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CURVATURE HOMOGENEITY OF AFFINE CONNECTIONS ON TWO-DIMENSIONAL MANIFOLDS

BY

OLDŘICH KOWALSKI (PRAHA), BARBARA OPOZDA (KRAKÓW) AND ZDENĚK VLÁŠEK (PRAHA)

Abstract. Curvature homogeneity of (torsion-free) affine connections on manifolds is an adaptation of a concept introduced by I. M. Singer. We analyze completely the relationship between curvature homogeneity of higher order and local homogeneity on two-dimensional manifolds.

1. Introduction. The theoretical foundations of this topic have been given by the second author in [4] and [5]. In this section we present the basic definition, some motivation and the main result.

DEFINITION 1.1. A smooth connection ∇ on a smooth manifold $\mathcal{M}$ is said to be *curvature homogeneous up to order* r if, for every $p, q \in \mathcal{M}$, there exists a linear isomorphism $F : T_p\mathcal{M} \to T_q\mathcal{M}$ such that $F^*(\nabla^k R)_q = (\nabla^k R)_p$ for all $k = 0, 1, \ldots, r$. Here R denotes the curvature tensor of ∇.

In fact, this definition originated in the paper by I. M. Singer [6] for the Riemannian situation. In the Riemannian case, ∇ is the Levi-Civita connection and Definition 1.1 must be completed by the assumption that F always preserves the scalar products.

The concept of curvature homogeneity on Riemannian (and also pseudo-Riemannian) manifolds has been studied in many papers. There are a lot of examples of curvature homogeneous Riemannian manifolds of order 0 which are not locally homogeneous. The 2-dimensional case is trivial (curvature homogeneity implies constant curvature) and the 3-dimensional case has been completely classified from the local point of view. For dimensions three and four, curvature homogeneity up to order 1 implies local homogeneity. (For dimension larger than four the problem remains open.) See, in particular,

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Chapter 12 of [1] for comprehensive information. Recently P. Bueken and L.Vanhecke [2] found an example of a 3-dimensional Lorentzian manifold which is curvature homogeneous up to order 1 but not locally homogeneous.

In contrast to the Riemannian situation, the affine case produces already in dimension *two* a rich theory. We are going to prove the following

MAIN THEOREM. *Let* ∇ *be a torsion-free analytic connection on an analytic two-dimensional manifold* $\mathcal{M}$. *If the Ricci tensor of* ∇ *is skew-symmetric*, *then the curvature homogeneity up to order* 3 *implies local homogeneity*, *and this bound cannot be improved. If the Ricci tensor of* ∇ *has nontrivial symmetrization*, *then the curvature homogeneity up to order* 2 *implies local homogeneity*, *and this bound cannot be improved.*

2. General results and formulas. In the following, all manifolds and connections are smooth, that is, of class C^∞, if not stated otherwise. We limit ourselves to the *torsion-free* connections. We shall start with the results which are valid in every dimension.

Let ∇ be a connection on $\mathcal{M}$ and let $p \in \mathcal{M}$. The Lie algebra of all endomorphisms of $T_p\mathcal{M}$ will be denoted by $\mathfrak{gl}(T_p\mathcal{M})$. Let $\mathfrak{g}(p;s)$ be the Lie subalgebra of $\mathfrak{gl}(T_p\mathcal{M})$ defined by

$$\mathfrak{g}(p;s) = \{A \in \mathfrak{gl}(T_p\mathcal{M}) \mid A\cdot R_p = A\cdot(\nabla R)_p = \ldots = A\cdot(\nabla^s R)_p = 0\}, \tag{2.1}$$

where A acts as a derivation on the tensor algebra of $T_p\mathcal{M}$. (See [6] for the original definition in the Riemannian case.)

We call the sequence $\mathfrak{g}(p;s)$ $(s = 0,1,2,\ldots)$ the *curvature sequence*. We say that the curvature sequence *stabilizes at level* $k \geq 0$ if $\mathfrak{g}(p;k+s) = \mathfrak{g}(p;k)$ for every $s > 0$.

