COLLOQUIUM MATHEMATICUM

VOL. 81 1999 NO. 2

SOME STRUCTURES RELATED TO METRIC PROJECTIONS IN ORLICZ SPACES

BY

BOR-LUH LIN (IOWA CITY, IA) AND
ZHONGRUI SHI (IOWA CITY, IA, AND HARBIN)

Abstract. We discuss k-rotundity, weak k-rotundity, C-k-rotundity, weak C-k-rotundity, k-nearly uniform convexity, k-β property, C-I property, C-II property, C-III property and nearly uniform convexity both pointwise and global in Orlicz function spaces equipped with Luxemburg norm. Applications to continuity for the metric projection at a given point are given in Orlicz function spaces with Luxemburg norm.

Let X be a Banach space, and D be a subset of X. The *metric projection* $P_D : X \to 2^D$ is defined by $P_D(x) = \{y \in D : \|x - y\| = \mathrm{dist}(x, D)\}$. D is a *proximinal* (resp. *Chebyshev*) *set* if $P_D(x)$ contains at least (resp. exactly) one point for all x in X. For a proximinal D, P_D is called *norm-norm* (resp. *norm-weak*) *upper semicontinuous* at x if for every normed (resp. weak) open set $W \supseteq P_D(x)$, there exists a normed neighborhood U of x such that $P_D(y) \subseteq W$ for all y in U. It is proved in [Wa95] that if X has the C-II (or C-III) property, then P_D is continuous for any Chebyshev convex set D. In this paper, we investigate some structures which imply the continuity of the metric projection at a given point for Orlicz function spaces with Luxemburg norm.

Let $B(X)$ and $S(X)$ be the unit ball and the unit sphere of the Banach space X respectively. A point $x \in S(X)$ is said to be a *locally C-I* (resp. *C-II*, *C-III*) *point* of $B(X)$ if the following implication holds for every sequence $\{x_n\} \subseteq B(X)$: if for any $\delta > 0$ there exists an integer m such that $\mathrm{conv}(\{x\} \cup \{x_n\}_{n \ge m}) \cap (1 - \delta)B(X) = \emptyset$, then $\lim_{n\to\infty} x_n = x$ (resp. $\{x_n\}$ is relatively compact, weakly compact) [Wa95]. We call such points LC-I, LC-II, and LC-III points respectively.

1991 *Mathematics Subject Classification*: 46B20, 46E30.

Key words and phrases: k-rotundity, weak k-rotundity, C-k-rotundity, weak C-k-rotundity, k-nearly uniform convexity, k-β property, C-I property, C-II property, C-III property, locally uniform convexity, nearly uniform convexity, Luxemburg norm, Orlicz function spaces.

The work was supported in part by the NSF and JYF of China.

Recall that the *Kuratowski measure of noncompactness* $\alpha(A)$ for $A \subset X$ is defined as

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by a finite family of sets of diameter less than } \varepsilon\}.$$

A *slice* of $B(X)$ is defined by $S(f,\eta) = \{x \in B(X) : f(x) > 1-\eta\}$ where $f \in S(X^*)$ and $\eta > 0$.

Let $\mathbb{R}$ be the set of all real numbers. A function $M : \mathbb{R} \to \mathbb{R}_+$ is called an *Orlicz function* if M is convex, even, $M(0) = 0$ and $M(\infty) = \infty$. The *complementary function* N of M in the sense of Young is defined by

$$N(v) = \sup_{u\in\mathbb{R}}\{uv - M(u)\}.$$

It is known that if M is an Orlicz function, then so is N. M is said to be *strictly convex* if $M((u+v)/2) < (M(u)+M(v))/2$ for all $u \neq v$. An interval (a,b) is said to be an *affine interval* of M if M is affine on (a,b) and M is strictly convex on $(b,b+\varepsilon)$ and $(a-\varepsilon,a)$ for some $\varepsilon > 0$. Denote all affine intervals of M by $\bigcup_{i=1}^{\infty}(a_i,b_i)$.

M is said to satisfy the $\triangle_2$-*condition for large* u (we simply write $M \in \triangle_2$) if for some K and $u_0 > 0$, $M(2u) \le KM(u)$ for $|u| \ge u_0$.

Let G be a bounded set in $\mathbb{R}^n$ and let (G,Σ,μ) be a finite non-atomic measure space. For a real-valued measurable function $x(t)$ over G, we call $\varrho_M(x) = \int_G M(x(t))\,d\mu(t)$ the *modular* of x. The *Orlicz function space* $L_{(M)}$ generated by M is the Banach space

$$L_{(M)} = \{x = x(t) : \exists\lambda > 0,\ \varrho_M(\lambda x) < \infty\}$$

equipped with the *Luxemburg norm*

$$\|x\| = \inf\{\lambda : \varrho_M(x/\lambda) \le 1\}.$$

For information on Orlicz spaces, see [KrRu61, Ch96].

First we recall some lemmas.

