Abstract. We discuss $k$-rotundity, weak $k$-rotundity, C-$k$-rotundity, and nearly uniform convexity. These properties are related to the continuity of metric projections in Orlicz spaces equipped with Luxemburg norm. Applications to continuity for the metric projection at a given point are given in Orlicz function spaces with Luxemburg norm.

Let $X$ be a Banach space, and $D$ be a subset of $X$. The metric projection $P_D : X \to 2^D$ is defined by $P_D(x) = \{y \in D : \|x - y\| = \text{dist}(x, D)\}$. $D$ is a proximinal (resp. Chebyshev) set if $P_D(x)$ contains at least (resp. exactly) one point for all $x$ in $X$. For a proximinal $D$, $P_D$ is called norm-norm (resp. norm-weak) upper semicontinuous at $x$ if for every normed (resp. weak) open set $W \supseteq P_D(x)$, there exists a normed neighborhood $U$ of $x$ such that $P_D(y) \subseteq W$ for all $y$ in $U$. It is proved in [Wa95] that if $X$ has the C-II (or C-III) property, then $P_D$ is continuous for any Chebyshev convex set $D$.

In this paper, we investigate some structures which imply the continuity of the metric projection at a given point for Orlicz function spaces with Luxemburg norm.

Let $B(X)$ and $S(X)$ be the unit ball and the unit sphere of the Banach space $X$ respectively. A point $x \in S(X)$ is said to be a locally C-I (resp. C-II, C-III) point of $B(X)$ if the following implication holds for every sequence $\{x_n\} \subseteq B(X)$: if for any $\delta > 0$ there exists an integer $m$ such that $\text{conv}(\{x\} \cup \{x_n\}_{n \geq m}) \cap (1 - \delta)B(X) = \emptyset$, then $\lim_{n \to \infty} x_n = x$ (resp. $\{x_n\}$ is relatively compact, weakly compact) [Wa95]. We call such points LC-I, LC-II, and LC-III points respectively.

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Recall that the Kuratowski measure of noncompactness \( \alpha(A) \) for \( A \subset X \) is defined as
\[
\alpha(A) = \inf \{ \varepsilon > 0 : A \text{ can be covered by a finite family of sets of diameter less than } \varepsilon \}.
\]

A slice of \( B(X) \) is defined by \( S(f, \eta) = \{ x \in B(X) : f(x) > 1 - \eta \} \) where \( f \in S(X^*) \) and \( \eta > 0 \).

Let \( \mathbb{R} \) be the set of all real numbers. A function \( M : \mathbb{R} \to \mathbb{R}_+ \) is called an Orlicz function if \( M \) is convex, even, \( M(0) = 0 \) and \( M(\infty) = \infty \). The complementary function \( N \) of \( M \) in the sense of Young is defined by
\[
N(v) = \sup_{u \in \mathbb{R}} \{ uv - M(u) \}.
\]

It is known that if \( M \) is an Orlicz function, then so is \( N \). \( M \) is said to be strictly convex if \( M((u + v)/2) < (M(u) + M(v))/2 \) for all \( u \neq v \). An interval \( (a, b) \) is said to be an affine interval of \( M \) if \( M \) is affine on \( (a, b) \) and \( M \) is strictly convex on \( (b, b + \varepsilon) \) and \( (a - \varepsilon, a) \) for some \( \varepsilon > 0 \). Denote all affine intervals of \( M \) by \( \bigcup_{i=1}^{\infty} (a_i, b_i) \).

\( M \) is said to satisfy the \( \Delta_2 \)-condition for large \( u \) (we simply write \( M \in \Delta_2 \)) if for some \( K \) and \( u_0 > 0 \), \( M(2u) \leq KM(u) \) for \( |u| \geq u_0 \).

Let \( G \) be a bounded set in \( \mathbb{R}^n \) and let \( (G, \Sigma, \mu) \) be a finite non-atomic measure space. For a real-valued measurable function \( x(t) \) over \( G \), we call \( \varrho_M(x) = \int_G M(x(t)) \, d\mu(t) \) the modular of \( x \). The Orlicz function space \( L_M \) generated by \( M \) is the Banach space
\[
L_M = \{ x = x(t) : \exists \lambda > 0, \varrho_M(\lambda x) < \infty \}
\]
equipped with the Luxembourg norm
\[
\| x \| = \inf \{ \lambda : \varrho_M(x/\lambda) \leq 1 \}.
\]

For information on Orlicz spaces, see [KrRu61, Ch96].

First we recall some lemmas.

**Lemma 1** [LiSh96]. In an Orlicz function space \( L_M \) equipped with Luxembourg norm, let \( x \in S(L_M) \). If \( M \) does not satisfy the \( \Delta_2 \)-condition, then \( \alpha(S(f, \eta)) \geq 1/4 \) for any slice \( S(f, \eta) \) of \( B(L_M) \) containing \( x \).