We have

THEOREM 2.1. *Let* ∇ *be an analytic connection on an analytic manifold* $\mathcal{M}$. *If* ∇ *is curvature homogeneous up to order* $m \geq 1$, *and* $\mathfrak{g}(p;m-1) = (0)$ *for some* $p \in \mathcal{M}$, *then* ∇ *is locally homogeneous.*

Proof. Because the curvature sequence $\mathfrak{g}(p;s)$ stabilizes at level $m-1$ for *all* $p \in \mathcal{M}$, the assertion follows as a special case of Theorem 1.1 in [5]. ∎

THEOREM 2.2. *Let* ∇ *be an analytic connection on an analytic manifold* $\mathcal{M}$. *Assume that* ∇ *is curvature homogeneous up to order* $m \geq 1$, *and* $\mathfrak{g}(p;m-1) = \mathfrak{g}(p;m)$ *for some* $p \in \mathcal{M}$. *If*, *moreover*, *the Lie algebra* $\mathfrak{g}(p;m-1)$ *is reductive in* $\mathfrak{gl}(T_p\mathcal{M})$, *then* ∇ *is locally homogeneous.*

Proof. It is an easy modification of Theorem 1.2 from [5]. ∎

Let us recall that a subalgebra $\mathfrak{h}$ is *reductive in a Lie algebra* $\mathfrak{g}$ if there is a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (where $\mathfrak{m}$ is a vector subspace) such that $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$.

From now on, let $\dim \mathcal{M} = 2$ and denote by Ric the Ricci tensor of ∇ on $\mathcal{M}$. Then the curvature tensor R is uniquely determined by Ric via the formula

$$R(X,Y)Z = \mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y, \tag{2.2}$$

where $X, Y, Z \in T_q\mathcal{M}$, $q \in \mathcal{M}$. Hence in Definition 1.1 one can replace $R, \nabla R, \ldots$ by $\mathrm{Ric}, \nabla \mathrm{Ric}, \ldots$ respectively, and the same is true for the definition of the curvature sequence (2.1). (See [5], p. 193, for more details.)

Choose a system (u,v) of local coordinates in a domain $\mathcal{U} \subset \mathcal{M}$ and denote by U, V the corresponding coordinate vector fields ∂_u, ∂_v. In $\mathcal{U}$, the connection ∇ is uniquely determined by six functions $A, \ldots, F$ given by the

$$\nabla_U U = AU + BV,\ \nabla_U V = CU + DV = \nabla_V U,\ \nabla_V V = EU + FV. \tag{2.3}$$

One can easily calculate

$$\begin{aligned}
\mathrm{Ric}(U,U) &= B_v - D_u + D(A-D) + B(F-C),\\
\mathrm{Ric}(U,V) &= D_v - F_u + CD - BE,\\
\mathrm{Ric}(V,U) &= C_u - A_v + CD - BE,\\
\mathrm{Ric}(V,V) &= E_u - C_v + E(A-D) + C(F-C).
\end{aligned} \tag{2.4}$$

Finally, the following fact will be useful:

PROPOSITION 2.3. *If ∇ is curvature homogeneous up to order 1 on $\mathcal{M}$ and $\mathrm{Ric}_p = 0$ or $(\nabla \mathrm{Ric})_p = 0$ at some $p \in \mathcal{M}$, then ∇ is locally homogeneous on $\mathcal{M}$.*

Proof. We easily see that ∇ is then either flat or locally symmetric. ∎

Because our main theorem requires curvature homogeneity up to order *at least* one, it is legitimate to make the following

CONVENTION. *We always assume that $\mathrm{Ric} \neq 0$ and $\nabla \mathrm{Ric} \neq 0$ on the whole $\mathcal{M}$ if not stated otherwise.*