LEMMA 1 [LiSh96]. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, let $x \in S(L_{(M)})$. If M does not satisfy the $\triangle_2$-condition, then $\alpha(S(f,\eta)) \ge 1/4$ for any slice $S(f,\eta)$ of $B(L_{(M)})$ containing x.*

LEMMA 2 [LiSh96]. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, let $x \in S(L_{(M)})$. If $\mu\{t \in G : x(t) \in \bigcup_{i=1}^{\infty}(a_i,b_i)\} > 0$, where $\bigcup_{i=1}^{\infty}(a_i,b_i)$ is the family of all affine intervals of M, then $\alpha(S(f,\eta)) \ge \theta > 0$ for any slice $S(f,\eta)$ of $B(L_{(M)})$ containing x, where θ is a constant that depends only on x.*

THEOREM 1. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, let $x \in S(L_{(M)})$. Then the following are equivalent*:

(1) *x is an LC-II point of $B(L_{(M)})$.*

(2) (i) *$M \in \triangle_2$,*

(ii) *$\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} = 0$, where $\bigcup_{i=1}^{\infty}(a_i, b_i)$ is all affine intervals of M,*

(iii) *if $\mu\{t \in G : |x(t)| = b\} > 0$ for some affine interval (a, b), then $N \in \triangle_2$ and $\mu\{t \in G : |x(t)| = c\} = 0$ for all affine intervals (c, d) of M.*

(3) *x is an LUR point of $B(L_{(M)})$, i.e., for all sequences $\{x_n\}$ in $B(L_{(M)})$, $\lim_{n\to\infty} \|x_n - x\| = 0$ whenever $\lim_{n\to\infty} \|x_n + x\| = 2$.*

Proof. (1)⇒(2). (i) Suppose that $M \notin \triangle_2$. Then (see the proof of Lemma 1 in [LiSh96]) there is a sequence $\{x_n\}$ satisfying

$$x_n = x|_{G\setminus G_n} + (x + u_n)|_{G_n}, \quad \lim_{n\to\infty} \|x_n\|_{(M)} = 1, \quad \alpha(\{x_n\}) \ge 1/4,$$

and $x_n \to x$ weakly. For every $\delta > 0$ there exists an integer N so that $\operatorname{conv}(\{x\} \cup \{x_n\}_{n\ge N}) \cap (1-\delta)B(X) = \emptyset$; but $\alpha(\{x_n\}) \ge 1/4$, which contradicts x being an LC-II point of $B(L_{(M)})$.

(ii) Suppose $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} > 0$. By Lemma 2, there exists a sequence $\{x_n\}$ in $B(L_{(M)})$ satisfying $\alpha(\{x_n\}) \ge \theta$ and $x_n \to x$ weakly, where θ depends only on x, which implies that x is not an LC-II point of $B(L_{(M)})$, a contradiction.

(iii) Suppose that $\mu B = \mu\{t \in G : |x(t)| = b\} > 0$ and $\mu C = \mu\{t \in G : |x(t)| = c\} > 0$ for some affine intervals (a, b) and (c, d) of M. Take $B_0 \subset B$ and $C_0 \subset C$ with $\mu B_0 > 0$, $\mu C_0 > 0$ and

$$[M(b) - M(a)]\mu B_0 = [M(d) - M(c)]\mu C_0$$

(i.e., $M(b)\mu B_0 + M(c)\mu C_0 = M(a)\mu B_0 + M(d)\mu C_0$). Set

$$z = x|_{G\setminus(B_0\cup C_0)} + \frac{a+b}{2}\operatorname{sign} x|_{B_0} + \frac{c+d}{2}\operatorname{sign} x|_{C_0}.$$

Then

$$\begin{aligned}\varrho_M(z) &= \varrho_M(x|_{G\setminus(B_0\cup C_0)}) + \frac{M(a)+M(b)}{2}\mu B_0 + \frac{M(c)+M(d)}{2}\mu C_0 \\ &= \varrho_M(x) = 1.\end{aligned}$$

As in the proof of Lemma 2, there exists a sequence $\{z_n\}$ in $B(L_{(M)})$ satisfying $\alpha(\{z_n\}) \ge \theta$ and $z_n \to z$ weakly, where θ depends only on z, hence only on x. Let $y = x|_{G\setminus(B_0\cup C_0)} + a \operatorname{sign} x|_{B_0} + d \operatorname{sign} x|_{C_0}$. Then

$$\varrho_M(y) = \varrho_M(x|_{G\setminus(B_0\cup C_0)}) + M(a)\mu B_0 + M(d)\mu C_0 = \varrho_M(x) = 1$$

and $z = (x+y)/2$. Since $\|x\|_{(M)} = \|y\|_{(M)} = \|z\|_{(M)} = 1$, there is $f \in L^*_{(M)}$ with $f(x) = f(z) = \|f\| = 1$. Since $z_n \to z$ weakly, for any $\delta > 0$ there exists an integer N so that $\mathrm{conv}(\{x\} \cup \{z_n\}_{n \geq N}) \cap (1-\delta)B(X) = \emptyset$; but $\alpha(\{z_n\}) \geq \theta$ contradicts x being an LC-II point of $B(L_{(M)})$.

