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ALMOST FREE SPLITTERS

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Abstract. Let R be a subring of the rationals. We want to investigate self splitting R-modules G, that is, such that $\mathrm{Ext}_R(G,G)=0$. For simplicity we will call such modules splitters (see [10]). Also other names like stones are used (see a dictionary in Ringel's paper [8]). Our investigation continues [5]. In [5] we answered an open problem by constructing a large class of splitters. Classical splitters are free modules and torsion-free, algebraically compact ones. In [5] we concentrated on splitters which are larger than the continuum and such that countable submodules are not necessarily free. The "opposite" case of $\aleph_1$-free splitters of cardinality less than or equal to $\aleph_1$ was singled out because of basically different techniques. This is the target of the present paper. If the splitter is countable, then it must be free over some subring of the rationals by Hausen [7]. In contrast to the results of [5] and in accordance with [7] we can show that all $\aleph_1$-free splitters of cardinality $\aleph_1$ are free indeed.

1. Introduction. Throughout this paper R will denote a subring of the rationals $\mathbb{Q}$ and we will consider R-modules in order to find out when they are splitters. "Splitters" were introduced in Schultz [10]. They also come up under different names as mentioned in the abstract.

Definition 1.1. An R-module G is a *splitter* if and only if $\mathrm{Ext}^1_R(G,G)=0$ or equivalently if $\mathrm{Ext}^1_{\mathbb{Z}}(G,G)=0$, which is the case if and only if any R-module sequence

$$0\to G\xrightarrow{\beta} X\xrightarrow{\alpha} G\to 0$$

splits.

Throughout we set $\mathrm{Ext}(A,B)=\mathrm{Ext}^1_R(A,B)$.

A short exact sequence

$$0\to B\xrightarrow{\beta} C\xrightarrow{\alpha} A\to 0$$

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represents 0 in $\mathrm{Ext}(A,B)$ if and only if there is a splitting map $\gamma : A \to C$ such that $\gamma\alpha = \mathrm{id}_A$. Here maps are acting on the right.

Recall an easy basic observation (see [4]):

If $\mathrm{Ext}(A,B) = 0, A' \subseteq A$ *and* $B' \subseteq B$, *then* $\mathrm{Ext}(A', B/B') = 0$ *as well.*

The first result showing freeness of splitters is much older than the notion of splitters and is due to Hausen [7]. It says that any countable, torsion-free abelian group is a splitter if and only if it is free over its nucleus. The nucleus is the largest subring R of $\mathbb{Q}$ which makes the abelian group canonically into an R-module. More precisely:

Definition 1.2. The *nucleus* R of a torsion-free abelian group $G \neq 0$ is the subring R of $\mathbb{Q}$ generated by all $1/p$ (p any prime) for which G is *p-divisible*, i.e. $pG = G$.

The fixed ring R mentioned at the beginning will be the nucleus $R = \mathrm{nuc}\, G$ of the associated abelian group G.

The following result reduces the study of splitters among abelian groups to those which are torsion-free and reduced modules over their nuclei.

Theorem 1.3 ([10]). *Let G be any abelian group and $G = D \oplus C$ a decomposition of G into the maximal divisible subgroup D and a reduced complement C. Then the following conditions are equivalent.*

(i) *G is a splitter.*
(ii) (a) *D is torsion* (*possibly* 0) *and C is a torsion-free* (*reduced*) *splitter with $pC = C$ for all p-primary components $D_p \neq 0$ of D*; *or*
(b) *D is not torsion and C is cotorsion.*

Many splitters are constructed in [5], in fact we are also able to prescribe their endomorphism rings. This shows that uncountable splitters are not classifiable in any reasonable way, a result very much in contrast to classical well-known (uncountable) splitters which are the torsion-free algebraically compact (or cotorsion) groups.

The classical splitters come up naturally among many others when considering Salce's work [9] on cotorsion theories: A *cotorsion theory* is a pair $(\mathfrak{F}, \mathfrak{C})$ of classes of R-modules which are maximal, closed under extensions, the torsion-free class $\mathfrak{F}$ is closed under subgroups, the cotorsion class $\mathfrak{C}$ is closed under epimorphic images and $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathfrak{F}$ and $C \in \mathfrak{C}$. The elements in $\mathfrak{F} \cap \mathfrak{C}$ are splitters and in the case of Harrison's classical cotorsion theory these are the torsion-free, algebraically compact groups. For the trivial cotorsion theory these are free R-modules.

Hausen's [7] theorem mentioned above can be slightly extended without much effort (see [5]).

Theorem 1.4. *If $R = \operatorname{nuc} G$ is the nucleus of the torsion-free group G and G is a splitter of cardinality $< 2^{\aleph_0}$, then G is an $\aleph_1$-free R-module.*

Recall that G is an *$\aleph_1$-free R-module* if any countably generated R-submodule is free. The algebraic key tool of this paper can be found in Section 2. We consider torsion-free R-modules M of finite rank which are minimal in rank and non-free. They are (by definition) *n-free-by-1* R-modules if $\operatorname{rk} M = n+1$; the name is self explanatory: They are pure extensions of a free R-module of rank n by an R-module of rank 1. Similar to simply presented groups, n-free-by-1 groups are easily represented by free generators and relations. Using these minimal R-modules we will show the following

Main Theorem 1.5. *Any $\aleph_1$-free splitter of cardinality $\aleph_1$ is free over its nucleus.*

The proof will depend on the existence of particular chains of $\aleph_1$-free R-module of cardinality $\aleph_1$ which we use to divide the $\aleph_1$-free R-modules of cardinality $\aleph_1$ into three types (I, II, III). This may be interesting independently and we would like to draw attention to Section 3. In Sections 4–7 we use our knowledge about these chains to show freeness of splitters. The proof is divided into two main cases depending on the continuum hypothesis CH (Section 5) and its negation (Section 4). In the appendix Section 8 we present a proof of the main result of Section 5 under the weaker set-theoretic assumption WCH $2^{\aleph_0} < 2^{\aleph_1}$, a weak form of CH which will be interesting (only) for splitters of cardinality $> \aleph_1$. The results in Section 6 and 7 on splitters of type II and III do not use the case distinction by additional axioms of set theory.

For general aspects of the discussed problem we also suggest to consult the work in [1], [2], [6], [13], [11], [12].

2. Solving linear equations. Let R be a subring of $\mathbb{Q}$. Then R-modules of minimal finite rank which are not free will lead to particular infinite systems of linear equations. Consider the Baer–Specker R-module R^ω of all R-valued functions $f : \omega \to R$ on ω, also denoted by $f = (f_m)_{m\in\omega}$.

Lemma 2.1. *Let $p = (p_m)_{m\in\omega}$, $k_i = (k_{im})_{m\in\omega} \in R^\omega$ $(i < n)$ where no p_m is a unit of R. Then we can find a sequence $s = (s_m)_{m\in\omega} \in R^\omega$ such that the following system $(\widehat{s})$ of equations has no solution $\overline{x} = (x_0, \ldots, x_n) \in R^{n+1}$, $y_m \in R$:*

$$(\widehat{s}) \qquad y_0 = x_n, \quad y_{m+1} p_m = y_m + \sum_{i<n} x_i k_{im} + s_m \quad (m \in \omega).$$

P r o o f. We use Cantor's argument which shows that there are more real numbers than rationals. First we enumerate all elements in R^{n+1} as

$$W = W = \{\overline{x}^m = (x_0^m, \ldots, x_n^m) : m \in \omega\}$$

and construct $s \in R^\omega$ inductively.

It is interesting to note that the set of bad elements

$$B = \{s \in R^\omega : \exists \overline{x} \in R^{n+1}, (y_m)_{m\in\omega} \in R^\omega \text{ solving } (\widehat{s})\} \subseteq R^\omega$$

is a submodule of R^ω but $|B|$ is uncountable in many cases. Hence enumerating B would not help.

Suppose $s_0, \ldots, s_{m-1} \in R$ are chosen and we must find s_m. We calculate $y_0, \ldots, y_m$ from $s_0, \ldots, s_{m-1}$ and $y_0 = x_n^m, x_1^m, \ldots, x_{n-1}^m$ and equation $(\widehat{s})$ up to $m-1$. The values are uniquely defined by torsion-freeness and in particular

$$z = y_m + \sum_{i<n} x_i^m k_{im} \in R \tag{2.1}$$

is uniquely defined. Recall that p_m is not a unit and either p_m does not divide z, then we set $s_m = 0$, or we can choose some $s_m \in R \setminus \{0\}$ such that p_m does not divide $z + s_m$. In any case

$$yp_m = z + s_m \quad \text{has no solution in } R \tag{2.2}$$

and s_m is defined.

Suppose that $(\widehat{s})$ has a solution $\overline{x} \in R^{n+1}$. Then $\overline{x} = \overline{x}^m$ for some m by our enumeration. We calculate y_{m+1} from $(\widehat{s})$ substituting $\overline{x}$, hence

$$y_{m+1}p_m = y_m + \sum_{i<m} x_i^m k_{im} + s_m = z + s_m$$

is solvable by (2.1), which contradicts (2.2). ∎

If $G' \subseteq G$ is a pure R-submodule of some R-module G which is of finite rank, not a free R-module, and such that all pure R-submodules of G' of smaller rank are free, then we will say that G' is *minimal non-free*. Such modules are "simply presented" in the sense that there are $x_i, y_m \in G'$ $(i < n,\ m \in \omega)$ such that

$$G' = \langle B, y_m R : m \in \omega\rangle \quad \text{with} \quad B = \bigoplus_{i<n} x_i R \tag{2.3}$$

and the *only relations* in G' are

$$y_{m+1}p_m = y_m + \sum_{i<n} x_i k_{im} \qquad (m \in \omega) \tag{2.4}$$

for some coefficients $p_m, k_{im} \in R$. The submodule B is pure in G'. If G is not $\aleph_1$-free, then the existence of minimal non-free submodules is immediate by Pontryagin's theorem. Non-freeness of G' implies that the Baer type of

$$G'/B = T \subseteq \mathbb{Q}$$

is strictly greater than the type of R (see Fuchs [4, Vol. 2, pp. 107–112]).

In more detail, B is a pure submodule of G', hence G'/B is torsion-free of rank 1 and since G' is not a free R-module, $G'/B = T$ cannot be isomorphic to R. If $\varphi : G' \to \mathbb{Q}$ is the canonical homomorphism taking B to 0 and y_0 to $1 \in \mathbb{Q}$, then $\operatorname{Im}\varphi \subseteq \mathbb{Q}$ represents the type of $\langle y_0 + B\rangle_* = G'/B$. There are $p_m \in \mathbb{N}$, not units in R, such that $T = \bigcup_{m\in\omega} q_m^{-1}\mathbb{Z} \subseteq \mathbb{Q}$ and $q_m = \prod_{i<m} p_i$. In order to derive the crucial equations as in the above definition we choose preimages $y_m \in G'$ of q_m^{-1} such that

$$y_0\varphi = q_0 = 1, \qquad y_m\varphi = q_m^{-1} \qquad (m \in \omega).$$

Using $q_{m+1} = q_m p_m$ we find elements $k_{im} \in R$ $(i < n)$ and $g_m \in G'$ such that (2.3) and (2.4) hold. We will constantly use the representations (2.3) and (2.4) which are basic for the following pushout.