3. The case of skew-symmetric Ricci tensor. Choose a coordinate neighborhood $\mathcal{U}$ of a basic point $p \in \mathcal{M}$ and any coordinate system (u,v) in $\mathcal{U}$. From the skew-symmetry of Ric and our convention we have

$$\mathrm{Ric}(U,U) = \mathrm{Ric}(V,V) = 0, \qquad \varrho = \mathrm{Ric}(U,V) = -\mathrm{Ric}(V,U),\ \varrho \neq 0. \tag{3.1}$$

This can be rewritten, by (2.4), in the form

$$\begin{aligned}
C_u &= A_v + BE - CD - \varrho,\\
D_u &= B_v + D(A-D) + B(F-C),\\
E_u &= C_v + E(D-A) + C(C-F),\\
F_u &= D_v + CD - BE - \varrho,
\end{aligned} \tag{3.2}$$

where

$$\varrho = D_v - F_u + CD - BE \neq 0.$$

For the first covariant derivatives of Ric we have (with the notation (2.3))

$$(3.3)\quad \begin{aligned} (\nabla_U \operatorname{Ric})(U,V) &= -(\nabla_U \operatorname{Ric})(V,U) = \varrho_u - (A+D)\varrho, \\ (\nabla_V \operatorname{Ric})(U,V) &= -(\nabla_V \operatorname{Ric})(V,U) \ = \varrho_v - (C+F)\varrho, \end{aligned}$$

$$(3.4)\quad (\nabla_X \operatorname{Ric})(U,U) = (\nabla_X \operatorname{Ric})(V,V) = 0 \quad \text{for } X = U, V.$$

Put now

$$(3.5)\quad r = \varrho(p) \neq 0$$

for the initial value of ϱ. Further, define

$$(3.6)\quad M = [\varrho_u/\varrho - (A+D)]r, \quad N = [\varrho_v/\varrho - (C+F)]r,$$

$$(3.7)\quad m = M(p), \quad n = N(p).$$

This notation gives, by (3.3) and (3.6),

$$(3.8)\quad (\nabla_U \operatorname{Ric})(U,V)(p) = m, \quad (\nabla_V \operatorname{Ric})(U,V)(p) = n.$$

Next, (3.3) can be rewritten in the form

$$(3.9)\quad (\nabla_U \operatorname{Ric})(U,V) \ = M(\varrho/r), \quad (\nabla_V \operatorname{Ric})(U,V) \ = N(\varrho/r).$$

We exclude the case $M = N = 0$ when ∇ is locally symmetric. Thus we assume in the sequel $N \neq 0$ in the given neighborhood and $n \neq 0$. (If $N = 0$, $M \neq 0$, we obtain the previous situation by interchanging u, v.)

From (3.6) we get

$$(3.10)\quad \varrho_u = \varrho((A+D) + M/r), \quad \varrho_v = \varrho((C+F) + N/r).$$

Next, set

$$(3.11)\quad H_{XY} = (\nabla^2_{XY} \operatorname{Ric})(U,V)$$

for $X, Y \in \{U, V\}$. Using (3.9) and (3.10) we easily obtain the following formulas for the tensor $\nabla^2 \operatorname{Ric}$:

$$(3.12)\quad \begin{aligned} H_{UU} &= \frac{\varrho}{r}\left[M_u + \frac{M^2}{r} - AM - BN\right], \\ H_{UV} &= \frac{\varrho}{r}\left[N_u + \frac{MN}{r} - CM - DN\right], \\ H_{VU} &= \frac{\varrho}{r}\left[M_v + \frac{MN}{r} - CM - DN\right], \\ H_{VV} &= \frac{\varrho}{r}\left[N_v + \frac{N^2}{r} - EM - FN\right], \end{aligned}$$

and

$$(3.13)\qquad \begin{aligned}(\nabla^2_{XY}\operatorname{Ric})(V,U) &= -(\nabla^2_{XY}\operatorname{Ric})(U,V),\\ (\nabla^2_{XY}\operatorname{Ric})(U,U) &= (\nabla^2_{XY}\operatorname{Ric})(V,V) = 0\end{aligned}$$

for $X, Y \in \{U, V\}$.