**Lemma 2** [LiSh96]. In an Orlicz function space \( L_M \) equipped with Luxembourg norm, let \( x \in S(L_M) \). If \( \mu \{ t \in G : x(t) \in \bigcup_{i=1}^{\infty} (a_i, b_i) \} > 0 \), where \( \bigcup_{i=1}^{\infty} (a_i, b_i) \) is the family of all affine intervals of \( M \), then \( \alpha(S(f, \eta)) \geq \theta > 0 \) for any slice \( S(f, \eta) \) of \( B(L_M) \) containing \( x \), where \( \theta \) is a constant that depends only on \( x \).
Theorem 1. In an Orlicz function space \( L(M) \) equipped with Luxemburg norm, let \( x \in S(L(M)) \). Then the following are equivalent:

1. \( x \) is an LC-II point of \( B(L(M)) \).
2. (i) \( M \in \Delta_2 \),
   (ii) \( \mu \{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} (a_i, b_i) \} = 0 \), where \( \bigcup_{i=1}^{\infty} (a_i, b_i) \) is all affine intervals of \( M \),
   (iii) if \( \mu \{ t \in G : |x(t)| = b \} > 0 \) for some affine interval \( (a, b) \), then \( N \in \Delta_2 \) and \( \mu \{ t \in G : |x(t)| = c \} = 0 \) for all affine intervals \( (c, d) \) of \( M \).
3. \( x \) is an LUR point of \( B(L(M)) \), i.e., for all sequences \( \{ x_n \} \) in \( B(L(M)) \),
   \( \lim_{n \to \infty} \| x_n - x \| = 0 \) whenever \( \lim_{n \to \infty} \| x_n + x \| = 2 \).

Proof. (1)\(\Rightarrow\)(2). (i) Suppose that \( M \notin \Delta_2 \). Then (see the proof of Lemma 1 in [LiSh96]) there is a sequence \( \{ x_n \} \) satisfying

\[
x_n = x|_{G \setminus G_n} + (x + u_n)|_{G_n}, \quad \lim_{n \to \infty} \| x_n \|_{(M)} = 1, \quad \alpha(\{ x_n \}) \geq 1/4,
\]

and \( x_n \to x \) weakly. For every \( \delta > 0 \) there exists an integer \( N \) so that

\[
\text{conv}(\{ x \} \cup \{ x_n \}_{n \geq N}) \cap (1 - \delta)B(X) = \emptyset; \quad \text{but } \alpha(\{ x_n \}) \geq 1/4, \text{ which contradicts } x \text{ being an LC-II point of } B(L(M)).
\]

(ii) Suppose \( \mu \{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} (a_i, b_i) \} > 0 \). By Lemma 2, there exists a sequence \( \{ x_n \} \) in \( B(L(M)) \) satisfying \( \alpha(\{ x_n \}) \geq \theta \) and \( x_n \to x \) weakly, where \( \theta \) depends only on \( x \), which implies that \( x \) is not an LC-II point of \( B(L(M)) \), a contradiction.

(iii) Suppose that \( \mu B = \mu \{ t \in G : |x(t)| = b \} > 0 \) and \( \mu C = \mu \{ t \in G : |x(t)| = c \} > 0 \) for some affine intervals \( (a, b) \) and \( (c, d) \) of \( M \). Take \( B_0 \subset B \) and \( C_0 \subset C \) with \( \mu B_0 > 0, \mu C_0 > 0 \) and

\[
[M(b) - M(a)]\mu B_0 = [M(d) - M(c)]\mu C_0
\]

(i.e., \( M(b)\mu B_0 + M(c)\mu C_0 = M(a)\mu B_0 + M(d)\mu C_0 \)). Set

\[
z = x|_{G \setminus (B_0 \cup C_0)} + \frac{a + b}{2} \text{sign } x|_{B_0} + \frac{c + d}{2} \text{sign } x|_{C_0}.
\]

Then

\[
\varrho_M(z) = \varrho_M(x|_{G \setminus (B_0 \cup C_0)}) + \frac{M(a) + M(b)}{2} \mu B_0 + \frac{M(c) + M(d)}{2} \mu C_0
\]

As in the proof of Lemma 2, there exists a sequence \( \{ z_n \} \) in \( B(L(M)) \) satisfying \( \alpha(\{ z_n \}) \geq \theta \) and \( z_n \to z \) weakly, where \( \theta \) depends only on \( z \), hence only on \( x \). Let \( y = x|_{G \setminus (B_0 \cup C_0)} + a \text{sign } x|_{B_0} + d \text{sign } x|_{C_0} \). Then

\[
\varrho_M(y) = \varrho_M(x|_{G \setminus (B_0 \cup C_0)}) + M(a)\mu B_0 + M(d)\mu C_0 = \varrho_M(x) = 1
\]
and \( z = (x + y)/2 \). Since \( \|x\|_{(M)} = \|y\|_{(M)} = \|z\|_{(M)} = 1 \), there is \( f \in L^*_M \) with \( f(x) = f(z) = \|f\| = 1 \). Since \( z_n \to z \) weakly, for any \( \delta > 0 \) there exists an integer \( N \) so that \( \text{conv}(\{x\} \cup \{z_n\}_{n \geq N}) \cap (1 - \delta)B(X) = \emptyset \); but \( \alpha(\{z_n\}) \geq \theta \) contradicts \( x \) being an LC-II point of \( B(L_M) \).

