By a *three-partite subamalgam* of Λ we mean the D-suborder

$$\Lambda^\bullet = \{\lambda = [\lambda_{ij}] : \varepsilon_1\lambda\varepsilon_1 - \varepsilon_2\lambda\varepsilon_2 \in \mathbb{M}_{n_1}(\mathfrak{p})\} \tag{1.4}$$

of Λ consisting of all matrices $\lambda = [\lambda_{ij}]$ of Λ such that the upper left corner $n_1 \times n_1$ submatrix $\varepsilon_1\lambda\varepsilon_1$ of λ is congruent modulo $\mathbb{M}_{n_1}(\mathfrak{p})$ to the right lower corner $n_1 \times n_1$ submatrix $\varepsilon_2\lambda\varepsilon_2$ of λ.

It was shown by the author in [?] and [?] that the weak positivity (resp. weak non-negativity) of a reduced Tits quadratic form $q_{\Lambda^\bullet} : \mathbb{Z}^{n_1+2n_3+2} \to \mathbb{Z}$ associated with $\Lambda^\bullet$ is a necessary and sufficient condition for $\Lambda^\bullet$ to be of finite (resp. tame) lattice type.

Our main results in this paper are: a characterization of the D-orders $\Lambda^\bullet$ of tame lattice type which are of polynomial growth (Theorem 1.5 below), and the structure theorem for lattice-tame D-orders $\Lambda^\bullet$ of non-polynomial growth (Theorem 6.2).

THEOREM 1.5. *Let K be an algebraically closed field and D a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D.*

Let Λ be a three-partite D-order of the form (1.2) *and let $\Lambda^\bullet$ be the subamalgam* (1.4) *of $\Lambda \subseteq \mathbb{M}_n(D)$, where $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 are as above. If the $\mathcal{X}$ part or the $\mathcal{Y}$ part in* (1.2) *consists of matrices with coefficients in $\mathfrak{p}$ then the following conditions are equivalent.*

(a) *The D-order $\Lambda^\bullet$* (1.4) *is of tame lattice type and of polynomial growth.*

(b) *Either $n_3 \geq 1$, the D-order Λ_1 in* (1.2) *is hereditary of the form* (1.3) *and the three-partite subamalgam D-orders $\Lambda^\bullet$ and* $\mathrm{rt}(\Lambda)^\bullet$ (1.7) *do not contain three-partite minor D-suborders dominated by any of the* 17 *three-partite subamalgam D-orders listed in Section* 7, *or else $n_3 = 0$ and there exists at most one pair (i,j) such that $1 \leq i < j \leq n_1$ and ${}_iD_j = \mathfrak{p}$ in Λ* (1.1).

(c) *Either $n_3 \geq 1$ and the two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated with $\Lambda^\bullet$ in* (3.3) *contains as a two-peak subposet with zero-relations neither the poset*

$$\mathcal{F}_0 : \quad \begin{tikzcd}[row sep=small, column sep=small] \circ \arrow[d] \arrow[dr] & \circ \arrow[d] \arrow[dl] \\ * & + \end{tikzcd}$$

nor any of the ten hypercritical forms $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ presented in Table 1.9 *below*; *or else $n_3 = 0$, the set $\mathfrak{Z}_{\Lambda^\bullet}$ of zero-relations is empty and the two-peak poset $I^{*+}_{\Lambda^\bullet}$* (*without zero-relations*) *is a two-peak subposet of*

$$(1.6) \quad \mathcal{L}^{*+}_m : \begin{tikzcd}[row sep=small, column sep=small] & & & \circ \arrow[dr] & & & & & * \\ \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[ur] \arrow[dr] & & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[urr] \arrow[drr] & & \\ & & & \circ \arrow[ur] & & & & & + \end{tikzcd} \quad (m \textit{ circle points}),\ m \geq 1.$$

We recall from [?] that given a matrix $\lambda \in \mathbb{M}_n(D)$ we define the *reflection transpose* of λ to be the transpose matrix $\mathrm{rt}(\lambda) \in \mathbb{M}_n(D)$ of λ with respect to the non-main diagonal. Given any D-order Λ we define the *reflection transpose* of Λ (resp. of $\Lambda^\bullet$) to be the D-order

$$(1.7) \qquad \mathrm{rt}(\Lambda) = \{\mathrm{rt}(\lambda) : \lambda \in \Lambda\} \qquad (\text{resp.}\, \mathrm{rt}(\Lambda^\bullet) = \{\mathrm{rt}(\lambda) : \lambda \in \Lambda^\bullet\}).$$

It is easy to see that $\mathrm{rt}(\Lambda^\bullet) = \mathrm{rt}(\Lambda)^\bullet$ and the map $\lambda \mapsto \mathrm{rt}(\lambda)$ defines the ring anti-isomorphisms $\Lambda \xrightarrow{\simeq} \mathrm{rt}(\Lambda)$ and $\Lambda^\bullet \xrightarrow{\simeq} \mathrm{rt}(\Lambda^\bullet)$.

If $1 \leq i_1 < \ldots < i_s \leq n_1$, we say that the order Δ is the $(i_1, \ldots, i_s)$-*minor D-suborder* of Λ_1 in (1.2) if Δ is obtained from Λ_1 by omitting the i_jth row and the i_jth column for $j = 1, \ldots, s$.

A three-partite order Ω is said to be a *three-partite minor D-suborder* of $\Lambda^\bullet$ if Ω is a minor D-suborder of $\Lambda^\bullet$ obtained by omitting rows and columns simultaneously in parts Λ_1 and Λ_2, that is, if we omit the ith row and the

ith column of $\Lambda^\bullet$, for some $1 \le i \le n_1$, then simultaneously we omit the (n_3+i)th row and the (n_3+i)th column of $\Lambda^\bullet$.

A three-partite subamalgam D-order $\Lambda^\bullet$ (1.4) is said to be *dominated* by a three-partite subamalgam D-order $\bar{\Lambda}^\bullet$ if $\Lambda^\bullet$ is a three-partite D-suborder $\bar{\Lambda}^\bullet$ of the same size (1.2) and $\Lambda_1 = \bar{\Lambda}_1$, $\Lambda_2 = \bar{\Lambda}_2$, $\Lambda_3 = \bar{\Lambda}_3$, $\mathcal{X} \subseteq \overline{\mathcal{X}}$, $\mathcal{Y} \subseteq \overline{\mathcal{Y}}$ (see [?], [?, p. 69]).

Let us recall from [?], [?, Section 15.12] and [?, Section 3] the definition of an order of tame lattice type. Let Ω be an arbitrary D-order in a semisimple D_0-algebra C, where D is a complete discrete valuation domain which is an algebra over an algebraically closed field K and $D/\mathfrak{p} \cong K$. Then Ω is said to be of *tame lattice type* (or the category $\mathrm{latt}(\Omega)$ is said to be of tame representation type) if for any number $r \in \mathbb{N}$ there exists a non-zero polynomial $h \in K[y]$ and a family of additive functors

$$(-) \otimes_A M^{(1)}, \ldots, (-) \otimes_A M^{(s)} : \mathrm{ind}_1(A) \to \mathrm{latt}(\Omega) \tag{1.8}$$

where $A = K[y, h^{-1}]$, $\mathrm{ind}_1(A)$ is the full subcategory of $\mathrm{mod}(A)$ consisting of one-dimensional A-modules and $M^{(1)}, \ldots M^{(s)}$ are A-Ω-bimodules satisfying the following conditions:

(P0) The left A-modules ${}_AM^{(1)}, \ldots, {}_AM^{(s)}$ are flat.

(P1) All but finitely many indecomposable Ω-lattices of D-rank r are isomorphic to lattices in $\mathrm{Im}(-) \otimes_A M^{(1)} \cup \ldots \cup \mathrm{Im}(-) \otimes_A M^{(s)}$.

(P2) $M^{(1)}_\Omega, \ldots, M^{(s)}_\Omega$ viewed as D-modules are torsion-free.

(P3) ${}_AM^{(1)}_\Omega, \ldots, {}_AM^{(s)}_\Omega$ are finitely generated as A-Ω-bimodules.

This means that the functors (1.8) form an almost parameterizing family (see [?, Definition 14.13]) for the category $\mathrm{ind}_r(\mathrm{latt}(\Omega))$ of indecomposable Ω-lattices of D-rank r.

Given an integer $r \ge 1$ we define $\boldsymbol{\mu}^1_{\mathrm{latt}(\Omega)}(r)$ to be the minimal number s of functors (1.8) satisfying the conditions above. The D-order Ω of tame lattice type is defined to be of *polynomial growth* [?, Section 3] if there exists an integer $g \ge 1$ such that $\boldsymbol{\mu}^1_{\mathrm{latt}(\Omega)}(r) \le r^g$ for all integers $r \ge 2$ (compare with [?], [?], [?], [?, p. 291], [?] and [?]).

It was proved in [?] that the tame-wild dichotomy holds for D-orders Ω under the assumption on D made above. The reader is referred to [?], [?, Section 3], [?, Section 7] for various definitions and discussion of D-orders of tame lattice type and of polynomial growth.