Let now P, Q, R, S, $PS-QR \neq 0$, be smooth functions on a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p and define a family $\Phi = \{\Phi_x : T_x\mathcal{M} \to T_p\mathcal{M} \mid x \in \mathcal{V}\}$ of linear isomorphisms by

$$(3.14)\qquad \Phi(U) = PU_p + QV_p, \quad \Phi(V) = SU_p + TV_p \quad (U_p, V_p \in T_p\mathcal{M}).$$

Set

$$(3.15)\qquad \mathcal{K} = PT - QS \neq 0.$$

We first see that the isomorphism field Φ preserves the Ricci tensor Ric if and only if

$$(3.16)\qquad \mathcal{K} = \varrho/r.$$

Further, Φ preserves *in addition* the tensor field $\nabla\operatorname{Ric}$ if and only if

$$(3.17)\qquad Pm + Qn = M, \quad Sm + Tn = N.$$

This follows at once from (3.3), (3.4), (3.8), (3.9) and (3.16). Hence we obtain

$$(3.18)\qquad Q = \frac{M - mP}{n}, \quad T = \frac{N - mS}{n}$$

and (3.15)–(3.17) imply

$$(3.19)\qquad P = \frac{n\varrho/r + MS}{N}.$$

We thus have

PROPOSITION 3.1. *The isomorphism field Φ preserves* Ric *and* $\nabla\operatorname{Ric}$ *if and only if S is arbitrary and P, Q, T are given by* (3.18) *and* (3.19) (*assuming $N \neq 0$*).

COROLLARY 3.2. *Every connected smooth affine* 2*-manifold* $(\mathcal{M}, \nabla)$ *with a skew-symmetric Ricci tensor is curvature homogeneous up to order* 1 *in a neighborhood of each point from an open dense subset.*

Proof. We call a point $p \in \mathcal{M}$ *regular* if either

(a) $(\mathcal{M}, \nabla)$ is locally symmetric around p, or
(b) $(\nabla\operatorname{Ric})_p \neq 0$ and $\operatorname{Ric}_p \neq 0$.

It is obvious that the subset of all regular points is open and dense. Hence the result follows easily. ∎

To express the second order Φ-invariance, set

$$i = H_{UU}(p), \quad j = H_{UV}(p), \quad k = H_{VV}(p). \tag{3.20}$$

From $(3.2)_{1,4}$ we get

$$C_u + F_u - A_v - D_v = -2\varrho. \tag{3.21}$$

Then the formulas (3.6) imply

$$N_u - M_v = 2r\varrho \tag{3.22}$$

and finally, from (3.12) we obtain

$$H_{UV} - H_{VU} = \frac{\varrho}{r}(N_u - M_v) = 2\varrho^2 = 2[\mathrm{Ric}(U,V)]^2. \tag{3.23}$$

At the initial point p we get

$$H_{VU}(p) = H_{UV}(p) - 2[\varrho(p)]^2 = j - 2r^2. \tag{3.24}$$

A routine calculation gives the following

PROPOSITION 3.3. *The isomorphism field Φ preserves the tensor fields* Ric, ∇ Ric *and* ∇^2 Ric *if and only if we have* (3.16), (3.17) *and*

$$\begin{aligned} iP^2 + (2j - 2r^2)PQ + kQ^2 &= (r/\varrho)H_{UU}, \\ iPS + jPT + (j - 2r^2)QS + kQT &= (r/\varrho)H_{UV}, \\ iS^2 + (2j - 2r^2)ST + kT^2 &= (r/\varrho)H_{VV}. \end{aligned} \tag{3.25}$$