Proposition 2.2. *Let $G_\alpha \subseteq G_{\alpha+1}$ be a countable free resolution of G' as in* (2.3) *and let the relations* (2.4) *be expressed in $G_{\alpha+1}$ by*

$$y''_{m+1}p_m = y''_m + \sum_{i<n} x''_i k_{im} + g_m$$

for some $g_m \in G_\alpha$; let z_m $(m \in \omega)$ be non-trivial elements of an $\aleph_1$-free R-module H^0 of cardinality $\aleph_1$ and

$$0 \to H^0 \to H_\alpha \xrightarrow{h} G_\alpha \to 0$$

be a short exact sequence. Then we can find an R-module

$$H' = \langle H_\alpha \oplus B', y'_m : m \in \omega\rangle$$

with $B' = \bigoplus_{i<n} x'_i R$ and $\overline{g}_m h = g_m$ such that the only relations in H' are

$$y'_{m+1}p_m = y'_m + \sum_{i<n} x'_i k_{im} + z_m + \overline{g}_m \qquad (m \in \omega).$$

The map h extends to h' by $x'_i h' = x''_i$, $y'_m h' = y''_m$ such that the new diagram with vertical maps inclusions commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0 & \longrightarrow & H_\alpha & \xrightarrow{h} & G_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0 & \longrightarrow & H' & \xrightarrow{h'} & G_{\alpha+1} & \longrightarrow & 0 \end{array}$$

Proof. Let

$$F_{\alpha+1} = H_\alpha \oplus \bigoplus_{i<n} \overline{x}_i R \oplus \bigoplus_{m\in\omega} \overline{y}_m R$$

and define

$$N_{\alpha+1} = \left\langle \Big(\overline{y}_{m+1}p_m - \overline{y}_m - \sum_{i<n} \overline{x}_i k_{im} - z_m - \overline{g}_m\Big)R : m \in \omega \right\rangle.$$

Hence $H' = F_{\alpha+1}/N_{\alpha+1}$ and let

$$x'_i = \overline{x}_i + N_{\alpha+1}, \quad y'_m = \overline{y}_m + N_{\alpha+1}, \quad x' = \overline{x} + N_{\alpha+1}.$$

First we see that

(a) $x \mapsto x'$ $(x \in H_\alpha)$ defines an embedding $H_\alpha \to H'$ and then we identify H_α with its image in H'.

It remains to show that $H_\alpha \cap N_{\alpha+1} = 0$ viewed in $F_{\alpha+1}$. If $x \in H_\alpha \cap N_{\alpha+1}$, then there are $k_m \in R$ for $m \leq l$ and some $l \in \omega$ such that

$$\sum_{m=0}^{l} \Big(\overline{y}_{m+1}p_m - \overline{y}_m - \sum_{i<n} \overline{x}_i k_{im} - z_m - \overline{g}_m\Big)k_m = x \in H_\alpha.$$

We get

$$x = -\sum_{m=0}^{l}(z_m + \overline{g}_m)k_m, \quad \sum_{m=0}^{l}\Big(\overline{y}_{m+1}p_m - \overline{y}_m - \sum_{i<n} \overline{x}_i k_{im}\Big)k_m = 0.$$

The coefficient of $\overline{y}_{l+1}$ is $p_l k_l = 0$, hence $k_l = 0$ and going down we get $k_m = 0$ for all $m \leq l$, hence $x = 0$ and (a) holds. From $N_{\alpha+1}$ we have the useful system of equations in H':

(b) $$y'_{m+1}p_m = y'_m + \sum_{i<n} x'_i k_{im} + z_m + \overline{g}_m \quad \text{with } \overline{g}_m \in H_\alpha.$$

In view of (a) we also have

(c) $$H' = \langle H_\alpha \oplus B', y'_m : m \in \omega \rangle \quad \text{with } B' = \bigoplus_{i<n} x'_i R \subseteq H'.$$

Next we claim that

(d) If $h' \restriction H_\alpha = h$, $x'_i h' = x_i$ and $y'_m h' = y_m$ $(i < n,\ m \in \omega)$, then $h' : H' \to G_{\alpha+1}$ is a well-defined homomorphism with

(e) $$\ker h' = H^0, \quad \operatorname{Im} h' = G_{\alpha+1}.$$

As h' is defined on non-free generators, we must check that the relations between them are preserved when passing to the proposed image. The relations are given by $N_{\alpha+1}$ or equivalently by (b). Using the definition (d) we see that the relations (b) are mapped summand-wise under h' as follows:

$$\begin{array}{ccccccccc} y'_{m+1}p_m & + & y'_m & + & \sum_{i<n} x'_i k_{im} & + & z_m & + & \overline{g}_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ y_{m+1}p_m & ? & y_m & + & \sum_{i<n} x_i k_{im} & + & 0 & + & g_m \end{array}$$

and inspection of (2.4) and the relations in $G_{\alpha+1}$ shows that ? is an equality sign. Hence h' is well defined. Notice that $H_\alpha h' = H_\alpha h = G_\alpha$, therefore h' induces a homomorphism

$$H'/H_\alpha \to G_{\alpha+1}/G_\alpha$$

and the last argument and $g_m \in G_\alpha$ show that this is an isomorphism. Hence when passing from h to the extended map h' the kernel cannot grow, we have $H^0 = \ker h' = \ker h$ and $\operatorname{Im} h' = G_{\alpha+1}$ is obvious, so (d) and (e) and the proposition are shown. ∎

3. The main reduction lemma—types I, II and III. The *Chase radical* νG of a torsion-free R-module G is the characteristic submodule

$$\nu G = \bigcap \{U \subseteq G : G/U \text{ is } \aleph_1\text{-free}\}.$$

Since $G/\nu G$ is also $\aleph_1$-free, the Chase radical is the smallest submodule with $\aleph_1$-free quotient. If U is a submodule of G we write $\nu_U G = G'$ for the Chase radical of G over U which is defined by $\nu(G/U) = G'/U$.

Given any $\aleph_1$-free R-module G of cardinality $|G| = \aleph_1$, we fix an $\aleph_1$-filtration

$$G = \bigcup_{\alpha<\omega_1} G^0_\alpha$$

which is an ascending, continuous chain of countable, free and pure R-submodules G^0_α of G with $G^0_0 = 0$.

We want to find a new ascending, continuous chain of pure R-submodules G_α (not necessarily countable) such that $G = \bigcup_{\alpha<\omega_1} G_\alpha$. However, we do require that

(3.1) $\qquad G/G_\alpha$ is $\aleph_1$-free if α is not a limit ordinal.

We will use the new chain to divide the $\aleph_1$-free R-modules of cardinality $\aleph_1$ into three types. This distinction helps to show that $\aleph_1$-free splitters of cardinality $\aleph_1$ are free.

Suppose $G_\beta \subseteq G$ is constructed for all $\beta < \alpha$. Next we want to define G_α. If α is a limit ordinal, then

$$G_\alpha = \bigcup_{\beta<\alpha} G_\beta.$$

Hence we may assume that $\alpha = \beta + 1$ and we must define $G_\alpha = G_{\beta+1}$. In order to ensure $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ we let

$$G_{\alpha 0} = (G_\beta + G^0_\alpha)_* \subseteq G \tag{3.2}$$

be the pure R-submodule generated by $G_\beta + G^0_\alpha$. In any case we want to ensure that (3.1) holds, hence $\nu_{G_{\alpha 0}} G \subseteq G_\alpha$. Therefore we construct an ascending, continuous chain of pure R-submodules

(3.3) $\{G_{\alpha j} : j < \omega_1\}$ with $G_{\alpha,j+1}/G_{\alpha j} = 0$ or minimal non-free for each $0 < j < \omega_1$

such that $G_\alpha = \bigcup_{j\in\omega_1} G_{\alpha j}$. Suppose that $G_{\alpha i}$ is defined for all $i < j < \omega_1$. If j is a limit ordinal we take $G_{\alpha j} = \bigcup_{i<j} G_{\alpha i}$, and if $j = i + 1$ we distinguish two cases:

(3.4) If $G/G_{\alpha i}$ is $\aleph_1$-free, then $G_{\alpha j} = G_{\alpha i}$ hence $G_\alpha = G_{\alpha i}$ and $|G_\alpha/G_{\alpha-1}| = \aleph_0$.

Otherwise $G/G_{\alpha i}$ is not $\aleph_1$-free, and by Pontryagin's theorem we can find a finite rank minimal non-free pure R-submodule $M/G_{\alpha i}$ of $G/G_{\alpha i}$. Since $G = \bigcup_{i\in\omega_1} G^0_i$ and $\emptyset \neq M \setminus G_{\alpha i} \leq G$, there is also a least ordinal $\gamma = \gamma(M) = \gamma(M/G_{\alpha i}) < \omega_1$ such that

$$(M \setminus G_{\alpha i}) \cap (G^0_{\gamma+1} \setminus G^0_\gamma) \neq \emptyset. \tag{3.5}$$

Among the candidates M we choose one with the smallest $\gamma(M)$ and take it for $M = G_{\alpha,i+1}$. This completes the construction of the $G_{\alpha i}$'s. Notice that either the construction of G_α stops as in case (3.4) or we arrive at the second possibility:

(3.6) $G_{\alpha,i+1}/G_{\alpha i}$ is minimal non-free for each $i < \omega_1$ and $|G_\alpha/G_{\alpha-1}| = \aleph_1$.

It remains to show that in case (3.6) the following holds:

(3.7) $\nu_{G_{\alpha 0}} G = G_\alpha$ or equivalently G/G_α is $\aleph_1$-free.

Suppose that G/G_α is not $\aleph_1$-free and let X be a non-free submodule of minimal finite rank in G/G_α which exists by Pontryagin's theorem. Representing X in G we have

$$G'' = \langle x_i, y_m, G_\alpha : i < n,\ m \in \omega\rangle_* \quad \text{with} \quad G''/G_\alpha = X$$

(see also Göbel–Shelah [5]). There are elements $g_m \in G_\alpha$ $(m \in \omega)$ such that

$$y_{m+1}p_m = y_m + \sum_{i<n} x_i k_{im} + g_m$$

for some $p_m, k_{im} \in R$ (p_m not units of R). We take

$$G' = \langle x_i, y_m, g_m : i < n,\ m \in \omega\rangle_* \subseteq G,$$

hence $X = G' + G_\alpha/G_\alpha$ was our starting point. Since G' is obviously countable, there is a $\gamma^* \in \omega_1$ with $G' \subseteq G^0_{\gamma^*}$. If $\{G_{\alpha j} : j < \omega_1\}$ is the chain constructed above, we also find $i \in \omega_1$ with $g_m \in G_{\alpha i}$ for all $m \in \omega$. If $i \le j \in \omega_1$, then $G' + G_{\alpha j}/G_{\alpha j}$ is an epimorphic image of X, hence minimal non-free or 0. The second case leads to the immediate contradiction:

$$G' \subseteq G_{\alpha j} \subseteq G_\alpha \quad \text{but} \quad X \neq 0.$$

Hence $G' + G_{\alpha j}/G_{\alpha j} \neq 0$ was a candidate for constructing $G_{\alpha, j+1}$ for any $i \le j \in \omega_1$. Has it been used? We must compare the γ-invariant $\gamma(G' + G_{\alpha j}/G_{\alpha j})$ with the various $\gamma(G_{\alpha,j+1}/G_{\alpha j})$. From $G' \subseteq G^0_{\gamma^*}$ we see that there is $\gamma^j < \gamma^*$ such that

$$(G' + G_{\alpha j} \setminus G_{\alpha j}) \cap (G^0_{\gamma^j+1} \setminus G^0_{\gamma^j}) \neq \emptyset.$$

By minimality of $\gamma_j =: \gamma(G_{\alpha,j+1}/G_{\alpha j})$ we must have $\gamma_j \le \gamma^j < \gamma^*$ and

$$(G_{\alpha,j+1} \setminus G_{\alpha j}) \cap (G^0_{\gamma_j+1} \setminus G^0_{\gamma_j}) \neq \emptyset$$

and $(G_{\alpha j} \cap G^0_{\gamma^*})$ $(j \in \omega_1)$ is a *strictly* increasing chain of length ω_1 of the countable module $G^0_{\gamma^*}$, which is impossible. Hence G/G_α is $\aleph_1$-free and (3.7) is shown.

We have a useful additional property of the constructed chain which reflects (3.7).

COROLLARY 3.1. *If* $0 \neq \alpha \in \omega_1$ *is not a limit ordinal*, *then* $G_\alpha = \nu_{G^0_\alpha} G$.

Proof. We concentrate on the case (3.6) and only note that the case (3.4) is similar.

Recall from (3.7) that G/G_α is $\aleph_1$-free, hence the statement of the corollary is equivalent to saying that any submodule X of G_α must be G_α if only $G^0_\alpha \subseteq X$ with G_α/X $\aleph_1$-free.