Our main theorem is proved in Section 4 by applying a technique developed in [?], [?] and [?]. In particular we apply the covering technique for bipartite stratified posets developed by the author in [?], and a reduction functor $\mathbb{H}$ (3.5) from $\mathrm{latt}(\Lambda^\bullet)$ to K-linear socle projective representations of a two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ (3.3) with zero-relations associated with $\Lambda^\bullet$ in

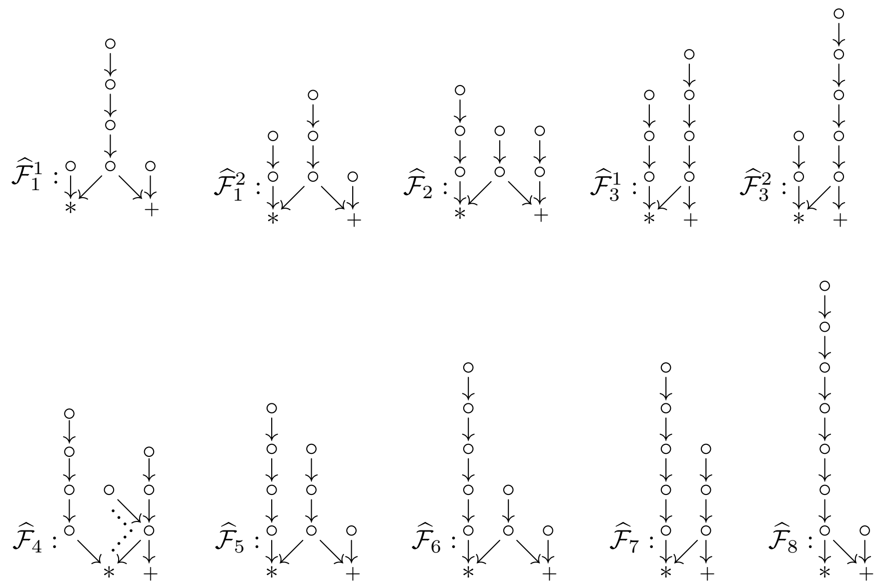
[?]. Then we apply a criterion for tame prinjective type and for polynomial growth of two-peak posets given in [?]–[?] (see also [?] and [?]).

In Section 2 we collect basic facts on K-linear socle projective representations of multi-peak posets with zero-relations we need in this paper.

In Section 3 we associate with $\Lambda^\bullet$ a two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations (see (3.3)) and we formulate the main properties of our reduction functor $\mathbb{H}$ (see Theorem 3.4).

Main results of this paper were presented at an AMS-IMS-SIAM Joint Summer Research Conference "Trends in the Representation Theory of Finite Dimensional Algebras" at the University of Washington, Seattle, in July 1997 (see [?, Theorem 4.2]). They were also presented at the Euroconference "Interactions between Ring Theory and Representations of Algebras", Murcia, 12–17 January 1998 (see [?] and [?, Theorem 8.7]).

Table 1.9. Hypercritical posets with zero-relations



The dotted line in $\widehat{\mathcal{F}}_4$ means a zero-relation.

2. Filtered socle projective representations of posets with zero-relations. We recall from [?] and [?, Chapter 13] that the study of tiled orders reduces to the study of representations of infinite posets having a unique maximal element. A similar idea applies in the study of some categories of abelian groups and of Cohen–Macaulay modules (see [?], [?] and [?]).

We shall prove the main theorems of the paper by reducing the problem for lattices over three-partite subamalgams of tiled D-orders to a corresponding problem for K-linear socle projective representations of two-peak posets (that is, having exactly two maximal elements) with zero-relations studied in [?] and [?], where $K = D/\mathfrak{p}$. Our reduction involves the reduction functors defined in [?] and [?], and the covering technique for bipartite stratified posets developed by the author in [?] (see also [?]).

Throughout we denote by $(I;\preceq)$ a finite *poset*, that is, a finite set I with partial order $\preceq$. We write $i \prec j$ if $i \preceq j$ and $i \neq j$. For simplicity we write I instead of $(I,\preceq)$. We denote by $\max I$ the set of all maximal elements of I, and I will be called an *r-peak poset* if $|\max I| = r$.

Given a poset I we denote by KI the *incidence algebra* of I (see [?]), that is, the subalgebra of the full matrix algebra $\mathbb{M}_I(K)$ consisting of all $I \times I$ square matrices $\lambda = [\lambda_{pq}]_{p,q\in I}$ such that $\lambda_{pq} = 0$ if $p \not\preceq q$ in $(I;\preceq)$.

For $i \preceq j$ we denote by $e_{ij} \in KI$ the matrix having 1 at the i-j-th position and zeros elsewhere. Given j in I we denote by $e_j = e_{jj}$ the standard primitive idempotent of KI corresponding to j.

In our definition of a main reduction functor we also need the notion of a poset with zero-relations (see [?]).

DEFINITION 2.1. A *zero-relation* in a poset I is a pair (i_0, j_0) of elements of I such that $i_0 \prec j_0$.

A *set of zero-relations* in I is a set $\mathfrak{Z}$ satisfying the following two conditions:

(Z1) $\mathfrak{Z}$ consists of zero-relations (i_0, j_0) of I;

(Z2) If $(i_0, j_0) \in \mathfrak{Z}$ and $i_1 \preceq i_0 \preceq j_0 \preceq j_1$ then $(i_1, j_1) \in \mathfrak{Z}$.

A *right multipeak* (or precisely an *r-peak*) *poset with zero-relations* is a pair $(I,\mathfrak{Z})$, where I is a poset, $r = |\max I|$, and $\mathfrak{Z}$ is a set of zero-relations satisfying the following condition:

(Z3) For every $i \in I \setminus \max I$ there exists $p \in \max I$ such that $(i,p) \notin \mathfrak{Z}$.

If $\mathfrak{Z}$ is empty we write I instead of $(I,\mathfrak{Z})$.

A right multipeak poset $(I',\mathfrak{Z}')$ with zero-relations is said to be a *peak subposet* of $(I,\mathfrak{Z})$ if I' is a subposet of I, $\mathfrak{Z}'$ is the restriction of $\mathfrak{Z}$ to I' and $\max I' = I' \cap (\max I)$. ■

Given a right r-peak poset $(I,\mathfrak{Z})$ with zero-relations we define the *incidence K-algebra* of $(I,\mathfrak{Z})$ to be the K-algebra (see [?])

$$K(I,\mathfrak{Z}) = \{\lambda = [\lambda_{ij}]_{i,j\in I} \in KI : \lambda_{ij} = 0 \text{ for } (i,j) \in \mathfrak{Z}\} \subseteq KI \tag{2.2}$$

consisting of all $I \times I$ square matrices $\lambda = [\lambda_{ij}]_{i,j\in I} \in \mathbb{M}_I(K)$ such that $\lambda_{ij} = 0$ if $i \not\preceq j$ in $(I;\preceq)$, or if $(i,j) \in \mathfrak{Z}$. The addition in $K(I,\mathfrak{Z})$ is the usual matrix addition, whereas the product of two matrices $\lambda = [\lambda_{ij}]_{i,j\in I}$

and $\lambda' = [\lambda'_{ij}]_{i,j\in I}$ in $K(I,\mathfrak{Z})$ is the matrix $\lambda'' = [\lambda''_{ij}]_{i,j\in I}$, where

$$\lambda''_{ij} = \begin{cases} \sum_{i\preceq s\preceq j} \lambda_{is}\lambda'_{sj} & \text{if } i \preceq j \text{ and } (i,j) \notin \mathfrak{Z}, \\ 0 & \text{if } i \not\preceq j \text{ or } (i,j) \in \mathfrak{Z}. \end{cases}$$

If $\mathfrak{Z}$ is empty we get $KI = K(I,\mathfrak{Z})$.

The incidence algebra $K(I,\mathfrak{Z})$ is basic and the standard matrix idempotents e_i, $i \in I$, form a complete set of primitive orthogonal idempotents of $K(I,\mathfrak{Z})$. It is easy to see that $K(I,\mathfrak{Z})$ is a factor K-algebra of KI modulo the ideal generated by all matrices $e_{ij} \in KI$ such that $(i,j) \in \mathfrak{Z}$. It follows that the global dimension of $K(I,\mathfrak{Z})$ is finite (see [?, Lemma 2.1]) and, in view of (Z3), the right socle of $K(I,\mathfrak{Z})$ is isomorphic to a direct sum of copies of the right ideals $e_pK(I,\mathfrak{Z})$, $p \in \max I$, called the *right peaks* of $K(I,\mathfrak{Z})$.

We denote by $\mathrm{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})$ the category of *socle projective right* $K(I,\mathfrak{Z})$-*modules*, that is, the full subcategory of $\mathrm{mod}\,K(I,\mathfrak{Z})$ consisting of modules X such that the socle $\mathrm{soc}(X)$ of X is projective and isomorphic to a direct sum of copies of the right ideals $e_pK(I,\mathfrak{Z})$, $p \in \max I$.