Proof. According to (3.23) it is sufficient to express the Φ-invariance of ∇^2 Ric using only the Φ-invariance of H_{UU}, H_{UV} and H_{VV}. Here we use the notation (3.20). Finally, we divide both sides of each equation by $\mathcal{K} = \varrho/r$ to obtain (3.25). ∎

Recall now the formulas (3.3) and (3.4). For any $x \in \mathcal{M}$ consider the linear form $\tau_x : Z \mapsto (\nabla_Z \mathrm{Ric})(X,Y)$ where $X, Y \in T_x\mathcal{M}$ are arbitrary but such that $X \wedge Y \neq 0$. Then τ_x is defined up to a nonzero factor. Because $\nabla \mathrm{Ric} \neq 0, \tau_x$ has a one-dimensional kernel, which is independent of the choice of X and Y. $\operatorname{Ker}\tau$ is a well-defined 1-dimensional distribution on $\mathcal{M}$, which we denote by $\mathcal{D}$. Define a special local coordinate system (u, v) such that $U = \partial/\partial u$ belongs to $\mathcal{D}$ everywhere. We have

$$\nabla \mathrm{Ric}(U,U,V) = 0, \quad \nabla \mathrm{Ric}(V,U,V) \neq 0 \tag{3.26}$$

on a neighborhood $\mathcal{U}$ of p. According to (3.9) we get $M = 0$, $N \neq 0$, $m = 0$, $n \neq 0$. Then (3.12) simplifies to

$$\begin{aligned} H_{UU} &= -\frac{\varrho}{r}BN, \quad & H_{UV} &= \frac{\varrho}{r}(N_u - DN), \\ H_{VU} &= -\frac{\varrho}{r}DN, \quad & H_{VV} &= -\frac{\varrho}{r}\left(N_v + \frac{N^2}{r} - FN\right), \end{aligned} \tag{3.27}$$

and (3.18), (3,19) simplify to

$$Q = 0, \quad T = \frac{N}{n}, \quad P = \frac{n\varrho}{rN}. \tag{3.28}$$

Now, let us calculate the (joint) isotropy group of Ric, ∇ Ric, ∇^2 Ric at the basic point p. From (3.5), (3.7) and (3.28) we get

$$P(p) = T(p) = 1, \quad Q(p) = 0, \quad S(p) = s \quad \text{(arbitrary parameter)}. \tag{3.29}$$

If we express (3.25) explicitly at the point p, and then use (3.5), (3.20) and (3.29), we see that the last two equations of (3.25) reduce to

$$is = 0, \quad s(j - r^2) = 0. \tag{3.30}$$

We now have two possibilities:

A. $i \neq 0$ or $j - r^2 \neq 0$. Then $s = 0$ and the joint isotropy subgroup of Ric, ∇ Ric and ∇^2 Ric at p reduces to {Id}. This means that $\mathfrak{g}(p; 2) = (0)$. Now, if ∇ is analytic and curvature homogeneous up to order 3 then, according to Theorem 2.1, it is locally homogeneous.

B. $i = 0$ and $j - r^2 = 0$. Assume from now on that ∇ is curvature homogeneous up to order two. Then using $M = Q = 0$ we get from $(3.25)_1$

$$H_{UU} = 0. \tag{3.31}$$

Further, from $(3.25)_2$ we deduce $jPT = (r/\varrho)H_{UV}$ and from (3.28) we get $H_{UV} = \varrho^2 j/r^2 = \varrho^2$. Finally, from (3.23) we obtain

$$H_{UV} = -H_{VU} = \varrho^2. \tag{3.32}$$

From the first equation of (3.27) we get $B = 0$ and hence $\nabla_U U = AU$, which means that the distribution $\mathcal{D}$ is totally geodesic. After a suitable change of local coordinates, $\overline{u} = \Phi(u, v)$, $\overline{v} = v$, we find that $\overline{U} = \partial/\partial\overline{u}$ still belongs to the distribution $\mathcal{D}$ and $\nabla_{\overline{U}}\overline{U} = 0$. Hence we can assume $A = 0$ in the whole neighborhood.