Let $\alpha > 0$ and suppose $G^0_\alpha \subseteq X \subseteq G_\alpha$ and $0 \neq G_\alpha/X$ is $\aleph_1$-free. First we claim that

$$G_\beta \subseteq X \quad \text{for all } \beta < \alpha.$$

If this is not the case, then let $\beta < \alpha$ be minimal with $G_\beta \not\subseteq X$. Recall that β cannot be a limit ordinal and we can write $\beta = \gamma + 1$ for some $\gamma < \beta$. We have $G_\beta = \bigcup_{j \in \omega_1} G_{\beta j}$, hence

$$i_\beta = \min\{j \in \omega_1 : G_{\beta j} \not\subseteq X\} \in \omega_1$$

exists. If $i_\beta = 0$, then $G^0_\beta \subseteq G^0_\alpha \subseteq X$ from $\alpha > \beta$ and $G_{\beta 0} \not\subseteq X$. We get $G_{\beta 0} = \langle G_\gamma, G^0_\beta \rangle_* \not\subseteq X$ and $G^0_\beta \subseteq X$ requires $G_\gamma \not\subseteq X$, contradicting minimality of β. Hence $i_\beta > 0$ and $i_\beta = j + 1$. We have $G_{\beta i_\beta} \not\subseteq X$ and $G_{\beta j} \subseteq X$ from $j < i_\beta$ and minimality of i_β. However $G_{\beta,j+1}/G_{\beta j}$ is minimal non-free, and $0 \neq G_{\beta,j+1} + X/X \subseteq G/X$ is an epimorphic image, hence non-free as well. Therefore G/X is not $\aleph_1$-free, a contradiction showing our first claim.

From the first claim we derive $\bigcup_{\beta<\alpha} G_\beta \subseteq X$. Now there must be a minimal

$$i_\alpha = \min\{j : G_{\alpha j} \not\subseteq X\} \in \omega_1,$$

which cannot be a limit ordinal, and again $i_\alpha > 0$, hence $i_\alpha = j+1$. We find $G_{\alpha j} \subseteq X, G_{\alpha,j+1} \not\subseteq X$ and G_α/X cannot be $\aleph_1$-free, a final contradiction. ∎

We now distinguish cases for G depending on the existence of particular filtrations. Let $G = \bigcup_{\alpha<\omega_1} G_\alpha$ be the filtration constructed from the $\aleph_1$-filtration $G = \bigcup_{\alpha<\omega_1} G^0_\alpha$.

If there is an ordinal $\beta < \omega_1$ (which we assume to be minimal) such that $G = G_\beta$, then let $C = G^0_\beta$, which is a countable, free and pure R-submodule of G. From Corollary 3.1 we see that $\nu_C G = G$. Hence, beginning with C we get a new $\aleph_1$-filtration (we use the same notation) $\{G_\alpha : \alpha \in \omega_1\}$ of countable, pure and free R-submodules of G such that $G_0 = C$ and *each* $G_{\alpha+1}/G_\alpha$ $(\alpha > 0)$ is minimal non-free. In this case we say that G and the filtration are of *type I*.

In the opposite case the chain only terminates at the limit ordinal ω_1, i.e. $G_\beta \neq G$ for all $\beta < \omega_1$. We have a proper filtration $G = \bigcup_{\alpha<\omega_1} G_\alpha$ such that Corollary 3.1 holds. If for each $\alpha \in \omega_1$ for some $i < \omega_1$ case (3.4) occurs, then the constructed chain $\{G_\alpha : \alpha \in \omega_1\}$ is an $\aleph_1$-filtration of countable, pure and free R-submodules with the properties of Corollary 3.1 and (3.1). We say that the chain and G are of *type II*.

If G is not of type I or of type II we say that G is of *type III*. In this case, there is a first $\alpha \in \omega_1$ such that $G_{\alpha+1}/G_\alpha$ is uncountable. We may assume that $\alpha = 0$. With the new enumeration we see that the following holds for type III:

(III) $G = \bigcup_{\alpha<\omega_1} G_\alpha$, $G_0 = 0$, $|G_1| = \aleph_1$ and (3.1) holds, $G_1 = \bigcup_{j\in\omega_1} G_{0j}$ is an $\aleph_1$-filtration of pure submodules of G_1 with *each* $G_{0,j+1}/G_{0j}$ minimal non-free.

We have

Reduction Lemma 3.2. *Any $\aleph_1$-free module G of cardinality $\aleph_1$ is either of type I, II or III.*

4. Splitters of cardinality $\aleph_1 < 2^{\aleph_0}$ are free. In this section we do not need the classification of $\aleph_1$-free R-modules of cardinality $\aleph_1$ given in Lemma 3.2. Moreover, we note that $\aleph_1$-freeness of splitters of cardinality $\aleph_1 < 2^{\aleph_0}$ follows by Theorem 1.4. In fact we will present a uniform proof showing freeness of splitters up to cardinality $\aleph_1 < 2^{\aleph_0}$, which extends Hausen's result [7] concerning countable splitters. We begin with a trivial observation:

Proposition 4.1. *Let* $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ *be an* $\aleph_1$*-filtration of pure and free* R*-submodules* G_α *of* G. *Then* $\operatorname{nuc} G_\alpha = R$ *for all* $\alpha \in \omega_1$.

Proof. Choose any basic element $b \in G_\alpha$ for some $\alpha \in \omega_1$. If $r \in \mathbb{Q}$ divides b in G, then r divides b in G_α by purity, hence $r \in R$ from $bR \oplus C = G_\alpha$ and $\operatorname{nuc} G = R$. ■

Corollary 4.2 ($\aleph_1 < 2^{\aleph_0}$). *If* G *is a splitter of cardinality* $\leq \aleph_1$ *and* $\operatorname{nuc} G = R$, *then there is an* $\aleph_1$*-filtration* $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ *of pure and free* R*-submodules* G_α *such that* $\operatorname{nuc} G_\alpha = R$ *for all* $\alpha \in \omega_1$.

Proof. From $\aleph_1 < 2^{\aleph_0}$ and Göbel–Shelah [5] (see Theorem 1.4) it follows that G is an $\aleph_1$-free R-module and G has an $\aleph_1$-filtration as in the hypothesis of Proposition 4.1. ■

Definition 4.3. Let G be a torsion-free abelian group with $\operatorname{nuc} G = R$ and X an R-submodule of G. Then X is *contra-Whitehead* in G if the following holds. There are $z_m \in G$ and $p_m, k_{im} \in R$ ($i < n,\ m \in \omega$) such that the system of equations

$$Y_{m+1}p_m \equiv Y_m + \sum_{i<n} X_i k_{im} + z_m \bmod X \qquad (m \in \omega)$$

has no solutions $y_m, a_i \in G$ (for Y_m, X_i respectively) with $\bigoplus_{i<n}(a_i + X)R$ free of rank n and pure in G/X. Otherwise we call X *pro-Whitehead* in G.

For $X \subseteq G$ as in the definition let $\mathfrak{W}$ be the set of all finite sequences $\overline{a} = (a_0, a_1, \ldots, a_n)$ such that

(i) $a_i \in G$ $(i \leq n)$,
(ii) $\bigoplus_{i<n}(a_i + X)R$ is pure in G/X,
(iii) $\langle (a_i + X)R : i \leq n\rangle_*$ is not a free R-module in G/X.

In particular $G'_{\overline{a}} = \bigoplus_{i<n} a_i R \oplus X$ is a pure submodule of $G_{\overline{a}} = \langle X, a_i R : i \leq n\rangle_*$ and of G, and the module $G_{\overline{a}}/X$ is an n-free-by-1 R-module. From (2.4) we find $p_{\overline{a}m} \in \mathbb{N}$ not units in R and elements $k_{\overline{a}im} \in R$ ($i < n$), $g_{\overline{a}m} \in G_{\overline{a}}$ such that

$$y_{\overline{a},m+1}p_{\overline{a}m} = y_{\overline{a}m} + \sum_{i<n} a_i k_{\overline{a}im} + g_{\overline{a}m} \qquad (m \in \omega). \tag{4.1}$$

The equations (4.1) are the basic systems of equations to decide whether G is a splitter or not. We will also consider an "inhomogeneous counterpart" of (4.1) and choose a sequence $\overline{z} = (z_m : m \in \omega)$ of elements $z_m \in G$. The $\overline{z}$-inhomogeneous counterpart of (4.1) is the system of equations

$$Y_{m+1}p_{\overline{a}m} \equiv Y_m + \sum_{i<n} X_i k_{\overline{a}im} + z_m \bmod X \qquad (m \in \omega). \tag{4.2}$$

According to the above definition we also say that $\overline{a} \in \mathfrak{W}$ is contra-Whitehead if (4.2) has no solutions y_m ($m \in \omega$) in G (hence in $G_{\overline{a}}$) for some $\overline{z}$ and

$X_i = a_i$. Otherwise we say that $\overline{a}$ is pro-Whitehead. If $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ is an $\aleph_1$-filtration of G, then we define $\mathfrak{W}_\alpha$ for $X = G_\alpha$ and let $S = \{\alpha \in \omega_1 :$ there exists $\overline{a} \in \mathfrak{W}_\alpha$ contra-Whitehead$\}$.

Proposition 4.4. *If $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ and S as above is stationary in ω_1, then G is not a splitter.*

Before proving this proposition we simplify our notation. If $\alpha \in S$ we choose $z^\alpha_m \in G$, $\overline{a} = (a^\alpha_0, a^\alpha_1, \dots, a^\alpha_n)$, $p_{\overline{a}m} = p_{\alpha m}$, $g_{\overline{a}m} = g^\alpha_m$, $k_{\overline{a}im} = k_{\alpha im}$, $y_{\overline{a}m} = y^\alpha_m$ so that equations (4.1) and (4.2) become, for $X = G_\alpha$,

$$y^\alpha_{m+1} p_{\alpha m} = y^\alpha_m + \sum_{i<n} a^\alpha_i k_{\alpha im} + g^\alpha_m \qquad (m \in \omega) \tag{4.3}$$

with $\overline{z}^\alpha$-inhomogeneous counterpart

$$Y_{m+1} p_{\alpha m} \equiv Y_m + \sum_{i<n} X_i k_{\alpha im} + z^\alpha_m \bmod G_\alpha \qquad (m \in \omega). \tag{4.4}$$

Hence (4.3) is a system of equations with solutions y^α_m, a^α_i, g^α_m in $G_{\alpha+1}$, while (4.4) with variables Y_m, X_i $(m \in \omega,\ i < n)$ has no solutions in G, as discussed in Definition 4.3 for $X = G_\alpha$. The set of limit ordinals is a cub, hence we may restrict S to this cub and assume that S consists of limit ordinals only. If $\alpha \in S$ we may also assume that

$$G_{\alpha+1} = \langle G_\alpha, a^\alpha_i R : i < n\rangle_* = \langle G_\alpha, a^\alpha_i R, y^\alpha_m R : m \in \omega_1,\ i < n\rangle.$$

Proof of Proposition 4.4. We will use the last remarks to construct $h : H \to G$ such that

$$0 \to H^0 \to H \xrightarrow{h} G \to 0 \tag{$*$}$$

does not split, hence $\operatorname{Ext}(G, H^0) \neq 0$. We will have $H^0 \cong G$, hence $\operatorname{Ext}(G, G)$
$\neq 0$ and G is not a splitter.