Definition 2.3 [?]. Let K be a field and let $(I,\mathfrak{Z})$ be a right multipeak poset with zero-relations. A *peak* $(I,\mathfrak{Z})$-*space* (or a *filtered socle projective representation* of $(I,\mathfrak{Z})$) over the field K is a system $\mathbf{M} = (M_j)_{j\in I}$ of finite-dimensional K-vector spaces M_j satisfying the following four conditions.

(a) For any $j \in I$ the K-space M_j is a K-subspace of

$$M^\bullet = \bigoplus_{p\in\max I} M_p.$$

(b) The inclusion $M_p \subseteq M^\bullet$ is the standard p-coordinate embedding for any $p \in \max I$.

(c) $\pi_j(M_i) \subseteq M_j$ for all $i \prec j$ in I, where $\pi_j : M^\bullet \to M^\bullet$ is the composed K-linear endomorphism

$$M^\bullet \xrightarrow{\pi'_j} \bigoplus_{j\preceq p\in\max I} M_p \hookrightarrow M^\bullet$$

of $M^\bullet$ and π'_j is the direct summand projection.

(d) If $p \in \max I$ and either $i \not\preceq p$ or $i \prec p$ and $(i,p) \in \mathfrak{Z}$ then $\pi_p(M_i) = 0$.

A *morphism* $f : \mathbf{M} \to \mathbf{M}'$ from $\mathbf{M}$ to $\mathbf{M}'$ is a system $f = (f_p)_{p\in\max I}$ of K-linear maps $f_p : M_p \to M'_p$, $p \in \max I$, such that

$$\Big(\bigoplus_{p\in\max I} f_p\Big)(M_j) \subseteq M'_j$$

for all $j \in I$. ∎

We denote by $(I,\mathfrak{Z})$-spr the *category of peak I-spaces* (or filtered socle projective representations of $(I,\mathfrak{Z})$) over the field K. The direct sum and indecomposability in $(I,\mathfrak{Z})$-spr are defined in an obvious way.

We recall from [**?**] that there exists a K-linear functor

$$\varrho : (I,\mathfrak{Z})\text{-spr} \xrightarrow{\simeq} \operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})$$

which is an equivalence of categories.

Following [**?**, Section 14.4], we say that the categories $\operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z}) \cong (I,\mathfrak{Z})$-spr are of *tame representation type* if for any vector $w \in \mathbb{N}^I$ there exists a non-zero polynomial $h \in K[y]$ and a family of additive functors

$$(2.4)\qquad (-)\otimes_S N^{(1)},\ldots,(-)\otimes_S N^{(s)} : \operatorname{ind}_1(S) \to \operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})$$

where $S = K[y,h^{-1}]$, $N^{(1)},\ldots N^{(s)}$ are S-$K(I,\mathfrak{Z})$-bimodules satisfying the following conditions:

(T0) The left S-modules ${}_SN^{(1)},\ldots,{}_SN^{(s)}$ are finitely generated.

(T1) All but finitely many indecomposable objects $\mathbf{M}$ in $(I,\mathfrak{Z})$-spr $\cong \operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})$ such that $\mathbf{dim}\,\mathbf{M} = w$ are isomorphic to modules in the union $\operatorname{Im}(-)\otimes_S N^{(1)} \cup \cdots \cup \operatorname{Im}(-)\otimes_S N^{(s)}$, where

$$\mathbf{dim}\,\mathbf{M} = (\dim_K M_j)_{j\in I}.$$

This means that for any vector $w \in \mathbb{N}^I$ the functors (2.4) form an almost parameterizing family (see [**?**, Definition 14.13]) for the category $\operatorname{ind}_w((I,\mathfrak{Z})\text{-spr})$ of indecomposable peak $(I,\mathfrak{Z})$-spaces $\mathbf{M}$ such that $\mathbf{dim}\,\mathbf{M} = w$.

Given a vector $w \in \mathbb{N}^I$ we define $\boldsymbol{\mu}^1_{\operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})}(w)$ to be the minimal number s of functors (2.4) satisfying the conditions above. The categories $\operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z}) \cong (I,\mathfrak{Z})$-spr of tame representation type are defined to be of *polynomial growth* [**?**] if there exists an integer $g \geq 1$ such that $\boldsymbol{\mu}^1_{\operatorname{mod}_{\mathrm{sp}}K(I,\mathfrak{Z})}(w) \leq (\|w\|+1)^g$ for all vectors $w \in \mathbb{N}^I$, where $\|w\| = \sum_{j\in I} w_j$ (compare with [**?**, p. 291], [**?**] and [**?**]).

It is easy to check that the definition above is equivalent to the one in [**?**, Definition 14.13] (see the proof of [**?**, Proposition 2.6]).

3. A reduction to two-peak poset representations. With any D-order $\Lambda^\bullet$ (1.4) we associate in (3.3) below (see [**?**, Section 4]) a two-peak poset $(I^{*+}_{\Lambda^\bullet},\mathfrak{Z})$ with zero-relations and we shall reduce the study of the category $\operatorname{latt}(\Lambda^\bullet)$ to the study of the category $(I^{*+}_{\Lambda^\bullet},\mathfrak{Z})$-spr.

Suppose that Λ, Λ_1, Λ_2 and Λ_3 are tiled D-orders in (1.2). In order to define $(I^{*+}_{\Lambda^\bullet},\mathfrak{Z}_{\Lambda^\bullet})$ we consider the poset $(I_\Lambda;\preceq)$ (see [**?**]), where

$$(3.1)\qquad I_\Lambda = \{1,\ldots,n\} \quad\text{and}\quad i \prec j \Leftrightarrow {}_iD_j = D.$$

First we associate with $\Lambda^\bullet$ the following combinatorial object:

$$I_{\Lambda^\bullet,\sigma} = (I_\Lambda, \preceq, I', C, I'', \sigma : I' \to I'') \tag{3.2}$$

where $(I_\Lambda; \preceq)$ is the poset (3.1), $C = I_{\Lambda_3} = \{n_1 + 1 \prec \ldots \prec n_1 + n_3 - 1 \prec n_1 + n_3\}$, $I' = I_{\Lambda_1} = \{1, 2, \ldots, n_1\}$ and $I'' = I_{\Lambda_2} = \{n_1 + n_3 + 1, \ldots, n-1, n\}$ are viewed as subposets of I_Λ such that $I_\Lambda = I' \cup C \cup I''$ is a splitting decomposition of I_Λ in the sense of [**?**, Section 8.1], and $\sigma : I' \to I''$ is the poset isomorphism defined by the formula $\sigma(j) = n_1 + n_3 + j$. It is clear that $I_{\Lambda^\bullet,\sigma}$ is a bipartite stratified poset in the sense of [**?**] and [**?**, Section 17.8], or a completed poset in the sense of [**?**] (see also [**?**]).

Let $C' = \{c' : c \in C\}$ be a chain isomorphic to C. We construct two one-peak enlargements

$$(C \cup I'')^* = C \cup I'' \cup \{*\} \quad \text{and} \quad (I' \cup C)^+ = I' \cup C' \cup \{+\}$$

of the posets $C \cup I''$ and $I' \cup C \equiv I' \cup C'$ by the unique maximal points $*$ and $+$, and by the new relations $i \prec *$ and $s \prec +$ for all $i \in C \cup I''$ and all $s \in I' \cup C'$.

We associate with $\Lambda^\bullet$ the two-peak poset with zero-relations

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) = ((C \cup I'')^* \underset{I''\equiv I'}{\cup} (I' \cup C)^+, \mathfrak{Z}_{\Lambda^\bullet}) \tag{3.3}$$

where $I^{*+}_{\Lambda^\bullet}$ is obtained from the disjoint union $(C \cup I'')^* \cup (I' \cup C)^+$ by making the identification $j \equiv \sigma(j)$ for any $j \in I' \subseteq (I' \cup C)^+$. The set $\mathfrak{Z}_{\Lambda^\bullet}$ consists of all pairs (c, c_1') such that $c \in C \subseteq (C \cup I'')^*$, $c_1' \in C' \subseteq (I' \cup C)^+$ and the relations $c \prec s$, $\sigma(s) \prec c_1$ hold in I_Λ for some $s \in I'$. Here we use the convention $+' = +$.

It is easy to see that $I^{*+}_{\Lambda^\bullet}$ is a poset and $\max I^{*+}_{\Lambda^\bullet} = \{*, +\}$. We call $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ the *poset with zero-relations associated with the D-order $\Lambda^\bullet$*.

The following reduction theorem was proved in [**?**, Section 3].