Now, on a neighborhood $\mathcal{V}'$ of p the equations (3.2) simplify to

$$\begin{aligned} &C_u + CD = -\varrho, \quad D_u + D^2 = 0, \\ &E_u - ED = C_v + C(C - F), \quad F_u = D_v + CD - \varrho. \end{aligned} \tag{3.33}$$

We first show that $D \neq 0$ on $\mathcal{V}'$. Indeed, from (3.32) we get $H_{VU} \neq 0$ and the rest follows from the third formula of (3.27). Hence the second equation of (3.33) gives the general solution

$$D(u, v) = 1/(u + f(v)), \tag{3.34}$$

where $f(v)$ is an arbitrary function. Substituting $A = 0$ and $M = 0$ in the first equation of (3.6), we obtain $\varrho_u/\varrho = D$ and hence

$$\varrho(u, v) = \varphi(v)(u + f(v)), \tag{3.35}$$

where $\varphi(v)$ is an arbitrary function.

Now, let us introduce new local coordinates $\overline{u}, \overline{v}$ by $\overline{u} = u + f(v)$, $\overline{v} = \int \varphi(v)\, dv$. Then $\mathrm{Ric}(\overline{U}, \overline{V}) = (1/\varphi(v))\,\mathrm{Ric}(U, V)$ and $\overline{\varrho} = \overline{u}$. Hence we can assume $\varphi(v) = 1$ and $f(v) = 0$, without loss of generality. Next, from (3.27) and (3.32) we get $H_{VU} = -(\varrho/r)DN = -\varrho^2$ and hence $N = \varrho r/D$, i.e.,

$$N(u, v) = ru^2. \tag{3.36}$$

From the first equation of (3.33) we get, by an easy calculation,

$$C(u, v) = -\tfrac{1}{3}u^2 + \psi(v)/u, \tag{3.37}$$

where $\psi(v)$ is another arbitrary function.

Finally, from $(3.25)_3$ we obtain $kT^2 = (r/\varrho)H_{VV}$, and using (3.27), (3.28) we get

$$k\frac{N^2}{n^2} = N_v + \frac{N^2}{r} - FN. \tag{3.38}$$

Hence

$$F = \frac{N_v}{N} + \frac{\lambda}{r}N = \lambda u^2, \tag{3.39}$$

where $\lambda = 1 - kr/n^2$ is a constant.

Now we make substitutions in the last equation of (3.33) using (3.34) and (3.35) (with $\varphi = 1$, $f = 0$), (3.37) and (3.39) to get

$$2\lambda u = \psi(v)/u^2 - \tfrac{4}{3}u. \tag{3.40}$$

Hence $\lambda = -2/3$, $\psi(v) = 0$ and we can write

$$C = -\tfrac{1}{3}u^2, \qquad D = 1/u, \tag{3.41}$$

$$F = -\tfrac{2}{3}u^2. \tag{3.42}$$

Finally, we calculate E from the third equation of (3.33):

$$E = -\tfrac{1}{36}u^5 + ue(v), \tag{3.43}$$

where $e(v)$ is a new arbitrary function.

We now want to prove that, in case B, if ∇ is curvature homogeneous up to order 2, then it is locally homogeneous in a coordinate neighborhood of p. For this purpose we calculate the corresponding affine Killing vector fields X, which are characterized by the equation

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] = \nabla_{[X,Y]} Z \tag{3.44}$$

to be satisfied for arbitrary vector fields Y, Z (see [3]). It is sufficient to satisfy (3.44) for $(Y, Z) \in \{(U, U), (U, V), (V, U), (V, V)\}$. Moreover, we easily check from the basic identities for the torsion and the Lie brackets that the choice $(Y, Z) = (V, U)$ gives the same condition as $(Y, Z) = (U, V)$.