Suppose \( \mu B = \mu \{ t \in G : |x(t)| = b \} > 0 \) for some affine interval \((a, b)\) of \( M \) and \( N \not\in \Delta_2 \). Since \( N \not\in \Delta_2 \), there exist \( u_n \not\to \infty \) such that

\[
2^n M \left( \frac{1}{2n} u_n \right) > \left( 1 - \frac{1}{n} \right) M(u_n).
\]

Without loss of generality, assume that \( x(t) = b \) on \( B \). Take subsets \( B_n \) in \( B \) such that \( B \supset B_1 \supset B_2 \supset \ldots \) and

\[
[M(u_n) - M(a)]\mu B = [M(b) - M(a)]\mu B.
\]

Then \( M(u_n)\mu B_n \geq [M(b) - M(a)]\mu B \). Set

\[
x_n = x|_{G \setminus B} + a|_{B \setminus B_n} + u_n|_{B_n}.
\]

Then

\[
\varrho_M(x_n) = \varrho_M(x|_{G \setminus B}) + M(a)(\mu B - \mu B_n) + M(u_n)\mu B_n
\]

\[
= \varrho_M(x|_{G \setminus B}) + M(b)\mu B = \varrho_M(x) = 1.
\]

Obviously

\[
\limsup_{\beta \to 0} \frac{\varrho_M(\beta x_n)}{\beta} \geq [M(b) - M(a)]\mu B > 0,
\]

by [An62], \( \{x_n\} \) is not weakly compact and so \( \alpha(\{x_n\}) \geq \theta > 0 \). For any \( \delta > 0 \), take \( K > 0 \) such that \( 2/K \leq \delta \). Set \( x_{n_0} = x \). Then for all \( K < n_1 < \ldots < n_k \) and any \( \sum_{i=0}^k \lambda_i = 1 \), \( \lambda_i \geq 0 \), we have

\[
\varrho_M \left( \sum_{i=0}^k \lambda_i x_{n_i} \right) = \varrho_M(x|_{G \setminus B}) + M(\lambda_0 b + \sum_{i=1}^k \lambda_i a)\mu(B \setminus B_{n_1})
\]

\[
+ M \left( \sum_{i=1}^k \lambda_i u_{n_i} + \lambda_0 b \right) \mu B_{n_k}
\]

\[
+ M \left( \sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_{k-1} a \right)_{B_{n_1} \setminus B_{n_k}}
\]

\[
\geq \varrho_M(x|_{G \setminus B}) + \left( \lambda_0 M(b) + \sum_{i=1}^k \lambda_i M(a) \right) \mu(B \setminus B_{n_1})
\]

\[
+ \sum_{i=1}^k (1 - 1/n_i)\lambda_i M(u_{n_i})\mu B_{n_k} + M(\lambda_0 b)\mu B_{n_k}
\]
\[ + M\left( \sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a \right) \mu(B_{n_k-1} \setminus B_{n_k}) \\
+ M\left( \left( \sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a \right) \bigg|_{B_{n_1} \setminus B_{n_k-1}} \right) \geq \varrho_{\mathcal{M}}(x|G\setminus B) + \left( \lambda_0 M(b) + \sum_{i=1}^{k} \lambda_i M(a) \right) \mu(B \setminus B_{n_1}) \]

\[ + \sum_{i=1, \lambda_i \geq 1/2^n}^{k} (1 - 1/n_i) \lambda_i M(u_{n_i}) \mu(B_{n_i} \setminus B_{n_{i+1}}) \]

\[ + \sum_{j=1}^{k} \lambda_0 b + (\lambda_{j+1} + \ldots + \lambda_k) a) \mu(B_{n_j} \setminus B_{n_{j+1}}) \geq \varrho_{\mathcal{M}}(x|G\setminus B) + \left( \lambda_0 M(b) + \sum_{i=1}^{k} \lambda_i M(a) \right) \mu(B \setminus B_{n_1}) \]

\[ + \sum_{j=1}^{k} \sum_{i=1, \lambda_i \geq 1/2^n}^{j} \lambda_i M(u_{n_i}) \mu(B_{n_j} \setminus B_{n_{j+1}}) \]