THEOREM 3.4. *Let K be an algebraically closed field, D a complete discrete valuation domain which is a K-algebra, and $\mathfrak{p}$ the unique maximal ideal of D. Assume that $D/\mathfrak{p} \cong K$. Let Λ be the D-order* (1.1) *with the three-partition* (1.2) *and $\Lambda_1 = \Lambda_2 \subseteq \mathbb{M}_{n_1}(D)$, $\Lambda_3 \subseteq \mathbb{M}_{n_3}(D)$ and n_1, n_3 as in Section* 1. *Let $\Lambda^\bullet$ be the subamalgam D-order* (1.4) *and let $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset with zero-relations* (3.3) *associated with $\Lambda^\bullet$. There exists an additive reduction functor*

$$\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \to (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \cong \mathrm{mod}_{\mathrm{sp}} K(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet}) \tag{3.5}$$

with the following properties:

(i) $\mathbb{H}$ *is full*, *reflects isomorphisms and preserves indecomposability.*

(ii) $\mathbb{H}$ *preserves and reflects tame representation type*, *wild representation type and the polynomial growth property*, *that is*, $\mathrm{latt}(\Lambda^\bullet)$ *is of tame representation type* (*resp. wild*, *or of polynomial growth*) *if and only if*

$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$-spr *is of tame representation type* (*resp. wild*, *or of polynomial growth*). ■

4. Proof of Theorem 1.5. Throughout this section K is an algebraically closed field and D is a complete discrete valuation domain which is a K-algebra such that $D/\mathfrak{p} \cong K$, where $\mathfrak{p}$ is the unique maximal ideal of D.

We recall from [**?**, Theorem 1.5] that under the assumption made in Theorem 1.5 the following three conditions are equivalent:

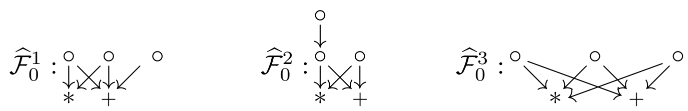
(A) The D-order $\Lambda^\bullet$ is of tame lattice type.

(B) Either $n_3 \geq 1$, the D-order Λ_1 in (1.2) is hereditary of the form (1.3) and the three-partite subamalgam D-orders $\Lambda^\bullet$ and $\mathrm{rt}(\Lambda)^\bullet$ (1.7) do not contain three-partite minor D-suborders dominated by any of the 17 three-partite subamalgam D-orders listed in Section 7, or else $n_3 = 0$ and the D-order Λ_1 in (1.2) does not contain minor D-suborders of one of the forms

$$\begin{aligned}
&\Delta_0 = \begin{pmatrix} D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} D & \mathfrak{p} & D \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}, \\
(4.1) \qquad &\Delta_2 = \begin{pmatrix} D & D & \mathfrak{p} \\ \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}, \quad \Delta_3 = \begin{pmatrix} D & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & D & D \\ \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}.
\end{aligned}$$

(C) The two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations associated with $\Lambda^\bullet$ in (3.3) contains as a two-peak subposet with zero-relations none of the ten hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_4$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ presented in Table 1.9, and contains none of the following three hypercritical posets:

(4.2)



We split the proof of Theorem 1.5 in two cases.

CASE 1: $n_3 = 0$. It follows from (3.3) that the sets C, C' and $\mathfrak{Z}_{\Lambda^\bullet}$ are empty, $I^{*+}_{\Lambda^\bullet}$ does not contain $\widehat{\mathcal{F}}^1_0$ as a two-peak subposet, and $i \prec *$ and $i \prec +$ for all $i \in I^{*+}_{\Lambda^\bullet} \setminus \{*, +\}$. Hence and from the equivalence (B)$\Leftrightarrow$(C) we easily conclude that in case $n_3 = 0$ the conditions (b) and (c) of Theorem 1.5 are equivalent, and the following two statements are valid:

• The D-order Λ_1 in (1.2) does not contain a minor D-suborder of the form Δ_0 shown in (4.1) if and only if $I^{*+}_{\Lambda^\bullet}$ does not contain $\widehat{\mathcal{F}}^3_0$ as a two-peak subposet.

• Λ_1 contains a minor D-suborder of one of the forms Δ_1, Δ_2, Δ_3 in (4.1) if and only if $I^{*+}_{\Lambda^\bullet}$ does not contain $\widehat{\mathcal{F}}^2_0$ as a two-peak subposet.

Hence we easily conclude that the following three statements are equivalent:

(1) Λ_1 contains no minor D-suborder isomorphic to Δ_0, Δ_1, Δ_2 or Δ_3.
(2) $I^{*+}_{\Lambda^\bullet}$ contains as a two-peak subposet none of $\widehat{\mathcal{F}}^1_0$, $\widehat{\mathcal{F}}^2_0$, $\widehat{\mathcal{F}}^3_0$.
(3) There exists $m \geq 2$ such that $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of the two-peak garland

$$(4.3) \qquad \mathcal{G}^{*+}_m : \quad \begin{array}{ccccccccccc} \circ & \longrightarrow & \circ \rightarrow & \cdots & \rightarrow \circ & \longrightarrow & \circ & \longrightarrow & * \\ & \boxtimes & & \cdots & & \boxtimes & & \boxtimes & \\ \circ & \longrightarrow & \circ \rightarrow & \cdots & \rightarrow \circ & \longrightarrow & \circ & \longrightarrow & + \end{array} \quad (2m \text{ points}), \ m \geq 1.$$

Note that $I^{*+}_{\Lambda^\bullet}$ is $\mathcal{G}^{*+}_3$ if Λ_1 is the D-order

$$\nabla_2 = \begin{pmatrix} D & \mathfrak{p} & D & D \\ \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}.$$

Hence according to [**?**, Theorem 5.2] and [**?**, Lemma 3.1] (see also [**?**]) the category $\mathcal{G}^{*+}_3$-spr is of tame representation type and of non-polynomial growth. Then in view of Theorem 3.4 and the equivalences (A)⇔(B)⇔(C) (see above) the D-order $\Lambda^\bullet$ is of tame lattice type and of polynomial growth if and only if $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of $\mathcal{G}^{*+}_m$ for some $m \geq 1$ and $I^{*+}_{\Lambda^\bullet}$ does not contain $\mathcal{G}^{*+}_3$. It follows that statement (a) of Theorem 1.5 holds if and only if $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of the garland $\mathcal{L}^{*+}_m$ (1.6) with $m \geq 3$. Hence the equivalence (a)⇔(c) follows and the proof of Theorem 1.5 is complete for $n_3 = 0$.

Case 2: $n_3 \geq 1$. First we show that the following four statements are equivalent:

(i) Λ_1 is hereditary of the form (1.3).
(ii) The poset $I' = I_{\Lambda_1}$ is linearly ordered.
(iii) The poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations does not contain the poset

$$\mathcal{F}_0 : \quad \begin{array}{cc} \circ & \circ \\ \downarrow & \downarrow \\ * & + \end{array}$$

as a two-peak subposet with zero-relations.

(iv) $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains as a two-peak subposet with zero-relations none of the posets $\widehat{\mathcal{F}}^1_0$, $\widehat{\mathcal{F}}^2_0$, $\widehat{\mathcal{F}}^3_0$ in (4.2).

The implications (i)⇔(ii)⇔(iii)⇒(iv) are immediate consequences of the construction $\Lambda^\bullet \mapsto (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ in (3.3).

To prove (iv)⇒(iii) assume that, on the contrary, $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ contains $\mathcal{F}_0$. Since $n_3 \ge 1$, neither of the chains C and C' in (3.3) is empty. Further, since by assumption the $\mathcal{X}$ part or the $\mathcal{Y}$ part of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$, it follows that either C or C' is incomparable with all elements of the subposet $I' \equiv I''$ of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$. Since $\mathcal{F}_0$ is a two-peak subposet of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$, its extension by a point of C or a point of C' is a two-peak subposet of $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ isomorphic to $\mathcal{F}_0^1$, contrary to our assumption in (iv). Consequently, (i)–(iv) are equivalent.

It now follows from [**?**, Theorem 1.5] that statements (b) and (c) of Theorem 1.5 are equivalent. Since $n_3 \ge 1$ the implication (a)⇒(b) is a direct consequence of (a)⇒(d) in [**?**, Theorem 1.5].

(c)⇒(a). Assume that $n_3 \ge 1$ and (c) holds. In view of Theorem 3.4, to prove (a) it is sufficient to show that (c) implies that the category $I^{*+}_{\Lambda^\bullet}$-spr is of tame representation type and of polynomial growth, because the reduction functor $\mathbb{H} : \mathrm{latt}(\Lambda^\bullet) \to I^{*+}_{\Lambda^\bullet}$-spr reflects tameness and the polynomial growth property. We split the proof in two cases.

Case 2(a): $n_3 \ge 1$ and the $\mathcal{Y}$ part of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$. It follows from (3.3) that C and C' are not empty, C is incomparable with all elements of $I' \equiv I''$, and $\mathfrak{Z}_{\Lambda^\bullet}$ is empty. Since $I^{*+}_{\Lambda^\bullet}$ does not contain a two-peak subposet isomorphic to $\mathcal{F}_0$, the posets $I' \cong I''$ are linearly ordered.