Suppose $\mu B = \mu\{t \in G : |x(t)| = b\} > 0$ for some affine interval (a, b) of M and $N \notin \triangle_2$. Since $N \notin \triangle_2$, there exist $u_n \nearrow \infty$ such that

$$2^n M\Big(\frac{1}{2^n} u_n\Big) > \Big(1 - \frac{1}{n}\Big) M(u_n).$$

Without loss of generality, assume that $x(t) = b$ on B. Take subsets B_n in B such that $B \supset B_1 \supset B_2 \supset \dots$ and

$$[M(u_n) - M(a)]\mu B_n = [M(b) - M(a)]\mu B.$$

Then $M(u_n)\mu B_n \geq [M(b) - M(a)]\mu B$. Set

$$x_n = x|_{G\setminus B} + a|_{B\setminus B_n} + u_n|_{B_n}.$$

Then

$$\begin{aligned}\varrho_M(x_n) &= \varrho_M(x|_{G\setminus B}) + M(a)(\mu B - \mu B_n) + M(u_n)\mu B_n \\ &= \varrho_M(x|_{G\setminus B}) + M(b)\mu B = \varrho_M(x) = 1.\end{aligned}$$

Obviously

$$\lim_{\beta\to 0} \sup_n \frac{\varrho_M(\beta x_n)}{\beta} \geq [M(b) - M(a)]\mu B > 0,$$

by [An62], $\{x_n\}$ is not weakly compact and so $\alpha(\{x_n\}) \geq \theta > 0$. For any $\delta > 0$, take $K > 0$ such that $2/K \leq \delta$. Set $x_{n_0} = x$. Then for all $K < n_1 < \dots < n_k$ and any $\sum_{i=0}^k \lambda_i = 1$, $\lambda_i \geq 0$, we have

$$\begin{aligned}\varrho_M\Big(\sum_{i=0}^k \lambda_i x_{n_i}\Big) &= \varrho_M(x|_{G\setminus B}) + M\Big(\lambda_0 b + \sum_{i=1}^k \lambda_i a\Big)\mu(B \setminus B_{n_1}) \\ &\quad + M\Big(\sum_{i=1}^k \lambda_i u_{n_i} + \lambda_0 b\Big)\mu B_{n_k} \\ &\quad + M\Big(\Big(\sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a\Big)\Big|_{B_{n_1}\setminus B_{n_k}}\Big) \\ &\geq \varrho_M(x|_{G\setminus B}) + \Big(\lambda_0 M(b) + \sum_{i=1}^k \lambda_i M(a)\Big)\mu(B \setminus B_{n_1}) \\ &\quad + \sum_{i=1,\, \lambda_i \geq 1/2^{n_i}}^k (1 - 1/n_i)\lambda_i M(u_{n_i})\mu B_{n_k} + M(\lambda_0 b)\mu B_{n_k}\end{aligned}$$

$$
\begin{aligned}
&+ M\Big(\sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a\Big)\mu(B_{n_{k-1}} \setminus B_{n_k})\\
&+ M\Big(\Big(\sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a\Big)\Big|_{B_{n_1}\setminus B_{n_{k-1}}}\Big)\\
\ge{}& \varrho_M(x|_{G\setminus B}) + \Big(\lambda_0 M(b) + \sum_{i=1}^{k} \lambda_i M(a)\Big)\mu(B \setminus B_{n_1})\\
&+ \sum_{i=1,\, \lambda_i \ge 1/2^{n_i}}^{k} (1 - 1/n_i)\lambda_i M(u_{n_i})\mu B_{n_k} + M(\lambda_0 b)\mu B_{n_k}\\
&+ \sum_{i=1,\, \lambda_i \ge 1/2^{n_i}}^{k-1} (1 - 1/n_i)\lambda_i M(u_{n_i})\mu(B_{n_{k-1}} \setminus B_{n_k})\\
&+ M(\lambda_0 b + \lambda_k a)\mu(B_{n_{k-1}} \setminus B_{n_k})\\
&+ M\Big(\Big(\sum_{i=1}^{k-1} \lambda_i u_{n_i} + \lambda_0 b + \lambda_k a\Big)\Big|_{B_{n_1}\setminus B_{n_{k-1}}}\Big)\\
\ge{}& \varrho_M(x|_{G\setminus B}) + \Big(\lambda_0 M(b) + \sum_{i=1}^{k} \lambda_i M(a)\Big)\mu(B \setminus B_{n_1})\\
&+ \sum_{j=1}^{k} \sum_{i=1,\, \lambda_i \ge 1/2^{n_i}}^{j} (1 - 1/n_i)\lambda_i M(u_{n_i})\mu(B_{n_j} \setminus B_{n_{j+1}})\\
&+ \sum_{j=1}^{k} M(\lambda_0 b + (\lambda_{j+1} + \ldots + \lambda_k)a)\mu(B_{n_j} \setminus B_{n_{j+1}})\\
\ge{}& \varrho_M(x|_{G\setminus B}) + \Big(\lambda_0 M(b) + \sum_{i=1}^{k} \lambda_i M(a)\Big)\mu(B \setminus B_{n_1})\\
&+ (1 - 1/n_1)\sum_{j=1}^{k} \sum_{i=1,\, \lambda_i \ge 1/2^{n_i}}^{j} \lambda_i M(u_{n_i})\mu(B_{n_j} \setminus B_{n_{j+1}})\\
&+ \sum_{j=1}^{k} M(\lambda_0 b + (\lambda_{j+1} + \ldots + \lambda_k)a)\mu(B_{n_j} \setminus B_{n_{j+1}})\\
\ge{}& (1 - 1/n_1)\sum_{i=1}^{k} \lambda_i \varrho_M(x_{n_i}) - (1 - 1/n_1)\sum_{i=1,\, \lambda_i < 1/2^{n_i}}^{k} \lambda_i \varrho_M(x_{n_i})\\
\ge{}& (1 - 1/n_1) - \sum_{i=1}^{k} 1/2^{n_i} = (1 - 1/K) - 1/2^K > 1 - \delta.
\end{aligned}
$$

Hence $\operatorname{conv}(\{x\}\cup\{x_n\}_{n\geq K})\cap(1-\delta)B(X)=\emptyset$; but $\alpha(\{x_n\})>0$ contradicts x being an LC-II point of $B(L_{(M)})$.