\[ + \sum_{j=1}^{k} M(\lambda_0 b + (\lambda_{j+1} + \ldots + \lambda_k) a) \mu(B_{n_j} \setminus B_{n_{j+1}}) \geq (1 - 1/n_1) \sum_{i=1}^{k} \lambda_i \theta_M(x_{n_i}) - (1 - 1/n_1) \sum_{i=1, \lambda_i < 1/2^n}^{k} \lambda_i \theta_M(x_{n_i}) \]

\[ \geq (1 - 1/n_1) - \sum_{i=1}^{k} 1/2^n = (1 - 1/K) - 1/2^K > 1 - \delta. \]
Hence \( \text{conv}(\{x\} \cup \{x_n, n \geq K\}) \cap (1-\delta)B(X) = \emptyset \); but \( \alpha(\{x_n\}) > 0 \) contradicts \( x \) being an LC-II point of \( B(L(M)) \).

(2)\( \Rightarrow \) (3). By [ChWa92], it follows that \( x \) is an LUR point of \( B(L(M)) \).

(3)\( \Rightarrow \) (1). Obvious.

For an integer \( k \), a point \( x \in S(X) \) is said to be:

- a locally \( k \)-rotund (LkR) point of \( B(X) \) if for any sequence \( \{x_n\} \) in \( B(X) \), \( \lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k + 1 \) implies \( \lim_{n \to \infty} \|x_n - x\| = 0 \);

- a locally weakly \( k \)-rotund (LWkR) point of \( B(X) \) if for any sequence \( \{x_n\} \) in \( B(X) \), \( \lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k + 1 \) implies \( w\)-\( \lim_{n \to \infty} x_n = x \);

- a locally \( C \)-\( k \)-rotund (LCkR) point of \( B(X) \) if for any sequence \( \{x_n\} \) in \( B(X) \), \( \lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k + 1 \) implies \( \{x_n\} \) is a relatively compact set;

- a locally \( k \)-nearly uniformly convex (LkNUC) point of \( B(X) \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all sequences \( \{x_n\} \) with \( \text{sep}(x_n) \geq \varepsilon \) there are \( \{n_1, \ldots, n_k\} \) with
  \[ \left\| x + x_{n_1} + \cdots + x_{n_k} \right\| \leq 1 - \delta; \]

- a locally \( k \)-\( \beta \) (Lk\( \beta \)) point of \( B(X) \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all sequences \( \{x_n\} \) with \( \text{sep}(x_n) \geq \varepsilon \) there are \( \{n_1, \ldots, n_k\} \) with \( \text{conv}(\{x, x_{n_1}, \ldots, x_{n_k}\}) \cap (1-\delta)B(X) \neq \emptyset \);

- a locally nearly uniformly convex (LNUC) point of \( B(X) \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all sequences \( \{x_n\} \) with \( \text{sep}(x_n) \geq \varepsilon \) we have \( \text{conv}(\{x\} \cup \{x_n\}) \cap (1-\delta)B(X) \neq \emptyset \).

It is easy to see that for all Banach spaces, we have the implications

\[
\begin{array}{c c c c c c c c c c c}
\text{LUR} & \rightarrow & \text{LkR} & \rightarrow & \text{LCkR} & \downarrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow & \text{LWkR} & \downarrow & \text{LC-II} & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}
\]

\( \text{LkNUC} \rightarrow \text{Lk}\beta \rightarrow \text{LNUC} \)

For these properties, we refer to [Ku91, KuLi94, KuLi93, Wa95].

**Corollary 1.** In an Orlicz function space \( L(M) \) equipped with Luxemburg norm, let \( x \in S(L(M)) \). Then the following are equivalent:

1. \( x \) is an LUR point of \( B(L(M)) \) [ChWa92];
2. \( x \) is an LkR point of \( B(L(M)) \) \((k \geq 1)\);
3. \( x \) is an LWkR point of \( B(L(M)) \) \((k \geq 1)\);
4. \( x \) is an LCkR point of \( B(L(M)) \) \((k \geq 1)\);
5. \( x \) is an LkNUC point of \( B(L(M)) \) \((k \geq 1)\);
(6) $x$ is an $Lk$-$\beta$ point of $B(L_{(M)})$ ($k \geq 1$);
(7) $x$ is an LNUC point of $B(L_{(M)})$;
(8) $x$ is an LC-I point of $B(L_{(M)})$;
(9) $x$ is an LC-II point of $B(L_{(M)})$;
(10) $M \in \triangle_2$, $\mu\{t \in G : \|x(t)\| \leq \sum_{i=1}^{\infty} (a_i, b_i)\} = 0$, where $\{(a_i, b_i)\}$ is the family of all affine intervals of $M$, and if $\mu\{t \in G : \|x(t)\| = b\} > 0$ for some affine interval $(a, b)$ of $M$, then $N \in \triangle_2$ and $\mu\{t \in G : \|x(t)\| = c\} = 0$ for all affine intervals $(c, d)$ of $M$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (9), (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7), and (1) $\Rightarrow$ (8) $\Rightarrow$ (9) are trivial by definitions.