Choose an isomorphism $\gamma : G \to H^0$ which carries the $\aleph_1$-filtration $\{G_\alpha : \alpha \in \omega_1\}$ to $H^0 = \bigcup_{\alpha\in\omega_1} H'_\alpha$ and z^α_m to $z'_{\alpha m}$. Inductively we want to define short exact sequences

$$0 \to H^0 \xrightarrow{\text{id}} H_\beta \xrightarrow{h_\beta} G_\beta \to 0 \qquad (\beta < \alpha) \tag{β}$$

which are increasing continuously. Let

$$0 \to H^0 \xrightarrow{\text{id}} H_0 \xrightarrow{h_0} 0 \to 0 \tag{0}$$

be defined for $H_0 = H^0$ with h_0 the zero map and suppose (β) is defined for all $\beta < \alpha < \omega_1$ with α a limit ordinal. We take unions and (α) is defined. If $\alpha \in \omega_1 \setminus S$, we extend (α) trivially to get $(\alpha + 1)$ and if $\alpha \in S$ we must work for $(\alpha + 1)$: We apply Proposition 2.2 to find $H_\alpha \subseteq H_{\alpha+1}$ with

$$H_{\alpha+1} = \langle H_\alpha, e_{\alpha m}, x_{\alpha m} : m \in \omega,\ i < n\rangle$$

and relations

$$e_{\alpha,m+1}p_{\alpha m} = e_{\alpha m} + \sum_{i<n} x_{\alpha i}k_{\alpha im} + y_{\alpha m} + z'_{\alpha m} \qquad (m \in \omega) \tag{4.5}$$

with $y_{\alpha m}h_\alpha = g^\alpha_m \in G_\alpha$. We want to extend the homomorphism $h_\alpha : H_\alpha \to G_\alpha$ to $h_{\alpha+1} : H_{\alpha+1} \to G_{\alpha+1}$, and set $e_{\alpha m}h_{\alpha+1} = y^\alpha_m$ and $x_{\alpha i}h_{\alpha+1} = a^\alpha_i$. By Proposition 2.2 the map $h_{\alpha+1}$ is a well defined homomorphism. It is clearly surjective with kernel H^0. Hence $(\alpha + 1)$ is well defined for all $\alpha \in \omega_1$ and $h = \bigcup_{\alpha\in\omega_1} h_\alpha$ shows $(*)$.

Finally we must show that $(*)$ does not split. Suppose that $\sigma : G \to H$ is a splitting map for $(*)$. Hence $\sigma h = \mathrm{id}_G$ and $H = H^0 \oplus \operatorname{Im}\sigma$ and $g^\alpha_m = y_{\alpha m}h_\alpha = y_{\alpha m}h$, so $(y_{\alpha m} - g^\alpha_m\sigma)h = 0$ implies $y_{\alpha m} - g^\alpha_m\sigma \in H^0$ for all $\alpha \in S$. The set

$$C = \{\alpha \in \omega_1 : \alpha \text{ a limit ordinal, } y_{\alpha m} - g^\alpha_e\sigma \in H'_\alpha\}$$

—by a back-and-forth argument—is a cub and hence $S \cap C$ is stationary in ω_1. We can find $\alpha \in C \cap S$ and consider the associated equations. In G we have (4.3):

$$y^\alpha_{m+1}p_{\alpha m} = y^\alpha_m + \sum_{i<n} a^\alpha_i k_{\alpha im} + g^\alpha_m$$

and σ moves these equations to H:

$$(y^\alpha_{m+1}\sigma)p_{\alpha m} = (y^\alpha_m\sigma) + \sum_{i<n}(a^\alpha_i\sigma)k_{\alpha im} + (g^\alpha_m\sigma),$$

which we subtract from (4.5). Hence

$$(e^\alpha_{m+1} - y^\alpha_{m+1}\sigma)p_{\alpha m} = (e^\alpha_m - y^\alpha_m\sigma) + \sum_{i<n}(x_{\alpha i} - a^\alpha_i\sigma)k_{\alpha im} + (y_{\alpha m} - g^\alpha_m\,\sigma) + z'_{\alpha m}.$$

Put

$$f_{\alpha m} = e^\alpha_m - g^\alpha_m\sigma, \qquad v_{\alpha i} = x_{\alpha i} - a^\alpha_i\sigma, \qquad w_{\alpha m} = y_{\alpha m} - g^\alpha_m\sigma$$

and note that

$$f_{\alpha m}h = e_{\alpha m}h - g^\alpha_m\sigma h = g^\alpha_m - g^\alpha_m = 0,$$

hence $f_{\alpha m} \in \ker h = H^0$. Similarly $w_{\alpha m}, v_{\alpha m} \in H^0$. The last equation turns into

$$f_{\alpha,m+1}p_{\alpha m} = f_{\alpha m} + \sum_{i<n} v_{\alpha im}k_{\alpha im} + w_{\alpha m} + z'_{\alpha m} \qquad (m \in \omega),$$

which, as just seen, is a system of equations in H^0. From $\alpha \in C$ we have $w_{\alpha m} \in H'_\alpha$. The isomorphism γ^{-1} moves the last equation back into G and $w_{\alpha m}\gamma^{-1} \in G_\alpha$. Using

$$f'_{\alpha m} = f_{\alpha m}\gamma^{-1}, \qquad w'_{\alpha m} = w_{\alpha m}\gamma^{-1}, \qquad v'_{\alpha i} = v_{\alpha i}\gamma^{-1}$$

we derive

$$f'_{\alpha,m+1}p_{\alpha m} = f'_{\alpha m} + \sum_{i<n} v'_{\alpha i}k_{\alpha im} + w'_{\alpha m} + z^m_\alpha \qquad (m \in \omega)$$

with $w'_{\alpha m} \in G_\alpha$ and z^m_α as in (4.4), which is impossible if $\alpha \in S$ is contra-Whitehead, where we have chosen z^m_α suitably. ■

THEOREM 4.5. *Let G be a splitter of cardinality $< 2^{\aleph_0}$ with* $\operatorname{nuc} G = R$. *If X is a pure, countable R-submodule of G which is pro-Whitehead in G, then G/X is an $\aleph_1$-free R-module.*

Proof. First we assume that $\operatorname{nuc}(G/X) = R$ and suppose for contradiction that G/X is not an $\aleph_1$-free R-module. By Pontryagin's theorem we can find an R-submodule $Y \subseteq G/X$ of finite rank which is not free. We may assume that Y is of minimal rank. Hence

$$Y = \langle B, y_m R : m \in \omega\rangle, \qquad B = \bigoplus_{i<n} \overline{x}_i R$$

with the only relations

$$y_{m+1}p_m = y_m + \sum_{i<n} \overline{x}_i k_{im} \qquad (m \in \omega)$$

as in Section 2 such that no $p_m \in R$ is a unit of R for $m \in \omega$. Choose $x_i \in G$ such that $x_i + X = \overline{x}_i$ for each $i < n$. We can also choose a sequence of elements $z_m \in G$ such that $z_m + X$ is not divisible by p_{m-1} from $\operatorname{nuc}(G/X) = R$ $(m \in \omega)$. If $\eta \in {}^\omega 2$, then let

$$\overline{z}^\eta = \langle \eta(e)z_e : e \in \omega\rangle = (z^\eta_e).$$

Recall that X is pro-Whitehead in G, hence the system of equations

$$(\eta) \qquad y^\eta_{m+1}p_m \equiv y^\eta_m + \sum_{i<n} x^k_i k_{im} + z^\eta_m \bmod X \qquad (m \in \omega)$$

has solutions $x^\eta_i, y^\eta_m \in G$ for each $\eta \in {}^\omega 2$. Note that

$$|\{\langle x^\eta_i : i < n\rangle^\wedge\langle y^\eta_0\rangle : \eta \in {}^\omega 2\}| \le |G| < 2^{\aleph_0}.$$

We can find $\eta \ne \nu \in {}^\omega 2$ such that $x^\eta_i = x^\nu_i$ for all $i < n$ and $y^\eta_0 = y^\nu_0$. From $\eta \ne \nu$ we find a branching point $j \in \omega$ such that $\eta(j) \ne \nu(j)$ but $\eta\restriction j = \nu\restriction j$. We may assume $\eta(j) = 1$ and $\nu(j) = 0$ and put $w_m = y^\eta_m - y^\nu_m$. Subtracting the equations (ν) from (η) we infer from $x^\eta_i - x^\nu_i = 0$ that

$$w_{m+1}p_m = w_m + (z^\eta_m - z^\nu_m) \bmod X$$

and $w_0 = y^\eta_0 - y^\nu_0 = 0$ as well. For $m \le j$ we have $z^\eta_m - z^\nu_m = 0$ and $z^\eta_j - z^\nu_j = z_j$, hence $w_m = 0$ for $m < j$ by torsion-freeness and

$$w_j p_{j-1} = z_j \bmod X,$$

which contradicts our choice of z_m's and p_m's.

If $\operatorname{nuc}(G/X) = \mathbb{Q}$, then G/X is divisible, hence X is dense and pure in G, we have $X \subseteq_* G \subseteq_* \widehat{X}$, where $\widehat{X}$ is the $\mathbb{Z}$-adic completion of X, and X is a free R-module of countable rank. Hence $G/X \subseteq_* \widehat{X}/X \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$ and there are $2^{\aleph_0}$ independent elements in $\widehat{X}/X$. These independent elements can be expressed as unique solutions of certain systems of equations—rewrite the $\mathbb{Z}$-adic limits accordingly. The equations must be solvable by pro-Whitehead. Hence $|G| = |\widehat{X}| = 2^{\aleph_0} > \aleph_1$, which is a contradiction.

So we find $p_m \in R$ and $z_m \in G$ such that p_{m-1} does not divide $z_m + X$ in G for all $m \in \omega$. The above argument applies again for $n = 0$ and leads to a contradiction. ∎

COROLLARY 4.6. *Any splitter of cardinality at most $\aleph_1 < 2^{\aleph_0}$ is free over its nucleus.*

Proof. Let $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ be an $\aleph_1$-filtration of the splitter G. By Corollary 4.2 we may assume that each G_α is a pure and free R-submodule of G with $R = \operatorname{nuc} G$. If S denotes the set

$$\{\alpha \in \omega_1 : G_\alpha \text{ is contra-Whitehead in } G\},$$

then S is not stationary in ω_1 by Proposition 4.4. We may assume that all G_α are pro-Whitehead in G and each $G_{\alpha+1}/G_\alpha$ is countable, hence free by Theorem 4.5. We see that G must be free as well. ∎

5. Splitters of type I under CH.

In view of Section 4 we may assume CH to derive a theorem in ZFC showing freeness for $\aleph_1$-free splitters of cardinality $\aleph_1$ of type I. The advantage of the set-theoretical assumption is—compared with the proof based on the weak continuum hypothesis WCH in Section 8—that the proof given here is by no means technical. Recall that G is of type I if $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ for some $\aleph_1$-filtration $\{G_\alpha : \alpha \in \omega_1\}$ of pure submodules G_α such that *each* $G_{\alpha+1}/G_\alpha$ $(\alpha > 0)$ is a minimal non-free R-module. In this section we want to show the following

PROPOSITION 5.1 (ZFC + CH). *Modules of type I are not splitters.*

Combining Proposition 5.1 and Corollary 4.6 we can remove CH and have the immediate consequence which holds in ZFC.

COROLLARY 5.2. *Any $\aleph_1$-free splitter of type I (and cardinality $\aleph_1$) is free over its nucleus.*

The proof of Proposition 5.1 is based on an observation strongly related to type I concerning splitting maps. Then we want to prove a step lemma for applications of CH. Finally we use CH to show $\operatorname{Ext}(G, G) \neq 0$ in Theorem 5.1. In Section 1 we noticed that if

$$0 \to B \xrightarrow{\beta} C \xrightarrow{\alpha} A \to 0$$

is a short exact sequence, hence representing an element in $\mathrm{Ext}(A, B)$, then this element is 0 if and only if there is a splitting map $\gamma : A \to C$ such that $\gamma\alpha = \mathrm{id}_A$. This simple fact is the key for the next two results.

OBSERVATION 5.3. *Let* $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ *be a filtration of type I. For* $\alpha \in \omega_1$, *let*

$$(\alpha) \qquad 0 \to H^0 \to H_\alpha \to G_\alpha \to 0$$

be a continuous, increasing chain of short exact sequences with union

$$(\omega_1) \qquad 0 \to H^0 \to H \to G \to 0$$

and let $H^0 \cong G$ *be* $\aleph_1$*-free. Then any splitting map of* (α) *for* $\alpha = 1$ *has at most one extension to a splitting map of* (ω_1).