(2)⇒(3). By [ChWa92], it follows that x is an LUR point of $B(L_{(M)})$.

(3)⇒(1). Obvious. ∎

For an integer k, a point $x\in S(X)$ is said to be:

• a *locally k-rotund* (LkR) *point* of $B(X)$ if for any sequence $\{x_n\}$ in $B(X)$, $\lim_{n_1,\ldots,n_k\to\infty}\|x+x_{n_1}+\ldots+x_{n_k}\|=k+1$ implies $\lim_{n\to\infty}\|x_n-x\|=0$;

• a *locally weakly k-rotund* (LWkR) *point* of $B(X)$ if for any sequence $\{x_n\}$ in $B(X)$, $\lim_{n_1,\ldots,n_k\to\infty}\|x+x_{n_1}+\ldots+x_{n_k}\|=k+1$ implies $w\text{-}\lim_{n\to\infty}x_n=x$;

• a *locally* C-*k-rotund* (LCkR) *point* of $B(X)$ if for any sequence $\{x_n\}$ in $B(X)$, $\lim_{n_1,\ldots,n_k\to\infty}\|x+x_{n_1}+\ldots+x_{n_k}\|=k+1$ implies $\{x_n\}$ is a relatively compact set;

• a *locally k-nearly uniformly convex* (LkNUC) *point* of $B(X)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for all sequences $\{x_n\}$ with $\operatorname{sep}(x_n)\geq\varepsilon$ there are $\{n_1,\ldots,n_k\}$ with

$$\left\|\frac{x+x_{n_1}+\ldots+x_{n_k}}{k+1}\right\|\leq 1-\delta;$$

• a *locally k-β* (L$k\beta$) *point* of $B(X)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for all sequences $\{x_n\}$ with $\operatorname{sep}(x_n)\geq\varepsilon$ there are $\{n_1,\ldots,n_k\}$ with $\operatorname{conv}(\{x,x_{n_1},\ldots,x_{n_k}\})\cap(1-\delta)B(X)\neq\emptyset$;

• a *locally nearly uniformly convex* (LNUC) *point* of $B(X)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that for all sequences $\{x_n\}$ with $\operatorname{sep}(x_n)\geq\varepsilon$ we have $\operatorname{conv}(\{x\}\cup\{x_n\})\cap(1-\delta)B(X)\neq\emptyset$.

It is easy to see that for all Banach spaces, we have the implications

$$\begin{array}{ccccc}
\text{LUR} & \Longrightarrow & \text{L}k\text{R} & \Longrightarrow & \text{LC}k\text{R} \\
\Big\Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{LW}k\text{R} & & \text{LC-II} \\
 & & & & \Uparrow \\
\text{L}k\text{NUC} & \Longrightarrow & \text{L}k\beta & \Longrightarrow & \text{LNUC}
\end{array}$$

For these properties, we refer to [Ku91, KuLi94, KuLi93, Wa95].

COROLLARY 1. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, let $x\in S(L_{(M)})$. Then the following are equivalent*:

(1) *x is an LUR point of $B(L_{(M)})$* [ChWa92];
(2) *x is an LkR point of $B(L_{(M)})$* $(k\geq 1)$;
(3) *x is an LWkR point of $B(L_{(M)})$* $(k\geq 1)$;
(4) *x is an LCkR point of $B(L_{(M)})$* $(k\geq 1)$;
(5) *x is an LkNUC point of $B(L_{(M)})$* $(k\geq 1)$;

(6) *x is an Lk-β point of $B(L_{(M)})$ $(k\ge 1)$;*
(7) *x is an LNUC point of $B(L_{(M)})$;*
(8) *x is an LC-I point of $B(L_{(M)})$;*
(9) *x is an LC-II point of $B(L_{(M)})$;*
(10) *$M\in\triangle_2$, $\mu\{t\in G:|x(t)|\in\bigcup_{i=1}^{\infty}(a_i,b_i)\}=0$, where $\{(a_i,b_i)\}$ is the family of all affine intervals of M, and if $\mu\{t\in G:|x(t)|=b\}>0$ for some affine interval (a,b) of M, then $N\in\triangle_2$ and $\mu\{t\in G:|x(t)|=c\}=0$ for all affine intervals (c,d) of M.*

Proof. (1)⇒(2)⇒(3), (1)⇒(2)⇒(4)⇒(9), (1)⇒(5)⇒(6)⇒(7), and (1)⇒(8)⇒(9) are trivial by definitions.

(7)⇒(9). By Theorem 4 of [Wa95], an LNUC point is an LC-II point in $B(X)$.

(10)⇒(1). This is proved in [ChWa92].

(9)⇒(10). This follows from Theorem 1.

(3)⇒(10). Since $\|x\|_{(M)}=1$, there is $c>0$ such that $\mu G_c=\mu\{t\in G:|x(t)|\le c\}>0$.

