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ON A GAP SERIES OF MARK KAC

BY

KATUSI FUKUYAMA (KOBE)

Abstract. Mark Kac gave an example of a function f on the unit interval such that f cannot be written as $f(t) = g(2t) - g(t)$ with an integrable function g, but the limiting variance of $n^{-1/2}\sum_{k=0}^{n-1} f(2^k t)$ vanishes. It is proved that there is no measurable g such that $f(t) = g(2t) - g(t)$. It is also proved that there is a non-measurable g which satisfies this equality.

1. Introduction. Let us recall the following result of Kac [3], which yields the central limit theorem for dyadic transformations.

THEOREM A. *Let f be a real-valued function with period 1 satisfying*

$$\int_0^1 f(t)\,dt = 0 \quad \text{and} \quad \int_0^1 f^2(t)\,dt = 1. \tag{1.1}$$

(1) *If f is of bounded variation or α-Hölder continuous for some $\alpha > 0$, then*

$$m\Big\{t \in [0,1] \;\Big|\; \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} f(2^k t) \le x\Big\} \to \Phi_{\sigma^2}(x), \tag{1.2}$$

where m denotes the Lebesgue measure and Φ_{σ^2} denotes the distribution function of the normal distribution with mean 0 and variance σ^2, i.e. $\Phi_{\sigma^2}(x) = \int_{-\infty}^{x} e^{-u^2/(2\sigma^2)}\,du/\sqrt{2\pi\sigma^2}$. Here, the limiting variance σ^2 is given by

$$\sigma^2 = 1 + 2\sum_{k=1}^{\infty}\int_0^1 f(t)f(2^k t)\,dt < \infty. \tag{1.3}$$

(2) *If f is of bounded variation or α-Hölder continuous for some $\alpha > 1/2$, then $\sigma^2 = 0$ if and only if f is of the form*

$$f(t) = g(2t) - g(t) \qquad \text{a.e.} \tag{1.4}$$

for some g which has period 1 and is square integrable on $[0,1]$.

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Earlier, Fortet [1] announced this result, but the proof was not complete. Kac succeeded in giving a rigorous proof, but he failed to prove part (2) for all $\alpha > 0$. Instead of completing the proof, he gave the example below to show that part (2) does not hold without assuming any condition on f.

EXAMPLE B. Put $c_1 = 1$ and $c_j = 1/\sqrt{j} - 1/\sqrt{j-1}$ for $j \ge 2$. Then the function $f(t) = \sum_{j=1}^{\infty} c_j \cos 2^j \pi t$ satisfies (1.2) with $\sigma^2 = 0$, but there is no integrable g satisfying (1.4).

Having given the above example, Kac [3; p. 43] stated: "The question whether the representation (1.4) can be achieved in this case by means of a g which is not integrable remains open".

In this paper, we give an answer to this question by showing the following theorem, which implies that there is no measurable g satisfying (1.4) for the function of Example B.

THEOREM 1. *Suppose that the Fourier coefficients $\widehat{f}(n)$ of f are absolutely summable in n and that $\widehat{f}(n) = 0$ if $n \ne \pm 2^k$ $(k = 0, 1, \ldots)$. If there is no square integrable g satisfying* (1.4), *then there is no measurable g satisfying* (1.4).

On the other hand, for any given function f, it is always possible to construct g satisfying (1.4), by using the Axiom of Choice. Of course this g is not measurable in our case.

2. Proof of Theorem 1. First we prove a lemma and a proposition. Set $S_n(t) = \sum_{k=0}^{n-1} f(2^k t)$ and $\|h\|_2 = \left(\int_0^1 |h(t)|^2\, dt\right)^{1/2}$.

LEMMA 1. *Let f be a square integrable function. Then there exists a square integrable g satisfying* (1.4) *if and only if*

$$\liminf_{n\to\infty} \|S_n\|_2 < \infty. \tag{2.1}$$

Proof. If we assume (1.4), then (2.1) is trivial. We prove the converse. By (2.1) we can take a sequence $\{n_j\}$ of integers such that $\sup_{j\in\mathbb{N}} \|S_{n_j}\|_2 < \infty$. Let g be the weak limit of $-S_{n_j}$ as $j \to \infty$. We see that $g(2t) - g(t)$ is the weak limit of $f(t) - f(2^{n_j} t)$ as $j \to \infty$. By the Riemann–Lebesgue lemma, $f(2^{n_j} t)$ converges weakly to 0 as $j \to \infty$. Since the weak limit is unique, we have $f(t) = g(2t) - g(t)$. ■

The following proposition plays the key role in the proof of the theorem.

PROPOSITION 1. *Assume the same conditions on f as in Theorem* 1. *If there is no square integrable g satisfying* (1.4), *then*

$$\|S_n\|_2 \to \infty \quad \textit{and} \quad m\{t \in [0,1] \mid S_n(t)/\|S_n\|_2 \le x\} \to \Phi_1(x). \tag{2.2}$$

P r o o f. Since there is no square integrable g satisfying (1.4), the first part of (2.2) follows from Lemma 1. The second part follows from the following theorem by Salem–Zygmund [4].