This shows that in this case $I^{*+}_{\Lambda^\bullet}$ is thin in the sense of [**?**, Definition 3.1]. By [**?**, Theorem 1.1] condition (c) implies that the category $I^{*+}_{\Lambda^\bullet}$-spr is of tame representation type. Moreover, by [**?**, Theorem 4.1] this category is of polynomial growth if and only if $I^{*+}_{\Lambda^\bullet}$ contains no two-peak subposets isomorphic to any of the following ones:

$$\mathcal{T}_1 : \quad \begin{array}{ccccc} \circ & & \circ & & \circ \\ \downarrow & \times & \downarrow & & \downarrow \\ \circ & & \circ & & \circ \\ & \searrow & \downarrow & \swarrow & \downarrow \\ & & p & & q \end{array}$$

$$\mathcal{T}_2 : \quad \begin{array}{ccccccccc} \circ & & \circ & & & & & & \\ \downarrow & \times & \downarrow & & & & & & \\ \circ & & \circ & & \circ & & \circ & & \circ \\ & \searrow & \downarrow & \swarrow & & \searrow & \downarrow & \swarrow & \\ & & p & & & & q & & \end{array}$$

$$\mathcal{T}_3 : \quad \begin{array}{ccccccccc} & & \circ & & & & & & \\ & \swarrow & \downarrow & \searrow & & & & & \\ \circ & & \circ & & \circ & & \circ & & \circ \\ & \searrow & \downarrow & \swarrow & & \searrow & \downarrow & \swarrow & \\ & & p & & & & q & & \end{array}$$

Since the subposet $I' = I_{\Lambda_1}$ of $I^{*+}_{\Lambda^\bullet}$ is linearly ordered, $I^{*+}_{\Lambda^\bullet}$ is a union of three chains and therefore it contains neither $\mathcal{T}_2$ nor $\mathcal{T}_3$. Further, by our assumption in (c), $I^{*+}_{\Lambda^\bullet}$ does not contain

$$\mathcal{F}_0: \begin{tikzcd} \circ \arrow[d] \arrow[dr] & \circ \arrow[d] \arrow[dl] \\ * & + \end{tikzcd}$$

as a subposet, and consequently it does not contain $\mathcal{T}_1$ as a two-peak subposet. It follows from [?, Theorem 4.1] that the category $I^{*+}_{\Lambda^\bullet}$-spr is of tame representation type and of polynomial growth and according to Theorem 3.4 the category $\mathrm{latt}(\Lambda^\bullet)$ is of tame representation type and of polynomial growth. This finishes the proof of (c)$\Rightarrow$(a) in Case 2(a).

Case 2(b): $n_3 \geq 1$ and the $\mathcal{X}$ part of Λ in (1.2) consists of matrices with coefficients in $\mathfrak{p}$. Let $\Gamma^\bullet = \mathrm{rt}(\Lambda^\bullet)$ be the reflection transpose of $\Lambda^\bullet$ (see (1.7)). Since the $\mathcal{X}$ part of Λ consists of matrices with coefficients in $\mathfrak{p}$, the corresponding $\mathcal{Y}$ part of Γ in its three-partition (1.2) consists of matrices with coefficients in $\mathfrak{p}$ and by the arguments in Case 2 applied to $\Gamma^\bullet$ the set $\mathfrak{Z}_{\Gamma^\bullet}$ is empty. It follows from Proposition 4.1 that $I^{*+}_{\Gamma^\bullet} = (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet}) \cong (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet$ and according to [?, (2.19)] there exists a reflection duality functor

$$D^\bullet : I^{*+}_{\Gamma^\bullet}\text{-spr} \xrightarrow{\simeq} (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet\text{-spr}.$$

Recall that $n_3 \geq 1$ and we assume that the two-peak poset $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ with zero-relations contains none of the ten hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_1, \widehat{\mathcal{F}}^2_1, \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}^1_3, \widehat{\mathcal{F}}^2_3, \widehat{\mathcal{F}}_4, \widehat{\mathcal{F}}_5, \widehat{\mathcal{F}}_6, \widehat{\mathcal{F}}_7, \widehat{\mathcal{F}}_8$ of Table 1.9. Since obviously the list above is closed under the reflection duality operation $(I, \mathfrak{Z}) \mapsto (I, \mathfrak{Z})^\bullet$ (2.17), Case 2 applies to I^{*+}_{Γ} and therefore the category $I^{*+}_{\Gamma^\bullet}$-spr is of tame representation type and of polynomial growth. Then (a) follows, because according to Proposition 5.5 of the following section the reflection duality functor $D^\bullet$ preserves and respects tame representation type of polynomial growth. This completes the proof of Theorem 1.5. ∎

5. Reflection duality functors and polynomial growth. Throughout this section we assume that $s \geq 1$, $(I, \mathfrak{Z})$ is an s-peak poset with zero-relations and $\max I = \{p_1, \ldots, p_s\}$.

Following [?, Definition 2.16] we associate with $(I, \mathfrak{Z})$ the reflection-dual s-peak poset $(I^\bullet, \mathfrak{Z}^\bullet)$ with zero-relations as follows. First we define a left-right s-peak poset with zero-relations $(\widehat{I}, \widehat{\mathfrak{Z}})$, where

$$\widehat{I} = \{p_1^-, \ldots, p_s^-\} \cup I$$

is a poset enlargement of I by a set $\{p_1^-, \ldots, p_s^-\}$ of minimal elements. The partial order $\preceq$ in $\widehat{I}$ is an extension of the partial order in I by the relations

$$p_h^- \prec j \Leftrightarrow \text{there exists } i \preceq j \text{ in } I \text{ such that } i \prec p_h \text{ in } I \text{ and } (i, p_h) \notin \mathfrak{Z}$$

for any $p_h \in \max I$. We define the set $\widehat{\mathfrak{Z}}$ of zero-relations in $\widehat{I}$ to be the set generated by the union of $\mathfrak{Z}$ and the set consisting of the following relations:

• (p_h^-, p_t) for all $h \neq t$, and

• (p_h^-, j), where $p_h^- \prec j$ holds in $(\widehat{I}, \preceq)$, whereas $j \prec p_h$ does not hold in $(I, \preceq)$.

Next we define the *reflection-dual s-peak poset* with zero-relations

$$(I, \mathfrak{Z})^\bullet = (I^\bullet, \mathfrak{Z}^\bullet) \tag{5.1}$$

to be the poset

$$I^\bullet = (\widehat{I} \setminus \max I)^{\mathrm{op}}$$

dual to $(\widehat{I} \setminus \max I, \preceq)$. We take for $\mathfrak{Z}^\bullet$ the dual of the restriction of $\widehat{\mathfrak{Z}}$ to $\widehat{I} \setminus \max I$.

Following [**?**, 2.6] and [**?**, Chapter 5] we have defined in [**?**, 2.19] a pair of *reflection duality functors*

$$(I, \mathfrak{Z})\text{-spr} \underset{D^\bullet}{\overset{D^\bullet}{\leftrightarrows}} (I, \mathfrak{Z})^\bullet\text{-spr}. \tag{5.2}$$

The aim of this section is to show that the reflection duality functors preserve and respect tame representation type and the polynomial growth property. For this purpose we consider the commutative diagram

$$\begin{tikzcd} (I, \mathfrak{Z})\text{-spr} \arrow[r, shift right, "D^\bullet"'] \arrow[d, shift right, "\nabla_-"'] & (I, \mathfrak{Z})^\bullet\text{-spr} \arrow[l, shift right, "D^\bullet"'] \arrow[dl, shift left, "D^\bullet"] \\ (I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir} \arrow[u, shift right, "\nabla_+"'] \arrow[ur, shift left, "D^\bullet"] & \end{tikzcd} \tag{5.3}$$

where $(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir} \cong \mathrm{mod}_{\mathrm{ti}}\, K(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}$ is the category of top-injective K-linear representations of $(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}$ (see [**?**, 2.4]) defined as follows. The objects of $(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}$-tir are systems $\mathbf{W} = (W_j)_{j \in I}$ of finite-dimensional K-vector spaces W_j satisfying the following four conditions.

(a) Each W_j is a factor space of

$$W^\bullet = \bigoplus_{p \in \max I} W_p.$$

(b) The epimorphism $W^\bullet \twoheadrightarrow W_j$ is the standard p-coordinate projection for any $p \in \max I$.

(c), (d) The conditions dual to (c) and (d) in Definition 2.3.