Suppose that $M\notin\triangle_2$. Then there exist $u_n\nearrow\infty$ such that

$$M((1+1/n)u_n)>2^nM(u_n).$$

On passing to a subsequence if necessary, there are disjoint subsets $G_n\subset G_c$ so that

$$M(u_n)\mu G_n=1/2^n,\qquad n=1,2,\ldots$$

Define $y=\sum_{n=1}^{\infty}u_n|_{G_n}$. Then $\varrho_M(y)=\sum_{n=1}^{\infty}M(u_n)\mu G_n=1$, $\|y\|_{(M)}=1$ and $\operatorname{dist}(y,E_M)=1$, where $E_M=\{x:\varrho_M(\lambda x)<\infty$ for all $\lambda\}$. By the Hahn–Banach theorem, there is a functional ϕ such that $\phi(y)=\|\phi\|=1$, and $\phi(z)=0$ for all z in E_M. Set $x_n=x|_{G\setminus\bigcup_{i>n}G_i}+y|_{\bigcup_{i>n}G_i}$. Then

$$\left\|\frac{x+x_{n_1}+\ldots+x_{n_k}}{k+1}\right\|_{(M)}\ge\|x|_{G\setminus\bigcup_{i>n_k}G_i}\|_{(M)}\to 1\qquad(n_1,\ldots,n_k\to\infty)$$

and

$$\varrho_M(x_n)=\varrho_M(x|_{G\setminus\bigcup_{i>n}G_i})+\varrho_M(y|_{\bigcup_{i>n}G_i})\to\varrho_M(x)\le 1.$$

But

$$\begin{aligned}\phi(x_n-x)&=\phi(y|_{\bigcup_{i>n}G_i})-\phi(x|_{\bigcup_{i>n}G_i})=\phi(y|_{\bigcup_{i>n}G_i})\\&=\phi(y|_{G_c})=1.\end{aligned}$$

So $x_n\not\to x$ weakly, contrary to x being an LWkR point of $B(L_{(M)})$.

We claim that $\mu\{t\in G:|x(t)|\in\bigcup_{i=1}^{\infty}(a_i,b_i)\}=0$.

In fact, if this measure is positive, then $\mu E>0$, where $E=\mu\{t\in G:x(t)\in(a+2\delta,b-2\delta)\}$ for some $\delta>0$. Split E into two parts E_1 and E_2

with $\mu E_1 = \mu E_2 = (\mu E)/2$. Define

$$z = x|_{G\setminus E} + (x+2\delta)|_{E_1} + (x-2\delta)|_{E_2}.$$

Then

$$\begin{aligned}\varrho_M(z) &= \varrho_M(x|_{G\setminus E}) + \varrho_M((x+2\delta)|_{E_1}) + \varrho_M((x-2\delta)|_{E_2})\\ &= \varrho_M(x|_{G\setminus E}) + \varrho_M(x|_{E_1}) + \varrho_M(x|_{E_2}) = 1,\\ \varrho_M\left(\frac{x+z}{2}\right) &= \varrho_M(x|_{G\setminus E}) + \varrho_M((x+\delta)|_{E_1}) + \varrho_M((x-\delta)|_{E_2})\\ &= \varrho_M(x|_{G\setminus E}) + \varrho_M(x|_{E_1}) + \varrho_M(x|_{E_2}) = 1.\end{aligned}$$

Moreover $x \neq z$. As in Lemma 2, there exists a sequence $\{z_n\}$ in $B(L_{(M)})$ such that $z_n \to z$ weakly and $\mathrm{sep}\{z_n\} \geq \theta > 0$, where θ depends only on z. For $k > 1$, since $z_n \to z$ weakly and $\|x+z\|_{(M)} = 2$, we have $\lim_{n_1,\dots,n_k\to\infty} \|x + z_{n_1} + \ldots + z_{n_k}\| = k+1$. This contradicts x being an LWkR point of $B(L_{(M)})$. For $k = 1$ we can take $x_n = z$ to get a contradiction.

From Theorem 1, it follows that if $\mu\{t \in G : |x(t)| = b\} > 0$ for some affine interval (a,b) of M, then $N \in \triangle_2$ and $\mu\{t \in G : |x(t)| = c\} = 0$ for all affine intervals (c,d) of M. ∎

Corollary 2. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, the following are equivalent*:

(1) $L_{(M)}$ *is locally UR* [ChWa92, Ka84];
(2) $L_{(M)}$ *is locally kR* $(k \geq 1)$;
(3) $L_{(M)}$ *is locally WkR* $(k \geq 1)$;
(4) $L_{(M)}$ *is locally CkR* $(k \geq 1)$;
(5) $L_{(M)}$ *is locally kNUC* $(k \geq 1)$;
(6) $L_{(M)}$ *is locally k-β* $(k \geq 1)$;
(7) $L_{(M)}$ *is locally NUC*;
(8) $L_{(M)}$ *has the C-I property*;
(9) $L_{(M)}$ *has the C-II property*;
(10) *$M \in \triangle_2$ and M is strictly convex on the real line.*

Corollary 3. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, suppose $M \in \triangle_2$ and let $x \in S(L_{(M)})$. If $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^\infty (a_i,b_i)\} = 0$ and either $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^\infty \{b_i\}\} = 0$, or $N \in \triangle_2$ and $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^\infty \{a_i\}\} = 0$, then every proximinal metric projection P_D is norm-norm upper semicontinuous at x.*

Moreover, if $M \in \triangle_2$ and $M \in SC$, then every proximinal metric projection P_D is norm-norm upper semicontinuous.

Next, we study the LC-III points.