Proof. We may assume that the splitting map $\sigma : G_1 \to H_1$ of (1) has two extensions $\sigma, \sigma' : G \to H$ which split. Since ω_1 is a limit ordinal, there is some $\beta < \omega_1$ minimal with $(\sigma - \sigma')\restriction G_\beta \neq 0$. Clearly β is not a limit ordinal and $\sigma - \sigma'$ induces a non-trivial map $\delta : G_\beta/G_{\beta-1} \to H_\beta$. The domain of this map is minimal non-free, while its range is $\aleph_1$-free, hence δ must be 0, a contradiction. ■

STEP LEMMA 5.4. *Let* $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ *be a filtration of type I and let*

$$0 \to H^0 \to H_\alpha \xrightarrow{h} G_\alpha \to 0$$

be a short exact sequence with $H^0 \cong G$. *If* $\sigma : G_\alpha \to H_\alpha$ *is a splitting map, then there is an extension of this sequence such that* σ *does not extend to a splitting map* σ' *of the new short exact sequence*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0 & \longrightarrow & H_\alpha & \underset{\sigma}{\overset{h}{\rightleftarrows}} & G_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0 & \longrightarrow & H' & \underset{\sigma'}{\overset{h'}{\not\rightleftarrows}} & G_{\alpha+1} & \longrightarrow & 0 \end{array}$$

Moreover, the vertical maps in the diagram are inclusions and if $G_{\alpha+1}/G_\alpha$ *is n-free-by-1, then* $B'_{\alpha+1}$ *is a free R-module of rank n and*

$$H' = \langle H_\alpha \oplus B'_{\alpha+1}, y''_{\alpha m} : m \in \omega\rangle$$

and $B'_{\alpha+1}$ *is mapped under* h' *mod* G_α *onto a free maximal pure R-sub-module of* $G_{\alpha+1}/G_\alpha$.

Proof. We will use special elements $s_m \in R$ $(m \in \omega)$ to kill extensions. It will help the reader to pose precise conditions on the choice of the s_m's only when needed, which will be at the end of the proof. Readers familiar with such proofs will know that we are working to produce a p-adic catastrophe.

First we use the fact that $G' = G_{\alpha+1}/G_\alpha$ is minimal non-free, say n-free-by-1. By (2.3) and (2.4) we have

$$G' = \Big\langle \bigoplus_{i<n} x_i R, y_m R : m \in \omega \Big\rangle$$

with the only relations

$$y_{m+1} p_m = y_m + \sum_{i<n} x_i k_{im} \qquad (m \in \omega)$$

and coefficients $p_m, k_{im} \in R$. By the last equations we can find $g_{\alpha m} \in G_\alpha$ and $x_{\alpha i}, y_{\alpha m} \in G_{\alpha+1}$ such that

$$(*) \qquad G_{\alpha+1} = \langle G_\alpha, x_{\alpha i} R, y_{\alpha m} R : i < n, m \in \omega \rangle \text{ with the relations}$$
$$y_{\alpha,m+1} p_m = y_{\alpha m} + \sum_{i<n} x_{\alpha i} k_{im} + g_{\alpha m} \qquad (m \in \omega).$$

The action of σ is known to us on G_α, hence we can choose a pure element $0 \neq z \in H^0$ and let $z_m = z s_m$. Then H^0/zR is $\aleph_1$-free by purity of z in an $\aleph_1$-free R-module. We also choose preimages $\overline{g}_{\alpha m} = g_{\alpha m}\sigma \in H_\alpha$, hence $\overline{g}_{\alpha m} h = g_{\alpha m}$. We are now in a position to apply Proposition 2.2. Let

$$H' = \langle H_\alpha \oplus B'_{\alpha+1}, y''_{\alpha m} : m \in \omega \rangle \quad \text{with} \quad B'_{\alpha+1} = \bigoplus_{i<n} x''_{\alpha i} R \subseteq H'$$

be the extension given by the proposition with the useful relations

$$(5.1) \qquad y''_{\alpha,m+1} p_m = y''_{\alpha m} + \sum_{i<n} x''_{\alpha i} k_{im} + z s_m + g_{\alpha m}\sigma \qquad (m \in \omega)$$

and an extended homomorphism $h' : H' \to G_{\alpha+1}$ with

$$h' \restriction H_\alpha = h, \quad x''_{\alpha i} h' = x_{\alpha i}, \quad y''_{\alpha m} h' = y_{\alpha m} \quad (i < n,\ m \in \omega)$$

such that $\ker h' = H^0$ and $\operatorname{Im} h' = G_{\alpha+1}$. It remains to show the non-splitting property of the lemma.

Suppose that $\sigma' : G_{\alpha+1} \to H'$ is an extension of $\sigma : G_\alpha \to H$ such that $\sigma' h' = \mathrm{id}_{G_{\alpha+1}}$. Now we want to derive a contradiction by choosing the s_m's suitably (independent of σ' !). We apply σ' to $(*)$ and get the equations in H':

$$(*\sigma') \qquad y_{\alpha,m+1}\sigma' p_m = y_{\alpha m}\sigma' + \sum_{i<n} x_{\alpha i}\sigma' k_{im} + g_{\alpha m}\sigma'.$$

If $d_{\alpha m} = y''_{\alpha m} - y_{\alpha m}\sigma'$ and $e_{\alpha i} = x''_{\alpha i} - x_{\alpha i}\sigma'$ then $d_{\alpha m} \in H^0$ since

$$d_{\alpha m} h' = (y''_{\alpha m} - y_{\alpha m}\sigma') h' = y''_{\alpha m} h' - y_{\alpha m}\sigma' h' = y_{\alpha m} - y_{\alpha m} = 0$$

and $\ker h' = H^0$. Similarly we argue with $e_{\alpha i}$ and get $d_{\alpha m}, e_{\alpha i} \in H^0$.

Subtracting $(*\sigma')$ from (5.1) now leads to a system of equations in H^0:

$$d_{\alpha,m+1}p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i}k_{im} + zs_m.$$

We consider the submodule

$$W = \langle d_{\alpha m} + zR, e_{\alpha i} + zR : i < n,\ m \in \omega\rangle_R \subseteq H^0/zR.$$

The last displayed equations tell us that W is an epimorphic image of a minimal non-free R-module, hence 0 or non-free of finite rank. On the other hand H^0/zR is $\aleph_1$-free as noted above, hence $W = 0$ or equivalently

$$\langle d_{\alpha m}, e_{\alpha i} : m \in \omega,\ i < n\rangle_R \subseteq zR \cong R.$$

The original equations

$$d_{\alpha,m+1}p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i}k_{im} + s_m \tag{5.2}$$

still hold, but this time require solutions $d_{\alpha m}, e_{\alpha i} \in R$. We get to an end: just choose rational numbers $s_m \in R$ such that (5.2) has no solutions. The existence of such s_m's follows from Lemma 2.1. Finally note that dealing with (5.2) is independent of the particular choices of the extensions of σ as required in the lemma. ■

Proof of Proposition 5.1. Let $H^0 \cong G = \bigcup_{\alpha\in\omega_1} G_\alpha$ be a module of type I. We must show that $\mathrm{Ext}(G, H^0) \neq 0$ and need a non-splitting short exact sequence

$$0 \to H^0 \to H \xrightarrow{h} G \to 0, \tag{5.3}$$

which we construct inductively as an ascending, continuous chain of short exact sequences

$$0 \to H^0 \to H_\alpha \xrightarrow{h_\alpha} G_\alpha \to 0$$

with union (5.3). Let

$$0 \to H^0 \to H_1 \xrightarrow{h_1} G_1 \to 0$$

be the first step with G_1 a free R-module of countable rank. By Observation 5.3 and CH we can enumerate all possible splitting maps $\sigma : G \to H$ of extensions h as in (5.3) of all h_1's by ω_1, and let $\{\sigma_\alpha : G \to H : \alpha \in \omega_1\}$ be such a list. Using Step Lemma 5.4 and the uniqueness in Observation 5.3 we can discard any σ_α at stage α when constructing

$$0 \to H^0 \to H_{\alpha+1} \xrightarrow{h_{\alpha+1}} G_{\alpha+1} \to 0.$$

The resulting extension (5.3) cannot split. ■

6. Splitters of type II. An R-module G is of type II if G has an $\aleph_1$-filtration $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ of pure submodules G_α such that G/G_α is $\aleph_1$-free for all non-limit ordinals $\alpha \in \omega_1$ (see Section 3). In this section we want to show our second main

THEOREM 6.1. *If G is of type II, then G is a splitter if and only if G is free over its nucleus R.*

REMARK. Theorem 6.1 includes the statement that strongly $\aleph_1$-free R-modules are never splitters, except if trivially the module is free. This was very surprising to us.

Proof (of Theorem 6.1). If $S = \{\alpha \in \omega_1 : G/G_\alpha \text{ is not } \aleph_1\text{-free}\}$, then S is a set of limit ordinals by (3.1), and if $\alpha \in S$ we may also assume that $G_{\alpha+1}/G_\alpha$ is minimal non-free (compare §3).

We get a Γ-invariant $\Gamma(G)$ defined by S modulo the ideal of thin sets (see e.g. [3]). If $\Gamma(G) = 0$, then we find a cub $C \subseteq \omega_1$ with $C \cap S = \emptyset$ and $G = \bigcup_{\alpha\in C} G_\alpha$. Let $\alpha_0 = \min C$. Then $G = G_{\alpha_0} \oplus F$ for some free R-module F, and G_{α_0} is a countable submodule of G which must be free over R by Hausen's [7] result (see also [5]). Hence G is free. Note that the hypothesis of G being $\aleph_1$-free is not used in this case! If $\Gamma(G) \neq 0$ we want to show that $\mathrm{Ext}(G, G) \neq 0$. Theorem 6.1 can be rephrased as

(6.1) *If G is of type II, then G is a splitter if and only if $\Gamma(G) = 0$.*

Now assume that S is stationary in ω_1. We want to construct some $H \xrightarrow{h} G \to 0$ with $\ker h = H^0$, G isomorphic to H^0 by γ, which does not split. If $G_\alpha\gamma = H'_\alpha$ ($\alpha \in \omega_1$), then

$$H^0 = \bigcup_{\alpha\in\omega_1} H'_\alpha$$

is a (canonical) $\aleph_1$-filtration of H^0 copied from G. First we pick elements $z_\alpha \in H^0$ such that $z_\alpha R \cong R$ and $H^0/z_\alpha R$ is $\aleph_1$-free, e.g. take any basis element from a layer $H'_{\alpha+2} \setminus H'_{\alpha+1}$ of the filtration of H^0. Then we define inductively a continuous chain of short exact sequences ($\alpha \in \omega_1$)

$$(\beta) \quad 0 \to H^0 \to H^\beta \xrightarrow{h_\beta} G_\beta \to 0,$$
$$\text{countable, free submodules } H_\beta \subseteq_* H^\beta, \text{ and ordinals } \beta < \beta' \le \omega_1$$

subject to various conditions. At the end we want in particular $H = \bigcup_{\alpha\in\omega_1} H_\alpha$
$= \bigcup_{\alpha\in\omega_1} H^\alpha$.

If $\beta = 0$, then $G_0 = 0$ and we take the zero map $h_0 : H^0 \to G_0 \to 0$ with kernel H^0.

Suppose (β) is constructed for all $\beta < \alpha$. If α is a limit, we take unions $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$, $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$ and $H^\alpha = \bigcup_{\beta<\alpha} H^\beta$, assuming that at inductive steps sequences extend (naturally) by inclusions. Then visibly (α) holds.

We may assume that (α) is known, and we want to construct $(\alpha+1)$.

If $\alpha \notin S$, then we extend (α) trivially: Put $H^{\alpha+1} = H^\alpha \oplus F_\alpha$ with F_α a free R-module of the same rank as the free R-module $G_{\alpha+1}/G_\alpha$. As $G_{\alpha+1} = F'_\alpha \oplus G_\alpha$, we may choose an isomorphism $h' : F_\alpha \to F'_\alpha$ and extend h_α to $h_{\alpha+1}$ by $h_{\alpha+1} = h_\alpha \oplus h'$. Clearly $\ker h_{\alpha+1} = \ker h_\alpha = H^0$ and $\operatorname{Im} h_{\alpha+1} = G_{\alpha+1}$.