It follows that for any $j \in I$ the epimorphism $W^\bullet \twoheadrightarrow W_j$ factorizes through an epimorphism $W_j^\bullet \twoheadrightarrow W_j$, where

$$W_j^\bullet = \bigoplus_{j \prec p \in \max I} W_p.$$

Morphisms in $(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}$-tir are defined in a natural way. We define the *reflection functor* $\nabla_- : (I,\mathfrak{Z})\text{-spr} \to (I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir}$ by associating with any object $\mathbf{M} = (M_j)_{j\in I}$ of $(I,\mathfrak{Z})$-spr the system $\nabla_-(\mathbf{M}) = (\overline{M}_j)_{j\in I}$ in $(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}$-tir, where $\overline{M}_{p^-} = M_p$ for $p \in \max I$, and $\overline{M}_j$ is the cokernel of the natural embedding $u_j : M_j \to M_j^\bullet = \bigoplus_{j\prec p\in\max I} M_p$. This means that the sequence

$$0 \to M_j \xrightarrow{u_j} M_j^\bullet \xrightarrow{v_j} \overline{M}_j \to 0 \tag{5.4}$$

is exact, where v_j is the cokernel epimorphism. The functor ∇_- is defined on morphisms in a natural way. The *reflection functor* ∇_+ is defined analogously by means of the kernel instead of the cokernel. It is easy to see that ∇_- and ∇_+ are equivalences of categories inverse to each other. It follows from the definitions that the compositions $D\circ\nabla_-$ and $\nabla_+\circ D$ with $D(-) = \mathrm{Hom}_K(-,K)$ are just the reflection duality functors $D^\bullet : (I,\mathfrak{Z})\text{-spr} \to (I,\mathfrak{Z})^\bullet\text{-spr}$ and $D^\bullet : (I,\mathfrak{Z})^\bullet\text{-spr} \to (I,\mathfrak{Z})\text{-spr}$, respectively. This means that the diagram (5.3) is commutative.

Now we are able to prove the main result of this section.

Proposition 5.5. *Let $(I,\mathfrak{Z})$ be an s-peak poset with zero-relations, $s\ge 1$, let K be an algebraically closed field and let*

$$(I,\mathfrak{Z})\text{-spr} \underset{D^\bullet}{\overset{D^\bullet}{\leftrightarrows}} (I,\mathfrak{Z})^\bullet\text{-spr}$$

be the reflection duality functors (5.2) *defined in* [**?**].

(a) *The K-linear reflection functors $\nabla_- : (I,\mathfrak{Z})\text{-spr} \to (I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir}$ and $\nabla_+ : (I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir} \to (I,\mathfrak{Z})\text{-spr}$ defined above are equivalences of categories inverse to each other. The diagram* (5.3) *is commutative.*

(b) *If $\mathbf{M}=(M_j)_{j\in I}$ is an object of $(I,\mathfrak{Z})$-spr and $\mathbf{dim}\,\mathbf{M}=(\dim_K M_j)_{j\in I}$ then*

$$\mathbf{dim}\, D^\bullet(\mathbf{M}) = \mathbf{s}^\bullet(\mathbf{dim}\,\mathbf{M})$$

where

$$\mathbf{s}^\bullet : \mathbb{Z}^I \to \mathbb{Z}^{I^\bullet} \cong \mathbb{Z}^I \tag{5.6}$$

is the group isomorphism defined by the formula

$$\mathbf{s}^\bullet(w)_j = \begin{cases} -w_j + \sum_{j\prec p\in\max I} w_p & \text{if } j \in I\setminus\max I,\\ w_p & \text{if } j = p^-,\ p\in\max I.\end{cases}$$

(c) *The functors ∇_- and ∇_+ are smooth, that is, they lower and lift almost parameterizing families in the sense of* [26, Definition 6.9].

(d) *The functors ∇_-, ∇_+ and the reflection duality $D^\bullet$ preserve the growth number $\boldsymbol{\mu}^1_{\mathrm{mod_{sp}}K(I,\mathfrak{Z})}(w)$* (see the end of Section 2). *In particular, if any of the categories $(I,\mathfrak{Z})$-spr, $(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}$-tir, $(I^\bullet,\mathfrak{Z}^\bullet)$-spr is tame* (*resp. of* *polynomial growth*) *then the remaining categories are tame* (*resp. of polynomial growth*).

P r o o f. Statement (a) was shown above, whereas (b) is a consequence of [**?**, Proposition 2.20 (iii)]. In order to prove (c) and (d) we need some notation.

Throughout we denote by $\operatorname{rep}_K(I,\mathfrak{Z})$ the category of K-linear representation of $(I,\mathfrak{Z})$, that is, the systems

$$(X_i,\, {}_jh_i)_{i,j\in I, i\prec j} \tag{5.7}$$

of finite-dimensional K-vector spaces X_j connected by K-linear maps ${}_jh_i : X_i \to X_j$ satisfying the following conditions:

- ${}_ih_i$ is the identity map on X_i for any $i \in I$,
- ${}_jh_i = 0$ if $(i,j) \in \mathfrak{Z}$,
- ${}_th_j \cdot {}_jh_i = {}_th_i$ if $i \preceq j \preceq t$.

It is well known that there exists a K-linear equivalence of categories

$$\operatorname{mod} K(I,\mathfrak{Z}) \xrightarrow{\simeq} \operatorname{rep}_K(I,\mathfrak{Z}) \tag{5.8}$$

defined as follows. If X is a module in $\operatorname{mod} K(I,\mathfrak{Z})$ we define the representation (5.7) in $\operatorname{rep}_K(I,\mathfrak{Z})$ by setting $X_i = Xe_i$ and we take for ${}_jh_i : X_i \to X_j$ the K-linear map defined by multiplication by $e_{ij} \in K(I,\mathfrak{Z})$. Conversely, if the system $(X_i, {}_jh_i)_{i,j\in I,i\prec j}$ in $\operatorname{rep}_K(I,\mathfrak{Z})$ is given we set $X = \bigoplus_{i\in I} X_i$ and we define the multiplication $\cdot : X \times K(I,\mathfrak{Z}) \to X$ by $x_i \cdot e_{ij} = {}_jh_i(x_i)$ for $x_i \in X_i$ and $i \preceq j$, $(i,j) \notin \mathfrak{Z}$. Throughout we identify the categories $\operatorname{mod} K(I,\mathfrak{Z})$ and $\operatorname{rep}_K(I,\mathfrak{Z})$ along the functor $X \mapsto (X_i, {}_jh_i)_{i,j\in I,i\prec j}$ (5.8).

Let $S = K[t,h^{-1}]$, where $h \in K[t]$ is non-zero, and let $R = K(I,\mathfrak{Z})$. The above correspondence allows us to identify any S-R-bimodule ${}_ST_R$ with the system

$${}_ST_R = ({}_ST_i, {}_jh_i^T)_{i,j\in I,i\prec j}$$

of S-modules ${}_ST_j$ connected by S-homomorphisms ${}_jh_i^T : {}_ST_i \to {}_ST_j$ satisfying the conditions stated above. If ${}_ST_R$ is isomorphic to a peak $(I,\mathfrak{Z})$-space and $R_1 = K(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}$ we define the S-R_1-bimodule

$$\nabla_-({}_ST_R) = ({}_S\overline{T}_i, {}_j\overline{h}_i^T)_{i,j\in I^\bullet,i\prec j} \tag{5.9}$$

where ${}_S\overline{T}_{p^-} = {}_ST_p$ for $p \in \max I$, and ${}_S\overline{T}_j$ is the cokernel of the natural embedding $u_j^T : {}_ST_j \to {}_ST_j^\bullet$ (see (5.4)) for $j \in I \setminus \max I$. This means that the sequence

$$0 \to {}_ST_j \xrightarrow{u_j^T} {}_ST_j^\bullet \xrightarrow{v_j^T} {}_S\overline{T}_j \to 0 \tag{5.10}$$

of S-modules is exact, where v_j^T is the cokernel epimorphism. The S-homomorphisms ${}_j\overline{h}_j^T$ are derived from ${}_jh_j^T$ in an obvious way.

We claim that if the one-dimensional $K[t]$-module $S_\lambda = K[t]/(t-\lambda)$ is an S-module then there is a natural R_1-module isomorphism (compare with [?] and [?, Theorem 6.10])

$$\nabla_-(S_\lambda \otimes_S T_R) \cong S_\lambda \otimes_S \nabla_-({}_S T_R). \tag{5.11}$$

Indeed, by tensoring the sequence (5.10) with S_λ over S we get the exact sequence

$$S_\lambda \otimes {}_S T_j \xrightarrow{S_\lambda \otimes u_j^T} S_\lambda \otimes {}_S T_j^\bullet \xrightarrow{S_\lambda \otimes v_j^T} S_\lambda \otimes {}_S \overline{T}_j \to 0.$$

It follows that $\nabla_-(S_\lambda \otimes_S T_R)_j \cong \operatorname{Coker}(S_\lambda \otimes u_j^T) \cong S_\lambda \otimes_S (\operatorname{Coker} u_j^T) \cong (S_\lambda \otimes_S \nabla_-({}_S T_R))_j$ and our claim follows.