(7) $\Rightarrow$ (9). By Theorem 4 of [Wa95], an LNUC point is an LC-II point in $B(X)$.
(10) $\Rightarrow$ (1). This is proved in [ChWa92].
(9) $\Rightarrow$ (10). This follows from Theorem 1.
(3) $\Rightarrow$ (10). Since $\|x\|_{(M)} = 1$, there is $c > 0$ such that $\mu G_c = \mu\{t \in G : \|x(t)\| \leq c\} > 0$.

Suppose that $M \notin \triangle_2$. Then there exist $u_n \not\to \infty$ such that

$$M((1 + 1/n)u_n) > 2^n M(u_n).$$

On passing to a subsequence if necessary, there are disjoint subsets $G_n \subset G_c$ so that

$$M(u_n)\mu G_n = 1/2^n, \quad n = 1, 2, \ldots.$$  

Define $y = \sum_{n=1}^{\infty} u_n|G_n$. Then $\varrho_M(y) = \sum_{n=1}^{\infty} M(u_n)\mu G_n = 1$, $\|y\|_{(M)} = 1$ and $\text{dist}(y, E_M) = 1$, where $E_M = \{x : \varrho_M(\lambda x) < \infty \text{ for all } \lambda\}$. By the Hahn–Banach theorem, there is a functional $\phi$ such that $\varrho(y) = \|\phi\| = 1$, and $\phi(z) = 0$ for all $z \in E_M$. Set $x_n = x|G \setminus \bigcup_{i>n} G_i + y|\bigcup_{i>n} G_i$. Then

$$\left\|x + x_{n_1} + \ldots + x_{n_k}\right\|_{(M)} = \|x|G \setminus \bigcup_{i>n_k} G_i\|_{(M)} \to 1 \quad (n_1, \ldots, n_k \to \infty)$$

and

$$\varrho_M(x_n) = \varrho_M(x|G \setminus \bigcup_{i>n_k} G_i) + \varrho_M(y|\bigcup_{i>n} G_i) \to \varrho_M(x) \leq 1.$$  

But

$$\varrho(x_n - x) = \varrho(y|\bigcup_{i>n_k} G_i) - \varphi(x|\bigcup_{i>n_k} G_i) = \varrho(y|\bigcup_{i>n} G_i) = 1.$$  

So $x_n \not\to x$ weakly, contrary to $x$ being an LWkR point of $B(L_{(M)})$.

We claim that $\mu\{t \in G : \|x(t)\| \in \bigcup_{i=1}^{\infty} (a_i, b_i)\} = 0$.

In fact, if this measure is positive, then $\mu E > 0$, where $E = \mu\{t \in G : x(t) \in (a + 2\delta, b - 2\delta)\}$ for some $\delta > 0$. Split $E$ into two parts $E_1$ and $E_2$.  

with \( \mu E_1 = \mu E_2 = (\mu E)/2 \). Define
\[
z = x|_{G \setminus E} + (x + 2\delta)|_{E_1} + (x - 2\delta)|_{E_2}.
\]
Then
\[
\varrho_M(z) = \varrho_M(x|_{G \setminus E}) + \varrho_M((x + 2\delta)|_{E_1}) + \varrho_M((x - 2\delta)|_{E_2}) = \varrho_M(x|_{G \setminus E}) + \varrho_M(x|_{E_1}) + \varrho_M(x|_{E_2}) + 1,
\]
\[
\varrho_M\left(\frac{x + z}{2}\right) = \varrho_M(x|_{G \setminus E}) + \varrho_M((x + \delta)|_{E_1}) + \varrho_M((x - \delta)|_{E_2}) = \varrho_M(x|_{G \setminus E}) + \varrho_M(x|_{E_1}) + \varrho_M(x|_{E_2}) = 1.
\]
Moreover \( x \neq z \). As in Lemma 2, there exists a sequence \( \{z_n\} \) in \( B(L(M)) \) such that \( z_n \to z \) weakly and \( \text{sep}\{z_n\} \geq \theta > 0 \), where \( \theta \) depends only on \( z \). For \( k > 1 \), since \( z_n \to z \) weakly and \( \|x + z\|_{(M)} = 2 \), we have \( \lim_{n_1, \ldots, n_k \to \infty} \|x + z_{n_1} + \ldots + z_{n_k}\| = k + 1 \). This contradicts \( x \) being an LWkR point of \( B(L(M)) \). For \( k = 1 \) we can take \( x_n = z \) to get a contradiction.

From Theorem 1, it follows that if \( \mu\{t \in G : |x(t)| = b\} > 0 \) for some affine interval \((a, b)\) of \( M \), then \( N \in \Delta_2 \) and \( \mu\{t \in G : |x(t)| = c\} = 0 \) for all affine intervals \((c, d)\) of \( M \).