If $\alpha \in S$, then we must work. We have $G_{\alpha+1}/G_\alpha = \langle B'_{\alpha+1}, y'_m : m \in \omega\rangle$ from (2.3) and (2.4). Hence

$$G_{\alpha+1} = \langle G_\alpha, B_{\alpha+1}, y_{\alpha m}R : m \in \omega\rangle, \qquad B_{\alpha+1} = \bigoplus_{i<n} x_{\alpha i}R \tag{6.2}$$

with relations

$$y_{\alpha,m+1}p_m = y_{\alpha m} + \sum_{i<n} x_{\alpha i}k_{im} + g_{\alpha m} \qquad (m \in \omega), \tag{6.3}$$

where $g_{\alpha m} \in G_\alpha$. Let $\overline{B}_{\alpha+1} = \bigoplus_{i<n} \overline{x}_{\alpha i}R$ be a copy of $B_{\alpha+1}$. Then we pose the following additional conditions on $(\alpha+1)$:

(a) $H^{\alpha+1}h_{\alpha+1} = H_{\alpha+1}h_{\alpha+1} = G_{\alpha+1}$,
(b) $\overline{B}_{\alpha+1} \subseteq H_{\alpha+1}$,
(c) $H^{\alpha+1}/H^\alpha \cong G_{\alpha+1}/G_\alpha$,
(d) $H^{\alpha+1}/H_\beta + H'_\gamma$ is free for all $\beta \le \alpha$, $\beta \notin S$ and $\gamma \in \omega_1$,
(e) $H^{\alpha+1}/H_\beta + H'_\gamma$ is $\aleph_1$-free for all $\beta \le \alpha$, $\gamma \in \omega_1$,
(f) $H'_{\alpha+1} \subseteq H_{\alpha+1} \cap H^0 = H'_{(\alpha+1)'}$.

We choose preimages $\overline{g}_{\alpha m} \in H_\alpha$ such that $\overline{g}_{\alpha m}h_\alpha = g_{\alpha m}$ and apply Proposition 2.2 to define the extension

$$H^\alpha \subseteq H^{\alpha+1} = \langle H^\alpha \oplus \overline{B}_{\alpha+1},\ \overline{y}_{\alpha m}R : m \in \omega\rangle$$

with relations

$$\overline{y}_{\alpha,m+1}p_m = \overline{y}_{\alpha m} + \sum_{i<n} \overline{x}_{\alpha i}k_{im} + z_{\alpha m}s_{\alpha m} + \overline{g}_{\alpha m} \tag{6.4}$$

where

$$\overline{B}_{\alpha+1} \cong \bigoplus_{i<n} x'_iR$$

as required in (b). Similarly, by Proposition 2.2 the map h_α extends to an epimorphism $h_{\alpha+1} : H^{\alpha+1} \to G_{\alpha+1}$. It is now easy to check that (c) holds and it is also easy to see that $\ker h_{\alpha+1} = H^0$. Next we extend $H_\alpha \subset H_{\alpha+1}$ carefully such that $(\alpha+1)$, (a), (b), (d), (e) and (f) hold.

$\operatorname{Im} h_{\alpha+1} = G_{\alpha+1}$ is a countable module of the $\aleph_1$-free R-module G, hence free and $h_{\alpha+1}$ must split. There is a splitting map $\sigma : G_{\alpha+1} \to H^{\alpha+1}$ such that $\sigma h_{\alpha+1} = \mathrm{id}_{G_{\alpha+1}}$, hence $H^{\alpha+1} = H^0 \oplus G_{\alpha+1}\sigma$. Let π_0 :

$H^{\alpha+1} \to H^0$ and $\pi_1 : H^{\alpha+1} \to G_{\alpha+1}\sigma$ be the canonical projections with $\pi_0 + \pi_1 = \mathrm{id}_{H^{\alpha+1}}$. Recall that $\overline{B}_{\alpha+1} \subseteq H^{\alpha+1}$. Choose $\beta \in \omega_1$ large enough such that $\beta \notin S$, $\beta \geq \alpha', \alpha+1$ and $(\overline{B}_{\alpha+1} + H_\alpha)\pi_0 \subseteq H'_\beta$. This is easy because $\omega_1 \setminus S$ is unbounded and $(\overline{B}_{\alpha+1} + H_\alpha)\pi_0$ is countable. Put $H_{\alpha+1} = H'_\beta \oplus (G_{\alpha+1}\sigma)$ and $\beta = (\alpha+1)'$. Note that

$$G_{\alpha+1} = H^{\alpha+1}h_{\alpha+1} \supseteq H_{\alpha+1}h_{\alpha+1} \supseteq G_{\alpha+1}\sigma h_{\alpha+1} = G_{\alpha+1}$$

and (a) follows.

If $\gamma \leq (\alpha+1)'$, then $H_{\alpha+1} + H_\gamma = H_{\alpha+1}$ and if $\gamma \geq (\alpha+1)'$, then $H_{\alpha+1} + H'_\gamma = H'_\gamma \oplus G_{\alpha+1}\sigma$ and (d) follows. We see immediately $H_\alpha\pi_1 \subseteq G_{\alpha+1}\sigma$ and $H_\alpha\pi_0 \subseteq H'_{(\alpha+1)'}$, hence

$$H_\alpha \subseteq H_\alpha\pi_0 + H_\alpha\pi_1 \subseteq H_{\alpha+1}$$

and $(\alpha+1)$ holds; similarly $\overline{B}_{\alpha+1} \subseteq H_{\alpha+1}$ for (c). From $H'_{(\alpha+1)'} \subseteq H^0$ and the modular law we have $H_{\alpha+1} \cap H^0 = H'_{(\alpha+1)'} \oplus (G_{\alpha+1}\sigma \cap H^0) = H'_{(\alpha+1)'}$ and (f) holds.

Finally we choose $H = \bigcup_{\alpha\in\omega_1} H^\alpha$, $h = \bigcup_{\alpha\in\omega_1} h_\alpha$ and

$$0 \to H^0 \to H \xrightarrow{h} G \to 0 \tag{6.5}$$

is established; it remains to show that (6.5) does not split. Suppose for contradiction that $\sigma : G \to H$ is a splitting map for h. We have $\aleph_1$-filtrations $H = \bigcup_{\alpha\in\omega_1} H_\alpha$ and $G = \bigcup_{\alpha\in\omega_1} G_\alpha$. Using the above properties of the H_α's, it follows by a back and forth argument that

$$E = \{\alpha \in \omega_1 : H_\alpha \cap H^0 = H'_\alpha,\ G_\alpha\sigma \subseteq H_\alpha\}$$

is a cub. On the other hand S is stationary in ω_1 and we find $\alpha \in S \cap E$. From (6.3) and (6.4) we have

$$y_{\alpha,m+1}\sigma p_m = y_{\alpha m}\sigma + \sum_{i<n} x_{\alpha i}\sigma k_{im} + g_{\alpha m}\sigma$$

and

$$\overline{y}_{\alpha,m+1}p_m = \overline{y}_{\alpha m} + \sum_{i<n} \overline{x}_{\alpha i}k_{im} + z_\alpha s_{\alpha m} + \overline{g}_{\alpha m}$$

with $g_{\alpha m}\sigma, \overline{g}_{\alpha m} \in H_\alpha$ and $G_\alpha\sigma \subseteq H_\alpha$. Put $d_{\alpha m} = \overline{y}_{\alpha m} - y_{\alpha m}\sigma$, $f_{\alpha m} = \overline{g}_{\alpha m} - g_{\alpha m}\sigma$, $e_{\alpha i} = \overline{x}_{\alpha i} - x_{\alpha i}\sigma$ and notice that $d_{\alpha m}h = e_{\alpha i}h = f_{\alpha m}h = 0$, hence $d_{\alpha m}, e_{\alpha i}, f_{\alpha m} \in H^0$.

Subtracting the last displayed equations we get

$$d_{\alpha,m+1}p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i}k_{im} + f_{\alpha m} + z_\alpha s_{\alpha m} \quad \text{in } H^0. \tag{j}$$

Recall that $f_{\alpha m} \in H_\alpha \cap H^0 \subseteq H'_{\alpha'}$ by (f) and modulo $T = H'_{\alpha'} + z_\alpha R$ the equations (j) say that $W = \langle d_{\alpha m}, e_{\alpha i} : i < n,\ m \in \omega\rangle + T/T$ is either minimal non-free or 0. On the other hand H^0/T is $\aleph_1$-free, hence $W = 0$ and

$\langle d_{\alpha m}, e_{\alpha i}, z_\alpha \rangle \subseteq H'_\alpha + z_\alpha R$. Recall from (e) that H^0/H'_α is $\aleph_1$-free. Hence (j) turns into

$$d_{\alpha,m+1}p_m \equiv d_{\alpha m} + \sum_{i<n} e_{\alpha i}k_{im} + z_\alpha s_{\alpha m} \bmod H'_\alpha.$$

Using $\aleph_1$-freeness of $H^0/z_\alpha R$ these equations tell us that we must have solutions $d_{\alpha m}, e_{\alpha i} \in R$ for

$$\text{(k)} \qquad d_{\alpha,m+1}p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i}k_{im} + s_{\alpha m}.$$

In Lemma 2.1 we selected particular $s_{\alpha m}$'s in R such that (k) has no solution in R. Now we are ready to make this choice which we should have done right at the beginning of the proof and hence derive a contradiction; we conclude $\mathrm{Ext}(G, G) \neq 0$. ∎

From Theorem 6.1 we see that non-free but strongly $\aleph_1$-free abelian groups are never splitters. We find this very surprising. Particular groups like the Griffith group $\mathbb{G}$ below which is a Whitehead group ($\mathrm{Ext}(\mathbb{G}, \mathbb{Z}) = 0$) under Martin's axiom and $\neg$CH is not a splitter. Recall a nice and easy construction of $\mathbb{G}$ which is sometimes Whitehead but always fails to be a splitter in general.

Let $P = \mathbb{Z}^{\aleph_1} = \prod_{\alpha\in\aleph_1} \alpha\mathbb{Z}$ be the cartesian product of $\mathbb{Z}$. If $\lambda \in \aleph_1$ is a limit ordinal choose an order preserving map $\delta_\lambda : \omega \to \lambda$ with $\sup(\omega\delta_\lambda) = \lambda$. Then along this ladder system we define branch elements

$$c_{\lambda n} = \sum_{i\geq n} (i\delta_\lambda)\frac{i!}{n!}$$

which are a "divisibility chain" of $c_{\lambda 0}$ modulo $\bigoplus_{\alpha\in\aleph_1} \alpha\mathbb{Z}$, hence

$$\mathbb{G} = \Big\langle \bigoplus_{\alpha\in\aleph_1} \alpha\mathbb{Z}, c_{\lambda n} : \lambda \in \aleph_1,\ \lambda \text{ a limit ordinal, } n \in \omega \Big\rangle$$

is a pure subgroup of P. We see that $|\mathbb{G}| = \aleph_1$ and $\mathbb{G}$ is $\aleph_1$-free by $\aleph_1$-freeness of P; see [4] (Vol. 1, p. 94, Theorem 19.2). Moreover $\Gamma_G \neq 0$ because $G_\beta = \mathbb{G} \cap \prod_{\alpha<\beta} \alpha\mathbb{Z}$ ($\beta \in \omega_1$) is an $\aleph_1$-filtration of $\mathbb{G}$ with $G_{\lambda+1}/G_\lambda$ divisible for all limit ordinals λ. Hence $\mathbb{G}$ is not free. It is easy to check that $\mathbb{G}$ is $\aleph_1$-separable, hence strongly $\aleph_1$-free; see also [3], p. 183, Theorem 1.3.