Hence we easily conclude that if

$$(-) \otimes_S N^{(1)}, \ldots, (-) \otimes_S N^{(s)} : \operatorname{ind}_1(S) \to \operatorname{mod}_{\mathrm{sp}} K(I, \mathfrak{Z}) \cong (I, \mathfrak{Z})\text{-spr}$$

is an almost parameterizing family (see (2.4) and [?, Definition 14.13]) for the category $\operatorname{ind}_w((I,\mathfrak{Z})\text{-spr})$ of indecomposable peak $(I,\mathfrak{Z})$-spaces $\mathbf{M}$ such that $\mathbf{dim}\,\mathbf{M} = w$ then, for $t = 1, \ldots, s$,

$$(-) \otimes_S \nabla_-({}_S N^{(t)}) : \operatorname{ind}_1(S) \to \operatorname{mod}_{\mathrm{ti}} K(I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}} \cong (I^\bullet, \mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir}$$

is an almost parameterizing family for the category $\operatorname{ind}_{\mathbf{s}^\bullet(w)}((I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir})$ of indecomposable top-injective K-linear representations $\mathbf{W}$ of $(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}$ such that $\mathbf{dim}\,\mathbf{W} = \mathbf{s}^\bullet(w)$, where $\mathbf{s}^\bullet$ is the isomorphism (5.6).

This shows that the category $(I,\mathfrak{Z})$-spr is of tame representation type if and only if the category $(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}$-tir is of tame representation type, and in this case $\boldsymbol{\mu}^1_{(I,\mathfrak{Z})\text{-spr}}(w) = \boldsymbol{\mu}^1_{(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir}}(\mathbf{s}^\bullet(w))$ (see the end of Section 2). Consequently, the polynomial growth property is preserved by the reflection functors ∇_+ and ∇_-.

The duality functor

$$D : (I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}}\text{-tir} \to (I^\bullet,\mathfrak{Z}^\bullet)\text{-spr}, \quad D(-) = \operatorname{Hom}_K(-,K),$$

also preserves tame representation type and the polynomial growth property, because in view of (b) the functor

$$D(-) = \operatorname{Hom}_K(-,K) : \operatorname{rep}_K(I^\bullet,\mathfrak{Z}^\bullet)^{\mathrm{op}} \to \operatorname{rep}_K(I^\bullet,\mathfrak{Z}^\bullet)$$

defines a regular K-variety isomorphism $({}_j h_i) \mapsto (({}_j h_i)^*)$ between the algebraic K-variety of top-injective representations

$$\mathbf{N} = (K^{w_j}, {}_j h_i : K^{w_i} \to K^{w_j})$$

such that $\mathbf{dim}\,\mathbf{N} = w$ (in the notation of [?, Section 14.5]) and the algebraic K-variety of socle projective representations

$$D(\mathbf{N}) = ((K^{w_j})^* \cong K^{w_j}, ({}_j h_i)^* : K^{w_j} \to K^{w_i})$$

such that $\mathbf{dim}\, D(\mathbf{N}) = \mathbf{s}^\bullet(w)$, where h^* means the K-dual map to h.

It follows that the duality D carries one-parameter families to one-parameter families and by applying [?, Lemma 14.30] we conclude that the functor D preserves tameness, the growth number $\boldsymbol{\mu}^1(w)$ (see the end of Section 2) and the polynomial growth property. ■

COROLLARY 5.12. *Let $\Lambda^\bullet$ be a subamalgam D-suborder* (1.4) *of the tiled order Λ* (1.2) *and let $\Gamma^\bullet = \mathrm{rt}(\Lambda^\bullet)$ be the reflection transpose order* (1.7) *of $\Lambda^\bullet$. Then $\Lambda^\bullet$ is of tame lattice type* (*resp. of polynomial growth*) *if and only if the D-order $\Gamma^\bullet$ is of tame lattice type* (*resp. of polynomial growth*).

Proof. Let $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ be the two-peak poset with zero-relations (3.3) and $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet$ its reflection-dual. By [?, Proposition 4.1] there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{latt}(\Lambda^\bullet) & \xrightarrow{\mathbb{H}} & (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \\ {\scriptstyle\cong}\big\downarrow{\scriptstyle D_\Lambda} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle \widetilde{D}^\bullet} \\ \mathrm{latt}(\Gamma^\bullet) & \xrightarrow{\mathbb{H}} & (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr} \end{array}$$

where $\mathbb{H}$ is the composed reduction functor (3.5), $D_\Lambda = \mathrm{Hom}_D(-, D)$ is the standard D-duality, and $\widetilde{D}^\bullet$ is the composed duality functor

$$(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})\text{-spr} \xrightarrow{D^\bullet} (I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})^\bullet\text{-spr} \cong (I^{*+}_{\Gamma^\bullet}, \mathfrak{Z}_{\Gamma^\bullet})\text{-spr}$$

induced by the reflection duality (5.2). Then the corollary is an immediate consequence of Theorem 3.4(ii) and Proposition 5.5. ■

6. Subamalgam D-orders of non-polynomial growth. We shall describe in Theorem 6.2(b) below the structure of D-orders $\Lambda^\bullet$ (1.4) which are of tame lattice type and of non-polynomial growth by means of the class of D-orders ∇_m, $m \geq 1$, defined inductively as follows:

(i) $\nabla_1 = \begin{bmatrix} D & \mathfrak{p} \\ \mathfrak{p} & D \end{bmatrix}$,

(ii) $\nabla_{m+1} = \left[\begin{array}{cc|ccccc} D & \mathfrak{p} & D & D & \cdots & D & D \\ \mathfrak{p} & D & D & D & \cdots & D & D \\ \hline \mathfrak{p} & \mathfrak{p} & & & & & \\ \vdots & \vdots & & & \nabla_m & & \\ \mathfrak{p} & \mathfrak{p} & & & & & \end{array}\right]$

for $m \geq 1$.

The following result is a consequence of the proof in Section 4.

COROLLARY 6.1. *Assume that D, ${}_iD_j$, n, n_1, n_3, Λ and $\Lambda^\bullet$ are as in Theorem* 1.5. *Let $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ be the poset with zero-relations associated with*

$\Lambda^\bullet$ *in* (3.3). *If* $n_3 = 0$ *then the sets* C, C', $\mathfrak{Z}_{\Lambda^\bullet}$ *are empty, and the* D*-order* $\Lambda^\bullet$ (1.4) *is of tame lattice type if and only if there exists* $m \geq 2$ *such that the* D*-order* Λ_1 *in* (1.2) *is contained in* ∇_m *as a minor* D*-suborder.*

Proof. One can easily conclude from the equivalences (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) proved in Section 4 that for $n_3 = 0$ the category $I^{*+}_{\Lambda^\bullet}$-spr is of tame representation type if and only if $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of a two-peak garland $\mathcal{G}^{*+}_m$ (4.3) for some $m \geq 3$. Hence the corollary easily follows from [?, Theorem 1.5], because $I^{*+}_{\Lambda^\bullet}$ is the garland $\mathcal{G}^{*+}_{m+1}$ if Λ_1 is the D-order ∇_m. ∎

THEOREM 6.2. *Assume that* D, ${}_iD_j$, n, n_1, n_3, Λ *and* $\Lambda^\bullet$ *are as in Theorem* 1.5. *Let* $(I^{*+}_{\Lambda^\bullet}, \mathfrak{Z}_{\Lambda^\bullet})$ *be the poset with zero-relations associated with* $\Lambda^\bullet$ *in* (3.3). *Then the following conditions are equivalent.*

(a) *The* D*-order* $\Lambda^\bullet$ (1.4) *is of tame lattice type and of non-polynomial growth.*

(b) $n_3 = 0$, *there exists* $m \geq 2$ *such that the* D*-order* Λ_1 *in* (1.2) *is contained in* ∇_m *as a minor* D*-suborder and* Λ_1 *contains the* D*-order*

$$\nabla_2 = \begin{pmatrix} D & \mathfrak{p} & D & D \\ \mathfrak{p} & D & D & D \\ \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D \end{pmatrix}$$

as a minor D*-suborder.*

(c) $n_3 = 0$, *there is no triple* (i, j, s) *of integers such that* $1 \leq i, j, s \leq n_1$ *and* ${}_iD_s = {}_jD_s = \mathfrak{p}$, *and there exist at least two different pairs* (i, j) *and* (s, t) *such that* $1 \leq i < j \leq n_1$, $1 \leq s < t \leq n_1$, ${}_iD_j = {}_sD_t = \mathfrak{p}$ *and* ${}_iD_s = {}_iD_t = {}_jD_s = {}_jD_t = D$ *in* (1.1).

(d) $n_3 = 0$, *the set* $\mathfrak{Z}_{\Lambda^\bullet}$ *of zero-relations is empty, the two-peak poset* $I^{*+}_{\Lambda^\bullet}$ (*without zero-relations*) *associated with* $\Lambda^\bullet$ *in* (3.3) *is a peak subposet of a two-peak garland* $\mathcal{G}^{*+}_m$ (4.3) *with* $m \geq 3$, *and* $I^{*+}_{\Lambda^\bullet}$ *contains a two-peak subposet isomorphic to the two-peak garland*

$$\mathcal{G}^{*+}_3 : \qquad \begin{array}{ccccc} \circ & \longrightarrow & \circ & \longrightarrow & * \\ & \nearrow\!\!\!\!\!\!\searrow & & \nearrow\!\!\!\!\!\!\searrow & \\ \circ & \longrightarrow & \circ & \longrightarrow & + \end{array}$$

Proof. It follows from (3.3) that, for $n_3 = 0$, the sets C, C' and $\mathfrak{Z}_{\Lambda^\bullet}$ are empty and the relations $i \prec *$ and $i \prec +$ hold for all $i \in I^{*+}_{\Lambda^\bullet} \setminus \{*, +\}$. Hence we easily conclude that $I^{*+}_{\Lambda^\bullet}$ contains as a two-peak subposet none of the hypercritical posets with zero-relations $\widehat{\mathcal{F}}^1_1$, $\widehat{\mathcal{F}}^2_1$, $\widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}^1_3$, $\widehat{\mathcal{F}}^2_3$, $\widehat{\mathcal{F}}_5$, $\widehat{\mathcal{F}}_6$, $\widehat{\mathcal{F}}_7$, $\widehat{\mathcal{F}}_8$ of Table 1.9.