**Corollary 2.** In an Orlicz function space \( L(M) \) equipped with Luxemburg norm, the following are equivalent:

1. \( L(M) \) is locally UR [ChWa92, Ka84];
2. \( L(M) \) is locally kR \((k \geq 1)\);
3. \( L(M) \) is locally WkR \((k \geq 1)\);
4. \( L(M) \) is locally CKR \((k \geq 1)\);
5. \( L(M) \) is locally kNUC \((k \geq 1)\);
6. \( L(M) \) is locally k-\( \beta \) \((k \geq 1)\);
7. \( L(M) \) is locally NUC;
8. \( L(M) \) has the C-I property;
9. \( L(M) \) has the C-II property;
10. \( M \in \Delta_2 \) and \( M \) is strictly convex on the real line.

**Corollary 3.** In an Orlicz function space \( L(M) \) equipped with Luxemburg norm, suppose \( M \in \Delta_2 \) and let \( x \in S(L(M)) \). If \( \mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} = 0 \) and either \( \mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{b_i\}\} = 0 \), or \( N \in \Delta_2 \) and \( \mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{a_i\}\} = 0 \), then every proximinal metric projection \( P_D \) is norm-norm upper semicontinuous at \( x \).

Moreover, if \( M \in \Delta_2 \) and \( M \in SC \), then every proximinal metric projection \( P_D \) is norm-norm upper semicontinuous.

Next, we study the LC-III points.
Lemma 3. For an Orlicz space $L(M)$, suppose $M \in \triangle_2$. Then

(1) for any $\varepsilon > 0$ there is $\eta > 0$ such that
\[
\varrho_M(x) < \eta \Rightarrow \|x\|_M < \varepsilon,
\]
\[
\|x\|_M > 1 - \eta \Rightarrow \varrho_M(x) > 1 - \varepsilon;
\]

(2) if $\varrho_M(x_n) \to \varrho_M(x)$ and $x_n \xrightarrow{w} x$ in measure, then $x_n \to x$ in norm.

For a proof, see [Ch86, Hu83, HuLa95].

Theorem 2. In an Orlicz function space $L(M)$ equipped with Luxemburg norm, let $x \in S(L(M))$. Then $x$ is a C-III point of $B(L(M))$ if and only if

(1) $M \in \triangle_2$;

(2) either $N \in \triangle_2$, or $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} (a_i, b_i)\} = 0$ and $\mu\{t \in G : |y(t)| \in \bigcup_{i=1}^{\infty} \{b_i\}\} = 0$.

Proof. Choose $c > 0$ such that $\mu G_c = \mu\{t \in G : |x(t)| \leq c\} > 0$. Suppose $M \not\in \triangle_2$. There exists [KrRu61] $y \in L(M)$ with $\text{supp} y \subset G_c$, $\|y\|_M = \text{dist}(y, E_M) = 1$, and $\phi \in L^*_M$ with $\phi(y) = \|\phi\| = \text{dist}(y, E_M) = 1$ and $\phi(z) = 0$ for all $z \in E_M$, and $G_n \subset G_c$, where $G_n = \{t \in G : |y(t)| \geq n\}$. Set
\[
x_n = x|_{G_n G_n} + y|_{G_n}.
\]
Then for $\theta > 0$, take $n_0$ such that $\|x|_{G \setminus G_{n_0}}\|_M > 1 - \theta$. Then for all $n_0 < n_1 < \ldots < n_k$ and for any $\sum_{i=0}^{k} \lambda_i = 1$, where $\lambda_i \geq 0,$
\[
\left\| \sum_{i=0}^{k} \lambda_i x_{n_i} \right\|_M \geq \|x|_{G \setminus G_{n_k}}\|_M > 1 - \theta.
\]
But $\{x_n\}$ is not relatively weakly compact. In fact, otherwise by the Shmul'yan Theorem $\{x_n\}$ is relatively weakly sequentially compact. By taking a subsequence if necessary we may assume that $x_n \xrightarrow{w^*} x'$ in the weak topology. Combining this with $x_n \xrightarrow{w} x$ in the $w^*$ topology, we get $x_n \xrightarrow{w} x$. A contradiction since $\phi(x_n - x) = \phi(y|_{G_n}) + \phi(x|_{G_n}) = \phi(y|_{G_n}) = 1$.