7. Splitters of type III. If G is of type III then we recall from Section 3 that $G = \bigcup_{\alpha\in\omega_1} G_\alpha$, $G_0 = 0$ with (3.4)–(3.6) and $G_1 = \bigcup_{j\in\omega_1} G_{0j}$ and $\{G_{0j} : j \in \omega_1\}$ is an $\aleph_1$-filtration of pure submodules G_{0j} such that each $G_{0,j+1}/G_{0j}$ is minimal non-free. Here we will show:

THEOREM 7.1. *Modules of type III are not splitters.*

Proof. Let $G' = \bigcup_{\alpha\in\omega_1} G'_\alpha$ be an isomorphic copy of G taking G_α to G'_α, and choose a sequence of elements $z_\alpha \in G'_{\alpha+2}$ ($\alpha \in \omega_1$) such that $G'_{\alpha+1} \cap z_\alpha R = 0$ and $G'_{\alpha+2}/G'_{\alpha+1} \oplus z_\alpha R$ is $\aleph_1$-free. This is possible by (III).

By a basic observation from Section 1 it is enough to show that $\operatorname{Ext}(G_1, G) \neq 0$. Inductively we will construct a non-trivial element in $\operatorname{Ext}(G_1, G)$. We consider the following diagram:

$$\begin{array}{cccccccccc}
(0) & 0 & \longrightarrow & G' & \longrightarrow & H^0 & \xrightarrow{h_0} & G_{00} = 0 & \longrightarrow & 0\\
 & & & \| & & \downarrow & & \downarrow & & \\
(\beta) & 0 & \longrightarrow & G' & \longrightarrow & H^\beta & \xrightarrow{h_\beta} & G_{0\beta} & \longrightarrow & 0\\
 & & & \| & & \downarrow & & \downarrow & & \\
(\omega_1) & 0 & \longrightarrow & G' & \longrightarrow & H & \xrightarrow{h} & G_1 & \longrightarrow & 0
\end{array}$$

The first row is the trivial extension with $G' = H^0$ and $h_0 = 0$. Vertical maps and maps between G' and H's are inclusions. The sequences (β) are increasing continuous and suppose (β) is constructed for all $\beta < \alpha$. Then $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$ and

$$0 \to G' \to \bigcup_{\beta<\alpha} H^\beta \xrightarrow{h_\alpha} G_{0\alpha} \to 0$$

if α is a limit. Next we want to construct $(\alpha+1)$ from (α) and recall that $G' = G_{0,\alpha+1}/G_{0\alpha}$ is minimal non-free generated as in (2.3), (2.4). We can write

(7.1) $$G_{0,\alpha+1} = \langle G_{0\alpha}, B_{\alpha+1}, y_{\alpha m}R : m \in \omega\rangle, \qquad B_{\alpha+1} = \bigoplus_{i<n} x_{\alpha i}R$$

with relations

(7.2) $$y_{\alpha,m+1}p_m = y_{\alpha m} + \sum_{i<n} x_{\alpha i}k_{im} + g_{\alpha m} \quad (m \in \omega), \quad g_{\alpha m} \in G_{0\alpha}.$$

Then we define $h_{\alpha+1} : H^{\alpha+1} \to G_{0,\alpha+1} \to 0$ by Proposition 2.2. Hence

$$H^{\alpha+1} = \langle H^\alpha \oplus \overline{B}_{\alpha+1}, \overline{y}_{\alpha m}R : m \in \omega\rangle$$

has the relations

(7.3) $$\overline{y}_{\alpha,m+1}p_m = \overline{y}_{\alpha m} + \sum_{i<n} \overline{x}_{\alpha i}k_{im} + z_\alpha s_{\alpha m} + \overline{g}_{\alpha n} \quad (m \in \omega),$$

where the $s_{\alpha m} \in R$ will be specified later on, and $\overline{g}_{\alpha m} \in H^\alpha$.

Suppose that (ω_1) splits and consequently $\sigma : G_1 \to H$ is a splitting map for h. Then let

$$d_{\beta m} = y_{\beta m} - \overline{y}_{\beta m}, \quad e_{\beta i} = x_{\beta i}\sigma - \overline{x}_{\beta i}, \quad f_{\beta m} = g_{\beta m}\sigma - \overline{g}_{\beta m}.$$

From splitting we get again

$$d_{\beta m}, e_{\beta i}, f_{\beta m} \in G' \quad (\beta \in \omega_1,\ m \in \omega,\ i < n).$$

Using $G' = \bigcup_{\alpha \in \omega_1} G'_\alpha$ for $\beta \in \omega_1$ we find $\alpha \in \omega_1$ such that

$$d_{\beta m}, e_{\beta i}, f_{\beta m} \in G'_\alpha \quad \text{for all } i < n,\ m \in \omega.$$

Consider a map $\tau : \omega_1 \longrightarrow \omega_1$ taking any $\alpha \in \omega_1$ to

$$\tau(\alpha) = \min\{\beta \in \omega_1 : \beta \text{ a limit ordinal}, \\ d_{\alpha m}, e_{\alpha i}, f_{\alpha m} \in G'_\beta,\ \alpha \le \beta,\ m \in \omega,\ i < n\}$$

and note that $C = \{\alpha \in \omega_1 : \tau(\alpha) = \alpha\}$ is a cub in ω_1 and a subset of

$$E = \{\alpha \in \omega : \alpha \text{ a limit ordinal}, \\ d_{\beta m}, e_{\beta i}, f_{\beta m} \in G'_\alpha \text{ for all } \beta < \alpha,\ i < n,\ m \in \omega\}.$$

Hence E is a cub in ω_1. Next we apply σ to (7.2) and subtract (7.3). Hence we get a system of equations in $H^{\alpha+1}$:

$$d_{\alpha,m+1} p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i} k_{im} + f_{\alpha m} + z_\alpha s_{\alpha m} \quad (m \in \omega) \tag{7.4}$$

If $\alpha \in E$, then modulo H^α the equations (7.4) turn into

$$d_{\alpha,m+1} p_m \equiv d_{\alpha m} + \sum_{i<n} e_{\alpha i} k_{im} + z_\alpha s_{\alpha m} \quad (m \in \omega)$$

and modulo $z_\alpha R$ an earlier argument and $\aleph_1$-freeness of $G'/G_\alpha \oplus z_\alpha R$ show that the last equation requires solutions $d_{\alpha m}, e_{\alpha i} \in R$ for

$$d_{\alpha,m+1} p_m = d_{\alpha m} + \sum_{i<n} e_{\alpha i} k_{im} + s_{\alpha m} \quad (m \in \omega).$$

By a special choice of $s_{\alpha m}$'s in Lemma 2.1 this is now excluded, a contradiction. Hence (ω_1) has no splitting map and Theorem 7.1 follows. ∎

8. Appendix: splitters of type I under $2^{\aleph_0} < 2^{\aleph_1}$. In Section 5 we have seen a proof that CH implies that modules of type I are never splitters. A slight but somewhat technical modification of the proof shows that this result can be extended to WCH, that is, $2^{\aleph_0} < 2^{\aleph_1}$. Due to Section 5 this is not needed for the main result of this paper dealing with modules of cardinality $\aleph_1$ but it will be interesting when passing to cardinals $> \aleph_1$. We outline the main steps, their proofs are suggested by the proofs in Section 5.

THEOREM 8.1 (ZFC + $2^{\aleph_0} < 2^{\aleph_1}$). *Modules of type I are not splitters.*

STEP LEMMA 8.2. *Let $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ be a filtration of type I and let*

$$0 \to K \to H_\alpha \xrightarrow{h} G_\alpha \to 0$$

be a short exact sequence with some $z \in K$ such that $zR \cong R$ and $K, K/zR$ are $\aleph_1$-free. Then there are two commuting diagrams ($\varepsilon = 0, 1$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & H_\alpha & \xrightarrow{h} & G_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_\varepsilon & \longrightarrow & H_\varepsilon & \xrightarrow{h_\varepsilon} & G_{\alpha+1} & \longrightarrow & 0 \end{array}$$

with vertical maps inclusions such that any third row with H'_ε $\aleph_1$-free,

$$0 \to K'_\varepsilon \to H'_\varepsilon \xrightarrow{h'_\varepsilon} G_\beta \to 0$$

and any splitting map σ of h cannot have two splitting extensions σ_ε of h'_ε:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & H_\alpha & \underset{\sigma}{\overset{h}{\rightleftarrows}} & G_\alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_\varepsilon & \longrightarrow & H_\varepsilon & \xrightarrow{h_\varepsilon} & G_{\alpha+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K'_\varepsilon & \longrightarrow & H'_\varepsilon & \underset{\sigma_\varepsilon}{\overset{h'_\varepsilon}{\not\rightleftarrows}} & G_\beta & \longrightarrow & 0 \end{array}$$

Moreover $H_\varepsilon = \langle H_\alpha \bigoplus \overline{B}^\varepsilon_{\alpha+1}, y^\varepsilon_{\alpha m} : m \in \omega \rangle$ and $\overline{B}^\varepsilon_{\alpha+1}$ is mapped under h_ε mod G_α onto a free maximal R-submodule of $G_{\alpha+1}/G_\alpha$ (cf. (2.3)).

Definition 8.3. If an extension σ_ε as in (8.2) exists for some $\varepsilon \in \{0, 1\}$ we say that σ *splits over* $(H_\varepsilon, h_\varepsilon)$.

Proof of Lemma 8.2. Compare the proof of Step Lemma 5.4 but note that at the end we must take once more differences of the elements $d_{\alpha m}, e_{\alpha i}$ for $\varepsilon = 1$ and $\varepsilon = 0$ respectively. Then we are able to apply Lemma 2.1 to get a contradiction from splitting. ∎

We then apply the Step Lemma and weak diamond Φ_{ω_1} to construct a short exact sequence

$$0 \to H^0 \to H \xrightarrow{h} G \to 0.$$

Let $\gamma : G \to H^0$ be a fixed isomorphism. Later we will use consequences of Φ_{ω_1} to show that h does not split.

Proof of Theorem 8.1. If $H'_\alpha = G_\alpha\gamma$, then $H^0 = \bigcup_{\alpha\in\omega_1} H'_\alpha$ is an $\aleph_1$-filtration if $G = \bigcup_{\alpha\in\omega_1} G_\alpha$ is the given filtration of type I.

Let $T = {}^{\omega_1 >}2$ be the tree of all branches $\eta : \alpha \to 2$ for some $\alpha \in \omega_1$. We call $\alpha = l(\eta)$ the *length* of η. Branches are ordered as usual, hence $\eta < \eta'$ if $\eta' \restriction \mathrm{Dom}\, \eta = \eta$. The empty set $\emptyset$ is the bottom element of the tree. If $\eta \in T$, then we construct triples (H_η, H^η, h_η) of R-modules $H_\eta \subseteq H^\eta$ with H_η free of countable rank and a homomorphism $h_\eta : H^\eta \to G$ subject to various natural conditions:

(i) $H^{\emptyset} = H^0$, $H_{\emptyset} = 0$ and $h_{\emptyset} = 0$, hence $0 \to H^0 \to H_{\emptyset} \xrightarrow{h_{\emptyset}} G_0 = 0 \to 0$ is short exact.

(ii) If $\eta < \eta'$, then $(H_\eta, H^\eta, h_\eta) \subseteq (H_{\eta'}, H^{\eta'}, h_{\eta'})$, that is, $H_\eta \subseteq H_{\eta'}$, $H^\eta \subseteq H^{\eta'}$ and $h_\eta \subseteq h_{\eta'}$.

(iii) $H^0 = \ker h_\eta$ and $\operatorname{Im} h_\eta = G_{l(\eta)} = H_\eta h_\eta$.

(iv) If $\alpha \in \omega_1$ is a limit and $\eta \in {}^{\alpha}2$, then we take unions

$$(H_\eta, H^\eta, h_\eta) = \bigcup_{\beta<\alpha} (H_{\eta\restriction\beta}, H^{\eta\restriction\beta}, h_{\eta\restriction\beta}) = \Big(\bigcup_{\beta<\alpha} H_{\eta\restriction\beta}, \bigcup_{\beta<\alpha} H^{\eta\restriction\beta}, \bigcup_{\beta<\alpha} h_{\eta\restriction\beta} \Big).$$

If $l(\eta) = \alpha$ we put further restrictions on those triples. In this case $G' = G_{\alpha+1}/G_\alpha$ is minimal non-free, and G' can be represented by (2.3), (2.4). There are elements $g_{\alpha m} \in G_\alpha$ and $x_{\alpha i}, y_{\alpha m} \in G_{\alpha+1}$ with

$$G_{\alpha+1} = \langle G_\alpha, B_{\alpha+1}, y_{\alpha m} R : m \in \omega \rangle, \qquad B_{\alpha+1} = \bigoplus_{i<n} x_{\alpha i}\, R$$

and relations

$$y_{\alpha,m+1} p_m = y_{\alpha m} + \sum_{i<n} x_{\alpha i} k_{im} + g_{\alpha m}.$$

We choose an isomorphic copy $\overline{B}_{\alpha+1} = \bigoplus_{i<n} \overline{x}_\alpha R$ of $B_{\alpha+1}$ and now continue defining the tree with triples.