Express the vector field X in the coordinate form

$$X = a(u, v)U + b(u, v)V. \tag{3.45}$$

If we use the identity $\nabla_U U = 0$, we easily see that (3.44) reduces to six linear partial differential equations for the unknown functions a, b:

$$
\begin{aligned}
&a_{uu} + 2Cb_u = 0,\\
&b_{uu} + 2Db_u = 0,\\
&a_{uv} - Da_v + Cb_v + Eb_u + C_u a + C_v b = 0,\\
&b_{uv} + Da_u + (F - C)b_u + D_u a + D_v b = 0,\\
&a_{vv} + 2Ca_v + 2Eb_v - Ea_u - Fa_v + E_u a + E_v b = 0,\\
&b_{vv} + 2Da_v + 2Fb_v - Eb_u - Fb_v + F_u a + F_v b = 0.
\end{aligned}
\tag{3.46}
$$

Here C, D, E and F are functions defined by (3.41), (3.43) and (3.42) respectively. From $(3.46)_1$ and $(3.46)_2$ we get easily

$$a(u,v) = \tfrac{1}{3}p(v)u^2 + r(v)u + s(v), \tag{3.47}$$

$$b(u,v) = -p(v)/u + q(v), \tag{3.48}$$

where $p(v), q(v), r(v)$ and $s(v)$ are unknown functions of v to be determined. Next we substitute the explicit expressions for C, D, E, F and a, b in the remaining equations $(3.46)_{3\text{–}6}$. Then each equation decomposes into a number of "coefficient equations" (involving only functions of v) which correspond to distinct powers of u. Here the highest power u^6 occurs only in two terms of $(3.46)_5$, with the total coefficient $-\frac{1}{36}p(v)$. Hence $p(v) = 0$ and we can simplify

$$a(u,v) = r(v)u + s(v), \qquad b(u,v) = q(v). \tag{3.49}$$

Then $(3.46)_3$ implies

$$r(v) = -\tfrac{1}{2}q'(v), \qquad s(v) = 0. \tag{3.50}$$

Now, substituting (3.50) (and the corresponding expressions for the derivatives of $r(v)$ and $s(v)$) in the whole system (3.46), we see that all equations are identically satisfied except for $(3.46)_5$ which gives a new (and final) condition

$$q'''(v) - 4e(v)q'(v) - 2e'(v)q(v) = 0. \tag{3.51}$$

Here $e(v)$ is the arbitrary function from (3.43).

For every particular solution $q(v)$ of (3.51) we get an affine Killing vector field

$$X = q'(v)u\frac{\partial}{\partial u} - 2q(v)\frac{\partial}{\partial v}. \tag{3.52}$$

If q_1, q_2, q_3 are three linearly independent particular solutions of (3.51), then the corresponding Wronskian is nonzero everywhere and so the matrix (q_i, q_i') has rank two everywhere. Hence, at each point of the given coordinate neighborhood, at least two of the corresponding Killing vector fields X_i are linearly independent, and the local homogeneity follows.

REMARK. The Lie bracket of two Killing vector fields (3.52) is again a Killing vector field. This is equivalent to the following statement: if q_1 and q_2 are two particular solutions of (3.51), then so is $q_1 q_2' - q_1' q_2$. This is a special case of a well-known theorem on linear homogeneous ODEs of third order.

Now we can conclude:

PROPOSITION 3.4. *Let* ∇ *be an analytic connection with skew-symmetric Ricci tensor on an analytic two-dimensional manifold* $\mathcal{M}$. *If* ∇ *is curvature homogeneous up to order three*, *then it is locally homogeneous.*

In the last part of this section we show that this proposition cannot be improved. To this end we present an example of an affine connection of the required type which is curvature homogeneous up to order 2 but nowhere locally homogeneous.