Assume that $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} (a_i, b_i)\} > 0$. Then $\mu B = \mu\{t \in G : x(t) \in (a + \theta, b - \theta)\} > 0$ for some affine interval $(a, b)$ and some $\theta > 0$. Split $B$ into two parts $B', B''$ with $\mu B' = \mu B'' = (\mu B)/2$. Define
\[
y = x|_{G \setminus B} + (x - \theta)|_{B'} + (x + \theta)|_{B''}.
\]
Then
\[
\varrho_M(y) = \varrho_M(x|_{G \setminus B}) + \varrho_M((x - \theta)|_{B'}) + \varrho_M((x + \theta)|_{B''})
\]
\[
= \varrho_M(x|_{G \setminus B}) + \varrho_M(x|_{B'}) + \varrho_M(x|_{B''}) = 1,
\]
and
\[
\varrho_M\left(\frac{x + y}{2}\right) = \varrho_M(x) = 1.
\]
If \( N \not\in \triangle_2 \), then there exists a real sequence \( \{u_n\} \) such that \( u_n \not\to \infty \) and
\[
2^n M\left( \frac{1}{2^n} u_n \right) > \left( 1 - \frac{1}{n} \right) M(u_n).
\]
Take decreasing subsets \( \{B_n\} \) of \( B \) such that
\[
\varrho_M(y_B) - M(a)\mu B = \varrho_M(x_B) - M(a)\mu B = [M(u_n) - M(a)]\mu B_n.
\]
Then \( M(u_n)\mu B_n \geq \varrho_M(x_B) - M(a)\mu B > 0 \). Set
\[
x_n = x_B|G_B + a|B_B + u_n|B_n.
\]
By [An62], \( \{x_n\} \) is not weakly compact. But
\[
\varrho_M(x_n) = \varrho_M(x_{G\setminus B}) + M(a)(\mu B - \mu B_n) + M(u_n)\mu B_n = \varrho_M(x) = 1.
\]
For any \( \delta > 0 \), take \( K \) such that \( 2/K \leq \delta \). Let \( x_{n_0} = x \). Then for all \( K < n_1 < \ldots < n_k \) and for any \( \sum_{i=0}^k \lambda_i = 1 \), where \( \lambda_i \geq 0 \), as in the proof of Theorem 1,
\[
\varrho_M\left( \sum_{i=0}^k \lambda_i x_{n_i} \right) \geq 1 - \delta.
\]
This contradicts \( x \) being a C-III point of \( B(L(M)) \).

By the same argument as for the second part of (iii) in Theorem 1 we can show that if \( x \) is a locally C-III point of \( B(L(M)) \) then \( \mu\{t \in G : |x(t)| = b\} > 0 \) for some affine interval \( (a, b) \) of \( M \) implies \( N \in \triangle_2 \).

Suppose \( \{x_n\} \) is a sequence in \( B(L(M)) \) such that for any \( \delta > 0 \) there exists an integer \( N \) with \( \text{conv}\(\{x\} \cup \{x_n\}_{n \geq N}\) \( \cap (1 - \delta)B(L(M)) = \emptyset \).

If \( N \in \triangle_2 \), then by (1), \( L(M) \) is reflexive. So \( B(L(M)) \) is weakly compact and \( \{x_n\} \) is relatively weakly compact.

If \( N \not\in \triangle_2 \), then we show that \( \lim_{n \to \infty} x_n = x \). By Lemma 3, it suffices to show that \( x_n \overset{M}{\to} x \) in measure. By (2), \( \mu\{t \in G : |x(t)| \in \bigcup_{i=1}^\infty (a_i, b_i)\} = 0 \) and \( \mu\{t \in G : |x(t)| = b\} = 0 \) for all affine intervals \( (a, b) \). Since \( \lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \ldots + x_{n_k}\|(M) = k + 1 \), we have \( \lim_{n \to \infty} \|x + x_n\|(M) = 2 \). From
\[
1 = \frac{\varrho_M(x) + \varrho_M(x_n)}{2} \geq \frac{\varrho_M(x + x_n)}{2} \to 1,
\]
it follows that \( x_n \overset{M}{\to} x \) in measure on \( \{t \in G : |x(t)| \not\in G \setminus \bigcup_{i=1}^\infty (a_i, b_i)\} \).

We claim: \( x_n \overset{a}{\to} x \) in measure on \( G_a = \{t \in G : |x(t)| = a\} \) for every left endpoint \( a \) of an affine interval \( (a, b) \). Without loss of generality, assume that \( G_a = \{t \in G : |x(t)| = a\} \).