LEMMA 3. *For an Orlicz space* $L_{(M)}$, *suppose* $M \in \triangle_2$. *Then*

(1) *for any* $\varepsilon > 0$ *there is* $\eta > 0$ *such that*

$$\varrho_M(x) < \eta \Rightarrow \|x\|_{(M)} < \varepsilon,$$
$$\|x\|_{(M)} > 1 - \eta \Rightarrow \varrho_M(x) > 1 - \varepsilon;$$

(2) *if* $\varrho_M(x_n) \to \varrho_M(x)$ *and* $x_n \overset{\mu}{\to} x$ *in measure*, *then* $x_n \to x$ *in norm.*

For a proof, see [Ch86, Hu83, HuLa95].

THEOREM 2. *In an Orlicz function space* $L_{(M)}$ *equipped with Luxemburg norm*, *let* $x \in S(L_{(M)})$. *Then* x *is a C-III point of* $B(L_{(M)})$ *if and only if*

(1) $M \in \triangle_2$;

(2) *either* $N \in \triangle_2$, *or* $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} = 0$ *and* $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{b_i\}\} = 0$.

Proof. Choose $c > 0$ such that $\mu G_c = \mu\{t \in G : |x(t)| \le c\} > 0$. Suppose $M \notin \triangle_2$. There exists [KrRu61] $y \in L_{(M)}$ with $\operatorname{supp} y \subset G_c$, $\|y\|_{(M)} = \operatorname{dist}(y, E_M) = 1$, and $\phi \in L^*_{(M)}$ with $\phi(y) = \|\phi\| = \operatorname{dist}(y, E_M) = 1$ and $\phi(z) = 0$ for all $z \in E_M$, and $G_n \subset G_c$, where $G_n = \{t \in G : |y(t)| \ge n\}$. Set

$$x_n = x|_{G\setminus G_n} + y|_{G_n}.$$

Then for $\theta > 0$, take n_0 such that $\|x|_{G\setminus G_{n_0}}\|_{(M)} > 1 - \theta$. Then for all $n_0 < n_1 < \ldots < n_k$ and for any $\sum_{i=0}^{k} \lambda_i = 1$, where $\lambda_i \ge 0$,

$$\Big\| \sum_{i=0}^{k} \lambda_i x_{n_i} \Big\|_{(M)} \ge \|x|_{G\setminus G_{n_k}}\|_{(M)} > 1 - \theta.$$

But $\{x_n\}$ is not relatively weakly compact. In fact, otherwise by the Shmul'yan Theorem $\{x_n\}$ is relatively weakly sequentially compact. By taking a subsequence if necessary we may assume that $x_n \overset{w}{\to} x'$ in the weak topology. Combining this with $x_n \overset{w^*}{\to} x$ in the w^* topology, we get $x_n \overset{w}{\to} x$. A contradiction since $\phi(x_n - x) = \phi(y|_{G_n}) + \phi(x|_{G_n}) = \phi(y|_{G_n}) = 1$.

Assume that $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} > 0$. Then $\mu B = \mu\{t \in G : x(t) \in (a + \theta, b - \theta)\} > 0$ for some affine interval (a, b) and some $\theta > 0$. Split B into two parts B', B'' with $\mu B' = \mu B'' = (\mu B)/2$. Define

$$y = x|_{G\setminus B} + (x - \theta)|_{B'} + (x + \theta)|_{B''}.$$

Then

$$\begin{aligned}\varrho_M(y) &= \varrho_M(x|_{G\setminus B}) + \varrho_M((x - \theta)|_{B'}) + \varrho_M((x + \theta)|_{B''}) \\ &= \varrho_M(x|_{G\setminus B}) + \varrho_M(x|_{B'}) + \varrho_M(x|_{B''}) = 1,\end{aligned}$$

and

$$\varrho_M\Big(\frac{x + y}{2}\Big) = \varrho_M(x) = 1.$$

If $N \notin \triangle_2$, then there exists a real sequence $\{u_n\}$ such that $u_n \nearrow \infty$ and

$$2^n M\Big(\frac{1}{2^n}u_n\Big) > \Big(1-\frac{1}{n}\Big)M(u_n).$$

Take decreasing subsets $\{B_n\}$ of B such that

$$\varrho_M(y|_B) - M(a)\mu B = \varrho_M(x|_B) - M(a)\mu B = [M(u_n) - M(a)]\mu B_n.$$

Then $M(u_n)\mu B_n \ge \varrho_M(x|_B) - M(a)\mu B > 0$. Set

$$x_n = x|_{G\setminus B} + a|_{B\setminus B_n} + u_n|_{B_n}.$$

By [An62], $\{x_n\}$ is not weakly compact. But

$$\varrho_M(x_n) = \varrho_M(x|_{G\setminus B}) + M(a)(\mu B - \mu B_n) + M(u_n)\mu B_n = \varrho_M(x) = 1.$$

For any $\delta > 0$, take K such that $2/K \le \delta$. Let $x_{n_0} = x$. Then for all $K < n_1 < \ldots < n_k$ and for any $\sum_{i=0}^k \lambda_i = 1$, where $\lambda_i \ge 0$, as in the proof of Theorem 1,

$$\varrho_M\Big(\sum_{i=0}^k \lambda_i x_{n_i}\Big) \ge 1 - \delta.$$

This contradicts x being a C-III point of $B(L_{(M)})$.