If $\varepsilon \in \{0,1\}$, then we require more from $(H_{\eta^\wedge\langle\varepsilon\rangle}, H^{\eta^\wedge\langle\varepsilon\rangle}, h_{\eta^\wedge\langle\varepsilon\rangle})$:

(S i) $H_\eta \oplus \overline{B}_{\alpha+1} \subseteq H_{\eta^\wedge\langle\varepsilon\rangle}$.

(S ii) $H_{\eta'} + H'_\beta \subseteq_* H^{\eta^\wedge\langle\varepsilon\rangle}$ for all $\eta' \in {}^{\alpha\geq}2$, and $\beta \in \omega_1$.

(S iii) $H'_\alpha \subseteq H_\eta \cap H^0 \subseteq H_{\alpha'}$ for some $\alpha' \in [\alpha, \omega_1)$.

Note that $H_\eta \cap H^0 = \ker(h_\eta \restriction H_\eta)$. The crucial condition is

(S iv) Suppose $\sigma : G_\alpha \to H_\eta$ is a homomorphism extending to $\sigma_\varepsilon : G_{\alpha+1} \to H^{\eta^\wedge\langle\varepsilon\rangle}$. Then not both of them can be splitting maps over $(H_{\eta^\wedge\langle\varepsilon\rangle}, h_{\eta^\wedge\langle\varepsilon\rangle})$ for $\varepsilon = 0, 1$.

Before we begin with the inductive construction, we observe from (iii) that $G_{\alpha+1}/G_\alpha \cong H_{\eta^\wedge\langle\varepsilon\rangle}/H_\eta \cong H^{\eta^\wedge\langle\varepsilon\rangle}/H^\eta$ for $\operatorname{Dom}\eta = \alpha$.

If $\eta \in {}^{\omega_1}2$ and $H = H(\eta) = \bigcup_{\alpha<\omega_1} H^{\eta\restriction\alpha}$, then $H' = \bigcup_{\alpha<\omega_1} H_{\eta\restriction\alpha} \subseteq H$ from (ii). (S iii) ensures $H^0 \subseteq H'$ and from (iii) we get $H/H_0 = H'/H_0$, hence $H' = H$. This will show that

$$(\eta 1) \qquad H(\eta) = \bigcup_{\alpha<\omega_1} H_{\eta\restriction\alpha} = \bigcup_{\alpha<\omega_1} H^{\eta\restriction\alpha} \quad (\eta \in {}^{\omega_1}2).$$

Similarly $h(\eta) = \bigcup_{\alpha<\omega_1} h_{\eta\restriction\alpha}$ is a well defined homomorphism $h(\eta) : H(\eta) \to G$ by (ii), it is onto with kernel H^0 by (iii), hence

$$(\eta 2) \qquad 0 \to H^0 \to H(\eta) \xrightarrow{h(\eta)} G \to 0 \quad (\eta \in {}^{\omega_1}2).$$

Condition ($\eta 1$) provides an $\aleph_1$-filtration used to apply weak diamond Φ for showing that ($\eta 2$) does not split for some η.

Next we will show that the tree with triples exists. This will follow by induction along the length α branches $\eta \in {}^{\alpha}2$. The case $\alpha = 0$ is (i) and already established. Suppose the construction is completed for all $\beta < \alpha$, and $\alpha < \omega_1$ is a limit ordinal. For $\eta \in {}^{\alpha}2$ we define (H_η, H^η, h_η) as in (iv). It is easy to verify that all conditions hold, in particular (S iii), because we take only countable unions. We come to the inductive step constructing $(H_{\eta^\wedge\langle\varepsilon\rangle}, H^{\eta^\wedge\langle\varepsilon\rangle}, h_{\eta^\wedge\langle\varepsilon\rangle})$ from (H_η, H^η, h_η) for $\alpha = \operatorname{Dom}\eta$.

First we apply the Step Lemma for $K = H^\eta \cap H^0 = \ker h_\eta$, $H_\alpha = H^\eta$, $H_\varepsilon = H^{\eta^\wedge\langle\varepsilon\rangle}$, $h = h_\eta$, $h_\varepsilon = h_{\eta^\wedge\langle\varepsilon\rangle}$ and note that the needed element z exists because $H^\eta \cap H^0 \subseteq H'_\alpha$ is free. We must still define $H_{\eta^\wedge\langle\varepsilon\rangle} \supseteq H_\eta$ carefully satisfying (ii), (S i)–(S iii) and the last equality in (iii): Write again h_ε for $h_{\eta^\wedge\langle\varepsilon\rangle}$ and H^ε for $H^{\eta^\wedge\langle\varepsilon\rangle}$. We know that $\operatorname{Im} h_\varepsilon = G_{\alpha+1}$ is a countable submodule of the $\aleph_1$-free module G, hence free, and h_ε must split. There is a splitting map $\varphi_\varepsilon : G_{\alpha+1} \to H^\varepsilon$ such that $\varphi_\varepsilon h_\varepsilon = \mathrm{id}_{G_{\alpha+1}}$, hence

$$H^\varepsilon = H^0 \oplus (G_{\alpha+1}\varphi_\varepsilon)$$

from the first part of (iii). Let $\pi_0^\varepsilon : H^\varepsilon \to H^0$ and $\pi_1^\varepsilon : H^\varepsilon \to H^0$ be the canonical projections, hence $\pi_0^\varepsilon + \pi_1^\varepsilon = \mathrm{id}_{H^\varepsilon}$.

Choose $\beta = (\alpha+1)' < \omega_1$ large enough such that $\overline{B}_{\alpha+1}\pi_0 \cup H_\eta\pi_0 \subseteq H'_\beta$, where $\overline{B}_{\alpha+1}$ is taken from the Step Lemma. We can choose β because $\overline{B}_{\alpha+1}$ and H_η are countable. Put

$$H_{\eta^\wedge\langle\varepsilon\rangle} = H'_\beta \oplus (G_{\alpha+1}\varphi_\varepsilon),$$

hence by the known half of (iii),

$$G_{\alpha+1} = H^\varepsilon h_\varepsilon \supseteq H_{\eta^\wedge\langle\varepsilon\rangle} h_\varepsilon \supseteq G_{\alpha+1}\varphi_\varepsilon h_\varepsilon = G_{\alpha+1}$$

and the other half of (iii) follows.

If $\gamma \leq \beta$, then $H_{\eta^\wedge\langle\varepsilon\rangle} + H'_\gamma = H_{\eta^\wedge\langle\varepsilon\rangle}$ and if $\gamma \geq \beta$, then $H_{\eta^\wedge\langle\varepsilon\rangle} + H'_\gamma = H'_\gamma \oplus (G_{\alpha+1}\varphi_\varepsilon)$ with quotient $H^{\eta^\wedge\langle\varepsilon\rangle}/H'_\gamma \oplus (G_{\alpha+1}\varphi_\varepsilon) \cong H^0/H'_\gamma$ which shows (S ii). Trivially $H_\eta\pi_1 \subseteq (G_{\alpha+1}\varphi_\varepsilon)$ and $H_\eta\pi_0 \subseteq H'_\beta$ by the choice of β, hence

$$H_\eta = H_\eta \,\mathrm{id}_{H^\varepsilon} \subseteq H_\eta\pi_0 + H_\eta\pi_1 \subseteq H_{\eta^\wedge\langle\varepsilon\rangle}$$

and (ii) holds. Similarly $\overline{B}_{\alpha+1} \subseteq H_{\eta^\wedge\langle\varepsilon\rangle}$ and (S i) is shown. From $H'_\beta \subseteq H^0$ and the modular law we have

$$H_{\eta^\wedge\langle\varepsilon\rangle} \cap H^0 = H'_\beta \oplus (G_{\alpha+1}\varphi_\varepsilon \cap H^0) = H'_\beta$$

and (S iii) holds.

The construction of the tree with triples is complete. We are ready to use the weak diamond $\Phi_{\aleph_1}(S)$ to show that G is not a splitter.

We will use $\Phi_{\omega_1}(S)$ as stated in Eklof–Mekler [3, p. 143, Lemma 1.7] and note that $G = \bigcup_{\alpha<\omega_1} G_\alpha$, $H(\eta) = \bigcup_{\alpha<\omega_1} H_{\eta\restriction\alpha}$ are $\aleph_1$-filtrations. $\Phi_{\omega_1}(S)$

must tell us which $\eta \in {}^{\omega_1}2$ we should pick. We define a partition P so that for $\alpha \in S$ a homomorphism $\sigma : G_\alpha \to H_\eta$ ($\eta \in {}^{\alpha}2$) has value $P_\alpha(\sigma) = 0$ if and only if σ does *not* split over $(H_{\eta^\wedge 0}, h_{\eta^\wedge 0})$. By the Step Lemma built into the construction, we observe that

(a) if $P(\sigma) = 1$, then σ cannot split over $(H_{\eta^\wedge 1}, h_{\eta^\wedge 1})$.

The prediction principle gives a branch $\eta \in {}^{\omega_1}2$ with the Φ-property

(b) If $\sigma : G \to H$ is any map, then $S' = \{\alpha \in S : P_\alpha(\sigma\restriction G_\alpha) = \eta(\alpha)\}$ is stationary in ω_1.

We pick that branch and build $H = H(\eta)$ and $h = h(\eta)$ suitably, hence $0 \to H^0 \to H \xrightarrow{h} G \to 0$ is short exact. After the branch is fixed we let $H^{\eta\restriction\alpha} = H^\alpha$, $h\restriction H^\alpha = h_\alpha$ and $H_{\eta\restriction\alpha} = H_\alpha$. Now we claim that the last sequence does not split. Suppose to the contrary that $\sigma : G \to H$ is a splitting map, hence $\sigma h = \mathrm{id}_G$. Notice that the set

$$C = \{\alpha < \omega_1 : H_\alpha \cap H^0 = H'_\alpha,\ G_\alpha\sigma \subseteq H_\alpha\}$$

is a cub. Since $S' \subseteq \omega_1$ is stationary, we find an $\alpha \in S' \cap C$ and also let $\sigma\restriction G_\alpha = \sigma$, hence

(c) $\sigma : G_\alpha \to H_\alpha \subset H^\alpha$, $P_\alpha(\sigma) = \eta(\alpha)$ and σ is a splitting map of $h_\alpha : H^\alpha \to G_\alpha$.

We also find some $\alpha < \beta \in C$. The difficulty is that $G_{\alpha+1}\sigma \subseteq H_{\alpha+1}$ does not follow, as in the case where S is not costationary. Hence we need the stronger Step Lemma (as usual).

If $\eta(\alpha) = P(\sigma) = 0$, then $(H_{\eta^\wedge 0}, h_{\eta^\wedge 0})$ is part of the construction of

$$0 \to H^0 \to H_{\alpha+1} \to G_{\alpha+1} \to 0$$

and σ does not split over $(H_{\eta^\wedge 0}, h_{\eta^\wedge 0})$, but σ is a global splitting map, hence σ splits at β over $(H_{\eta^\wedge 0}, h_{\eta^\wedge 0})$, a contradiction.

Necessarily $\eta(\alpha) = P(\sigma) = 1$ and by (a), σ does not split over $(H_{\eta^\wedge 1}, h_{\eta^\wedge 1})$, but this time $(H_{\eta^\wedge 1}, h_{\eta^\wedge 1})$ was used in the construction of $H \xrightarrow{h} G$ and a contradiction follows. This shows that σ is not a splitting map, and G is not a splitter. ∎

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