THEOREM C. *Suppose that a sequence* $\{\nu_j\}$ *of positive integers satisfies the Hadamard gap condition*:

$$\nu_{j+1}/\nu_j > q > 1 \quad \text{for all } j \in \mathbb{N},$$

and that arrays $\{a_{n,j}\}_{j \le j_n, n\in\mathbb{N}}$ *and* $\{b_{n,j}\}_{j \le j_n, n\in\mathbb{N}}$ *of real numbers satisfy*

$$A_n = \left(\frac{1}{2}\sum_{j=1}^{j_n}(a_{n,j}^2 + b_{n,j}^2)\right)^{1/2} \to \infty \quad \text{and} \quad \max_{j\le j_n}(|a_{n,j}|, |b_{n,j}|) = o(A_n).$$

Then

$$m\left\{t \in [0,1] \;\middle|\; \frac{1}{A_n}\sum_{j=1}^{j_n}(a_{n,j}\cos 2\pi\nu_j t + b_{n,j}\sin 2\pi\nu_j t) \le x\right\} \to \Phi_1(x).$$

Let $a_{n,j}$ and $-b_{n,j}$ be the real and imaginary parts of $2(\widehat{f}((j-n+1)\vee 0) + \ldots + \widehat{f}(j))$ respectively. It is clear that

$$S_n(t) = \sum_{j=0}^{\infty}(a_{n,j}\cos 2\pi 2^j t + b_{n,j}\sin 2\pi 2^j t)$$

and $\|S_n\|_2 = \big(\frac{1}{2}\sum_{j=0}^{\infty}(a_{n,j}^2 + b_{n,j}^2)\big)^{1/2} \to \infty$. Clearly, $|a_{n,j}|$ and $|b_{n,j}|$ are bounded by $\sum|\widehat{f}(n)| < \infty$. Take $\{j_n\}$ satisfying $\sum_{j=0}^{j_n}(a_{n,j}^2 + b_{n,j}^2)/\|S_n\|_2^2 \to 1$, and divide S_n into two parts:

$$S_n(t) = \Big(\sum_{j\le j_n} + \sum_{j>j_n}\Big)(a_{n,j}\cos 2\pi 2^j t + b_{n,j}\sin 2\pi 2^j t).$$

If we normalize by dividing by $\|S_n\|_2$, thanks to Theorem C, the first part converges in law to the normal distribution. The second part converges to 0 in L^2-sense. Combining these, we have the conclusion. ■

Proof of Theorem 1. By Proposition 1, we have $\|S_n\|_2 \to \infty$. Suppose that f is represented by a measurable g in the form (1.4). Then $S_n(t) = g(2^n t) - g(t)$ and therefore, for $\varepsilon > 0$, we have

$$\begin{aligned} m\{|S_n|/\|S_n\|_2 > \varepsilon\} &\le m\{|g(2^n t)| > \varepsilon\|S_n\|_2/2\} + m\{|g(t)| > \varepsilon\|S_n\|_2/2\} \\ &= 2m\{|g(t)| > \varepsilon\|S_n\|_2/2\} \to 0, \end{aligned}$$

which contradicts the second formula of (2.2). ■

3. Construction of g. Let us first introduce an equivalence relation $\sim$ on $[0,1)$ by $s \sim t$ if and only if there exist $n, m \ge 0$ such that $2^n s \equiv 2^m t \pmod 1$. It is clear that each equivalence class E satisfies $E \subset \mathbb{Q}$ or $E \subset \mathbb{Q}^{\mathrm{c}}$.

If we regard each element of E as a vertex, and if we consider that we have an edge connecting t and s if $2t \equiv s \pmod 1$, then E has the structure of a graph. Since $t \notin \mathbb{Q}$ implies $2^n t \not\equiv t \pmod 1$, if $E \subset \mathbb{Q}^c$ then E has no cycle and is a binary graph.

Now we are in a position to construct g. Take a representative $t_0 \in E$ and put $g(t_0)$ arbitrary. Set

$$g(t) = \begin{cases} g(t_0) + S_n(t_0) & \text{if } t = 2^n t_0 \pmod 1, \\ g(t_0) - S_n(t) & \text{if } 2^n t = t_0 \pmod 1, \end{cases}$$

where $n \in \mathbb{N}$. Since E has no cycle, the function g is well defined on E and it satisfies $f(t) = g(2t) - g(t)$ for any $t \in E$. Thus we can define g such that $f(t) = g(2t) - g(t)$ for any $t \in \mathbb{Q}^c$. If we define $g(t) = 0$ for $t \in \mathbb{Q}$, we have g satisfying (1.4).

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Department of Mathematics
Kobe University
Rokko, Kobe, 657-8501 Japan
E-mail: fukuyama@math.kobe-u.ac.jp

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