By the same argument as for the second part of (iii) in Theorem 1 we can show that if x is a locally C-III point of $B(L_{(M)})$ then $\mu\{t \in G : |x(t)| = b\} > 0$ for some affine interval (a, b) of M implies $N \in \triangle_2$.

Suppose $\{x_n\}$ is a sequence in $B(L_{(M)})$ such that for any $\delta > 0$ there exists an integer N with $\operatorname{conv}(\{x\} \cup \{x_n\}_{n\ge N}) \cap (1-\delta)B(L_{(M)}) = \emptyset$.

If $N \in \triangle_2$, then by (1), $L_{(M)}$ is reflexive. So $B(L_{(M)})$ is weakly compact and $\{x_n\}$ is relatively weakly compact.

If $N \notin \triangle_2$, then we show that $\lim_{n\to\infty} x_n = x$. By Lemma 3, it suffices to show that $x_n \xrightarrow{\mu} x$ in measure. By (2), $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^\infty (a_i, b_i)\} = 0$ and $\mu\{t \in G : |x(t)| = b\} = 0$ for all affine intervals (a, b). Since $\lim_{n_1,\ldots,n_k\to\infty} \|x + x_{n_1} + \ldots + x_{n_k}\|_{(M)} = k+1$, we have $\lim_{n\to\infty} \|x + x_n\|_{(M)} = 2$. From

$$1 = \frac{\varrho_M(x) + \varrho_M(x_n)}{2} \ge \varrho_M\Big(\frac{x + x_n}{2}\Big) \to 1,$$

it follows that $x_n \xrightarrow{\mu} x$ in measure on $\{t \in G : |x(t)| \notin G \setminus \bigcup_{i=1}^\infty [a_i, b_i]\}$.

We claim: $x_n \xrightarrow{\mu} x$ in measure on $G_a = \{t \in G : |x(t)| = a\}$ for every left endpoint a of an affine interval (a, b). Without loss of generality, assume that $G_a = \{t \in G : x(t) = a\}$.

We first show that for any $\varepsilon > 0$, $\mu\{t \in G_a : x_n(t) \le a - \varepsilon\} \to 0$ as $n \to \infty$. Indeed, if for some $\varepsilon_0 > 0$ and $\sigma_0 > 0$ and a subsequence of $\{x_n\}$ (again denoted by $\{x_n\}$) we have $\mu G_n = \mu\{t \in G_a : x_n(t) \le a - \varepsilon_0\} \ge \sigma_0 > 0$

for all n, then there exists a $\delta_0 > 0$ such that

$$M\left(\frac{a+a-\varepsilon_0}{2}\right) \le \frac{1}{2}(1-\delta_0)[M(a)+M(a-\varepsilon_0)]$$

(because $c \ne d$ for all affine intervals (c,d)). Hence

$$\begin{aligned}\varrho_M\left(\frac{x+x_n}{2}\right) &\le \tfrac{1}{2}[\varrho_M(x|_{G\setminus G_n})+\varrho_M(x_n|_{G\setminus G_n})] + M\left(\frac{a+a-\varepsilon_0}{2}\right)\mu G_n \\ &\le \tfrac{1}{2}[\varrho_M(x|_{G\setminus G_n})+\varrho_M(x_n|_{G\setminus G_n})] \\ &\quad + \tfrac{1}{2}(1-\delta_0)[M(a)+M(a-\varepsilon_0)]\mu G_n \\ &\le \tfrac{1}{2}[\varrho_M(x)+\varrho_M(x_n)] - \tfrac{1}{2}\delta_0[M(a)+M(a-\varepsilon_0)]\mu G_n \\ &\le 1-\tfrac{1}{2}\delta_0[M(a)+M(a-\varepsilon_0)]\mu G_n < 1.\end{aligned}$$

By Lemma 3, $\lim_{n\to\infty}\|x+x_n\|_{(M)} < 2$, a contradiction.

Next we show that for any $\varepsilon > 0$, $\mu\{t \in G_a : x_n(t) \ge a+\varepsilon\} \to 0$ as $n \to \infty$. Indeed, suppose that for some $\varepsilon_0 > 0$ and $\sigma_0 > 0$ and a subsequence $\{x_n\}$ (again labeled $\{x_n\}$) we have $\mu G_n = \mu\{t \in G_a : x_n(t) \ge a+\varepsilon_0\} \ge \sigma_0$ for all n. Since

$$\begin{aligned}G = \Big\{t \in G : |x(t)| \notin \bigcup_{i=1}^{\infty}[a_i,b_i]\Big\} \cup \Big\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i,b_i)\Big\} \\ \cup \Big\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{b_i\}\Big\} \cup \Big\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{a_i\}\Big\},\end{aligned}$$

by the Fatou Lemma, we see that for all $G' \subset G$,

$$\liminf_{n\to\infty}\varrho_M(x_n|_{G'}) \ge \varrho_M(x|_{G'}).$$

Hence for n large enough,

$$\begin{aligned}\varrho_M(x_n) &= \varrho_M(x_n|_{G\setminus G_n}) + \varrho_M(x_n|_{G_n}) \\ &\ge \varrho_M(x_n|_{G\setminus G_n}) + M(a+\varepsilon_0)\mu G_n \\ &= \varrho_M(x_n|_{G\setminus G_n}) + M(a)\mu G_n + [M(a+\varepsilon_0)-M(a)]\mu G_n \\ &\ge \varrho_M(x) + [M(a+\varepsilon_0)-M(a)]\sigma_0 > 1,\end{aligned}$$

a contradiction.