We first show that for any \( \varepsilon > 0 \), \( \mu\{t \in G_a : x_n(t) \leq a - \varepsilon\} \to 0 \) as \( n \to \infty \). Indeed, if for some \( \varepsilon_0 > 0 \) and \( \sigma_0 > 0 \) and a subsequence of \( \{x_n\} \) (again denoted by \( \{x_n\} \)) we have \( \mu G_n = \mu\{t \in G_a : x_n(t) \leq a - \varepsilon_0\} \geq \sigma_0 > 0 \)
for all $n$, then there exists a $\delta_0 > 0$ such that

$$M\left( \frac{a + a - \varepsilon_0}{2} \right) \leq \frac{1}{2}(1 - \delta_0)[M(a) + M(a - \varepsilon_0)]$$

(because $c \neq d$ for all affine intervals $(c,d)$). Hence

$$\varrho_M\left( \frac{x + x_n}{2} \right) \leq \frac{1}{2}[\varrho_M(x|G \cap G_n) + \varrho_M(x_n|G \cap G_n)] + M\left( \frac{a + a - \varepsilon_0}{2} \right)\mu G_n$$

$$\leq \frac{1}{2}[\varrho_M(x|G \cap G_n) + \varrho_M(x_n|G \cap G_n)] + \frac{1}{2}(1 - \delta_0)[M(a) + M(a - \varepsilon_0)]\mu G_n$$

$$\leq \frac{1}{2}[\varrho_M(x) + \varrho_M(x_n)] - \frac{1}{2}\delta_0[M(a) + M(a - \varepsilon_0)]\mu G_n$$

$$\leq 1 - \frac{1}{2}\delta_0[M(a) + M(a - \varepsilon_0)]\mu G_n < 1.$$  

By Lemma 3, $\lim_{n \to \infty} \|x + x_n\|_{(M)} < 2$, a contradiction.

Next we show that for any $\varepsilon > 0$, $\mu \{ t \in G_a : x_n(t) \geq a + \varepsilon \} \to 0$ as $n \to \infty$. Indeed, suppose that for some $\varepsilon_0 > 0$ and $\sigma_0 > 0$ and a subsequence $\{x_n\}$ (again labeled $\{x_n\}$) we have $\mu G_n = \mu \{ t \in G_n : x_n(t) \geq a + \varepsilon_0 \} \geq \sigma_0$ for all $n$. Since

$$G = \left\{ t \in G : |x(t)| \not\in \bigcup_{i=1}^{\infty}[a_i, b_i] \right\} \cup \left\{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i) \right\}$$

$$\cup \left\{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(b_i) \right\} \cup \left\{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i) \right\},$$

by the Fatou Lemma, we see that for all $G' \subset G$,

$$\liminf_{n \to \infty} \varrho_M(x_n|G') \geq \varrho_M(x|G').$$

Hence for $n$ large enough,

$$\varrho_M(x_n) = \varrho_M(x_n|G \cap G_n) + \varrho_M(x_n|G_n)$$

$$\geq \varrho_M(x_n|G \cap G_n) + M(a + \varepsilon_0)\mu G_n$$

$$= \varrho_M(x_n|G \cap G_n) + M(a)\mu G_n + [M(a + \varepsilon_0) - M(a)]\mu G_n$$

$$\geq \varrho_M(x) + [M(a + \varepsilon_0) - M(a)]\sigma_0 > 1,$$

a contradiction.

We now show that $x_n \overset{\mu}{\rightarrow} x$ in measure on $\{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{a_i\} \}$. Indeed, for every $\varepsilon > 0$ and $\sigma > 0$, take $i_0$ such that $\mu \{ t \in G : |x(t)| \in \bigcup_{i=i_0}^{\infty}\{a_i\} \} < \varepsilon/2$. From the claim we deduce that for $n$ large enough,

$$\mu \left\{ t \in G : |x(t)| \in \bigcup_{i=1}^{i_0}\{a_i\} \text{ and } |x_n(t) - x(t)| \geq \sigma \right\} < \frac{\varepsilon}{2}.$$  

From the decomposition of $G$ as above we get $x_n \overset{\mu}{\rightarrow} x$ in measure on $G$.  


By Lemma 3, we know that \( x_n \to x \) in norm, so \( \{x_n\} \) is relatively weakly compact.

**Remark.** By the same argument we can show that an element in \( S(L_{(M)}) \) is a locally C-III point of \( B(L_{(M)}) \) iff it is a locally W\( k \)R point of \( B(L_{(M)}) \).

**Corollary 4.** In an Orlicz function space \( L_{(M)} \) equipped with Luxemburg norm, the following are equivalent:

1. \( L_{(M)} \) is locally W\( k \)R;
2. \( L_{(M)} \) has the C-III property;
3. \( M \in \triangle_2 \) and either \( M \in SC \) or \( N \in \triangle_2 \).

**Corollary 5.** In an Orlicz function space \( L_{(M)} \) equipped with Luxemburg norm, suppose \( M \in \triangle_2 \) and let \( x \in S(L_{(M)}) \). If \( \mu \{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} (a_i, b_i) \} = 0 \) and \( \mu \{ t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} \{b_i\} \} = 0 \), then every proximal metric projection \( P_D \) is norm-weak upper semicontinuous at \( x \).

Moreover, if \( M \in \triangle_2 \), and either \( M \in SC \) or \( N \in \triangle_2 \), then every proximal metric projection \( P_D \) is norm-weak upper semicontinuous on \( L_{(M)} \).

**REFERENCES**


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