(a)$\Rightarrow$(d). Assume that $\Lambda^\bullet$ is of tame lattice type and of non-polynomial growth. It follows that $n_3 = 0$, because according to Theorem 1.5 in case $n_3 \geq 1$ every D-order $\Lambda^\bullet$ of tame lattice type is of polynomial growth.

On the other hand, from the equivalences (1)⇔(2)⇔(3) proved in Section 4 together with Theorem 3.4 it follows that for $n_3 = 0$ the D-order $\Lambda^\bullet$ is of tame lattice type if and only if $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of $\mathcal{G}^{*+}_m$ (4.3) for some $m \ge 3$. Since $\Lambda^\bullet$ is of tame lattice type and of non-polynomial growth, Theorem 1.5 yields that $I^{*+}_{\Lambda^\bullet}$ is a two-peak subposet of the two-peak garland (1.6). Hence we easily conclude that $I^{*+}_{\Lambda^\bullet}$ contains $\mathcal{G}^{*+}_3$ and the implication (a)⇒(d) follows.

The equivalence (c)⇔(d) follows immediately from (3.3). Further, since $I^{*+}_{\Lambda^\bullet}$ is $\mathcal{G}^{*+}_{m+1}$ if Λ_1 is ∇_m, the equivalence (b)⇔(d) follows.

(d)⇒(a). Assume that (d) holds. Then $I^{*+}_{\Lambda^\bullet}$ contains $\mathcal{G}^{*+}_3$ and by Theorem 1.5 the D-order $\Lambda^\bullet$ is not of polynomial growth. Hence (a) easily follows. ■

As a consequence of Theorem 6.2 we get the following result on a minimal embedding of non-polynomial growth (compare with [**?**, Problem 1.7(γ_2)] and [**?**]).

Corollary 6.3. *Assume that D, ${}_iD_j$, n, n_1, n_3, Λ and $\Lambda^\bullet$ are as in Theorem 1.5. Then the D-order $\Lambda^\bullet$ (1.4) of tame lattice type is of non-polynomial growth if and only if there exist an idempotent $\widehat{e} \in \Lambda^\bullet$ and a D-algebra isomorphism $\widehat{e}\Lambda^\bullet\widehat{e} \cong \widehat{\nabla}^\bullet_2$, where*

$$\widehat{\nabla}^\bullet_2 = \begin{pmatrix} \nabla_2 & \mathbb{M}_4(D) \\ & \equiv \\ \mathbb{M}_4(\mathfrak{p}) & \nabla_2 \end{pmatrix} \tag{6.4}$$

and the congruence $x \equiv y$ means $x - y \in \mathbb{M}_4(\mathfrak{p})$.

If this is the case then there exists a D-linear functor F : $\mathrm{latt}(\widehat{\nabla}_2) \to \mathrm{latt}(\Lambda^\bullet)$ which is a fully faithful left exact embedding.

Proof. (⇐) By Theorem 6.2 the D-order $\widehat{\nabla}_2$ is of tame lattice type and of non-polynomial growth. Then according to [**?**, Lemma 4.3(c)] the existence of an idempotent $\widehat{e} \in \Lambda^\bullet$ and a D-algebra isomorphism $\widehat{e}\Lambda^\bullet\widehat{e} \cong \widehat{\nabla}^\bullet_2$ yields the non-polynomial growth property of $\Lambda^\bullet$.

(⇒) Assume that $\Lambda^\bullet$ of tame lattice type is of non-polynomial growth. By Theorem 6.2, $n_3 = 0$ and Λ_1 contains ∇_2 as a minor D-order. Hence there exists an idempotent $e_1 \in \Lambda_1$ such that $e_1\Lambda_1 e_1 \cong \nabla_2$. Let $e_2 \in \Lambda_2$ be the idempotent corresponding to e_1 via the ring isomorphism $\Lambda_1 \cong \Lambda_2$. It is clear that the element

$$\widehat{e} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in \Lambda^\bullet = \begin{pmatrix} \Lambda_1 & \mathbb{M}_{n_1}(D) \\ & \equiv \\ \mathbb{M}_{n_1}(\mathfrak{p}) & \Lambda_2 \end{pmatrix}$$

is an idempotent such that there exists a required D-algebra isomorphism $\widehat{e}\Lambda^\bullet\widehat{e} \cong \widehat{\nabla}^\bullet_2$. This proves the first part of the corollary.

To finish the proof we note that if $\widehat{e} \in \Lambda^\bullet$ is an idempotent such that $\widehat{e}\Lambda^\bullet\widehat{e} \cong \widehat{\nabla}_2^\bullet$ then according to [**?**, Lemma 4.3] the functor $L_{\widehat{e}} : \mathrm{latt}(\widehat{e}\Lambda^\bullet\widehat{e}) \to \mathrm{latt}(\Lambda^\bullet)$ defined by the formula $L_{\widehat{e}}(-) = \mathrm{Hom}_{\widehat{e}\Lambda^\bullet\widehat{e}}(\widehat{e}\Lambda^\bullet, -)$ is a fully faithful left exact embedding. ∎

PROBLEM 6.5. *Is the existence of a fully faithful left exact D-linear embedding $F : \mathrm{latt}(\widehat{\nabla}_2) \to \mathrm{latt}(\Lambda^\bullet)$ in Corollary* 5.3 *sufficient for the non-polynomial growth of a tame lattice type D-order $\Lambda^\bullet$?*

7. Minimal three-partite subamalgams of tiled D-orders of wild lattice type. We present here the tables of Section 1 of [**?**] containing minimal three-partite subamalgams of tiled D-orders of wild lattice type.

We use the notation introduced in Section 1, that is, we exhibit three-partite tiled D-orders Ω_j, and the three-partition is indicated by vertical and horizontal lines. The subamalgam $\Omega_j^\bullet$ is obtained from Ω_j by identifying modulo $\mathfrak{p} = \mathrm{rad}(D)$ the upper left corner block with the lower right corner block.

TYPE $\widehat{\mathcal{F}}_1^1$

$$\Omega_1 = \left(\begin{array}{cccc|c|cccc}
D & D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 1$$

$$\Omega_2 = \left(\begin{array}{cccc|cc|cccc}
D & D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 2$$

TYPE $\widehat{\mathcal{F}}_1^2$

$$\Omega_3 = \left(\begin{array}{ccc|cc|ccc}
D & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) \quad n_3 = 2$$

$$\Omega_4 = \left(\begin{array}{ccc|ccc|ccc}
D & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 3$$

Type $\widehat{\mathcal{F}}_2$

$$\Omega_5 = \left(\begin{array}{cc|ccc|cc}
D & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 3$$

$$\Omega_6 = \left(\begin{array}{cc|cccc|cc}
D & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4$$

$$\Omega_7 = \left(\begin{array}{cc|ccccc|cc}
D & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5$$

Type $\widehat{\mathcal{F}}_3^1$

$$\Omega_8 = \left(\begin{array}{cccc|ccc|cccc}
D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 3$$

TYPE $\widehat{\mathcal{F}}_3^2$

$$\Omega_9 = \left(\begin{array}{ccccc|cc|ccccc}
D & D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 2$$

TYPE $\widehat{\mathcal{F}}_4$

$$\Omega_{10} = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4$$

$$\Omega_{11} = \left(\begin{array}{ccc|ccccc|ccc}
D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5$$

TYPE $\widehat{\mathcal{F}}_5$

$$\Omega_{12} = \left(\begin{array}{ccc|cccc|ccc}
D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 4$$

$$\Omega_{13} = \left(\begin{array}{ccc|ccccc|ccc}
D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5$$

Type $\widehat{\mathcal{F}}_6$

$$\Omega_{14} = \left(\begin{array}{cc|ccccc|cc}
D & D & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5$$

$$\Omega_{15} = \left(\begin{array}{cc|cccccc|cc}
D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 6$$

Type $\widehat{\mathcal{F}}_7$

$$\Omega_{16} = \left(\begin{array}{ccc|ccccc|ccc}
D & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & D & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 5$$

Type $\widehat{\mathcal{F}}_8$

$$\Omega_{17} = \left(\begin{array}{ccccccc|c|ccccccc}
D & D & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & D & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & D & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & \mathfrak{p} & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D & D \\
\hline
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D & D \\
\mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & D
\end{array}\right) n_3 = 1$$

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