We now show that $x_n \xrightarrow{\mu} x$ in measure on $\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{a_i\}\}$. Indeed, for every $\varepsilon > 0$ and $\sigma > 0$, take i_0 such that $\mu\{t \in G : |x(t)| \in \bigcup_{i>i_0}\{a_i\}\} < \varepsilon/2$. From the claim we deduce that for n large enough,

$$\mu\Big\{t \in G : |x(t)| \in \bigcup_{i=1}^{i_0}\{a_i\} \text{ and } |x_n(t)-x(t)| \ge \sigma\Big\} < \frac{\varepsilon}{2}.$$

From the decomposition of G as above we get $x_n \xrightarrow{\mu} x$ in measure on G.

By Lemma 3, we know that $x_n \to x$ in norm, so $\{x_n\}$ is relatively weakly compact. ■

REMARK. By the same argument we can show that an element in $S(L_{(M)})$ is a locally C-III point of $B(L_{(M)})$ iff it is a locally WCkR point of $B(L_{(M)})$.

COROLLARY 4. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, the following are equivalent*:

(1) *$L_{(M)}$ is locally WCkR*;
(2) *$L_{(M)}$ has the C-III property*;
(3) *$M \in \triangle_2$ and either $M \in SC$ or $N \in \triangle_2$.*

COROLLARY 5. *In an Orlicz function space $L_{(M)}$ equipped with Luxemburg norm, suppose $M \in \triangle_2$ and let $x \in S(L_{(M)})$. If $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}(a_i, b_i)\} = 0$ and $\mu\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty}\{b_i\}\} = 0$, then every proximinal metric projection P_D is norm-weak upper semicontinuous at x.*

Moreover, if $M \in \triangle_2$, and either $M \in SC$ or $N \in \triangle_2$, then every proximinal metric projection P_D is norm-weak upper semicontinuous on $L_{(M)}$.

REFERENCES

[An62] T. Ando, *Weakly compact sets in Orlicz spaces*, Canad. J. Math. 14 (1962), 170–196.

[Ch86] S. T. Chen, *Some rotundities in Orlicz spaces with Orlicz norm*, Bull. Polish Acad. Sci. Math. 34 (1986), 585–596.

[Ch96] —, *Geometry of Orlicz spaces*, Dissertationes Math. 356 (1996).

[ChWa92] S. T. Chen and F. T. Wang, *Uniformly rotund points of Orlicz spaces*, J. Harbin Normal Univ. 8 (1992), no. 3, 5–10.

[Hu83] H. Hudzik, *Uniform convexity of Musielak–Orlicz spaces with Luxemburg norm*, Comment. Math. (Prace Mat.) 23 (1983), 21–23.

[HuLa95] H. Hudzik and T. Landes, *Packing constant in Orlicz spaces equipped with the Luxemburg norm*, Boll. Un. Mat. Ital. A (7) 9 (1995), 225–237.

[Ka84] A. Kamińska, *The criteria for local uniform rotundity of Orlicz spaces*, Studia Math. 79 (1984), 201–215.

[Ku91] D. Kutzarova, *k-β and k-nearly uniformly convex Banach spaces*, J. Math. Anal. Appl. 162 (1991), 322–338.

[KuLi93] D. Kutzarova, B.-L. Lin and W. Y. Zhang, *Some geometrical properties of Banach spaces related to nearly uniform convexity*, in: Banach Spaces (Merida, 1992), Contemp. Math. 144, Amer. Math. Soc., Providence, RI, 1993, 165–171.

[KuLi94] D. Kutzarova and B.-L. Lin, *Locally k-nearly uniformly convex Banach spaces*, Math. Balkanica 8 (1994), 203–210.

[KrRu61] M. A. Krasnosel'skiĭ and B. Ya. Rutitskiĭ, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[LiSh96] B.-L. Lin and Z. R. Shi, *On denting points and drop properties in Orlicz spaces*, J. Math. Anal. Appl. 201 (1996), 252–273.

[Na19] Q. Y. Na, *On fully convex and locally fully convex Banach spaces*, Acta Math. Sci. 10 (1990), 327–343.

[NaWa87] C. X. Nan and J. H. Wang, *Locally fully k-rotund and weakly locally fully k-rotund spaces*, J. Nanjing Univ. Math. Biquarterly 2 (1989).

[NaWa871] —, —, *On the Lk-UR and L-kR spaces*, Math. Proc. Cambridge Philos. Soc. 104 (1988), 521–526.

[Pa83] J. R. Partington, *On nearly uniformly convex Banach spaces*, ibid. 93 (1983), 127–129.

[Wa95] J. H. Wang, *Some results on the continuity of metric projections*, Math. Appl. 8 (1995), 80–85.

[WaWa91] J. H. Wang and M. Wang, *Compact locally fully convex spaces*, Ke Xue Tong Bao 36 (1991), 796.

[Yu85] X. T. Yu, *On LKUR spaces*, Chinese Ann. of Math. Ser. B 6 (1985), 465–469.

Department of Mathematics
The University of Iowa
Iowa City, IA 52242, U.S.A.
E-mail: bill@math.uiowa.edu

Current address of Z. R. Shi:
Department of Mathematics
Harbin University of Science and Technology
150080 Harbin, China
E-mail: zshi@public.hr.hl.cn

Received 24 November 1998;
revised 15 February 1999