Consider the plane $\mathbb{R}^2[u,v]$ with affine connection ∇ such that in (2.3) we have

$$A = B = D = 0, \qquad C = F = e^v u, \qquad E = \tfrac{1}{2} e^v u^2. \tag{3.53}$$

We first look for the affine Killing vector fields. We repeat the previous procedure. From $(3.46)_1$ and $(3.46)_2$ we calculate

$$a(u,v) = -\tfrac{1}{3} p(v) e^v u^3 + r(v) u + s(v), \qquad b(u,v) = p(v) u + q(v). \tag{3.54}$$

By a similar observation as before we see from $(3.46)_5$ that $p(v) = 0$. Hence

$$a(u,v) = r(v) u + s(v), \qquad b(u,v) = q(v). \tag{3.55}$$

Now, substituting the expressions for C, D, E, a, b into $(3.46)_3$ we get

$$q'(v) + q(v) + r(v) = 0, \qquad r'(v) + e^v s(v) = 0, \tag{3.56}$$

the equation $(3.46)_4$ is identically satisfied, and from $(3.46)_5$ we deduce

$$\begin{aligned} 2r'(v) + 2q'(v) + r(v) + q(v) &= 0, \\ r''(v) + e^v (s(v) + s'(v)) &= 0, \\ s''(v) &= 0. \end{aligned} \tag{3.57}$$

No new conditions come from $(3.46)_6$.

From the first equations of (3.56) and (3.57) we deduce at once

$$r(v) + q(v) = c e^{-v/2}, \qquad q'(v) = -c e^{-v/2}, \qquad r'(v) = \tfrac{1}{2} c e^{-v/2}, \tag{3.58}$$

where c is a constant. From the second equation of (3.56) we see that $s(v) = -\frac{1}{2} e^{-3v/2}$, which contradicts the condition $s''(v) = 0$ unless $c = 0$. Hence $s(v) = 0$ and $r(v) = -q(v) = \text{constant}$.

Our affine manifold admits just a 1-dimensional space of affine Killing vector fields generated by

$$X = u\frac{\partial}{\partial u} - \frac{\partial}{\partial v}. \tag{3.59}$$

Hence $(\mathbb{R}^2, \nabla)$ is not locally homogeneous.

Next, we inspect the curvature homogeneity. If we put $\varrho = -e^v$, then all equations (3.2) are satisfied and Ric is skew-symmetric. Take the origin $(0,0)$ for p. Then we easily calculate that

$$\begin{aligned} &r = -1, \quad M = 0, \quad m = 0, \quad N = 2e^v u - 1,\\ &n = -1, \quad T = 1 - 2e^v u, \quad PT = e^v, \quad Q = 0. \end{aligned} \tag{3.60}$$

Further, (3.27) implies

$$H_{UU} = 0, \quad i = 0, \quad H_{UV} = 2e^{2v}, \quad j = 2, \quad H_{VU} = 0. \tag{3.61}$$

We see that $(3.25)_{1\text{–}2}$ are satisfied. Next, H_{VV} is determined by the last formula of (3.27) and we easily see that $k = r = -1$. To satisfy $(3.25)_3$ we calculate

$$S = \frac{ue^v(3 - 2ue^v)}{2(1 - 2ue^v)}. \tag{3.62}$$

We know that our space is curvature homogeneous up to order 2 in any domain where S is correctly defined (see Proposition 3.3). We can choose for this domain the connected component of the origin in $\mathbb{R}^2$ cut out by the graph $u = \frac{1}{2}e^{-v}$. This gives the desired example.

4. The case where Sym(Ric) **has rank one.** Here we have, in a convenient system (u, v) of local coordinates,

$$\begin{aligned} &\varrho = \mathrm{Ric}(U,V) = -\mathrm{Ric}(V,U) \neq 0,\\ &\mathrm{Ric}(U,U) = 0, \quad \mu = \mathrm{Ric}(V,V) \neq 0. \end{aligned} \tag{4.1}$$

From (2.4) we get

$$\begin{aligned} B_v - D_u + D(A - D) + B(F - C) = 0,\\ C_u - A_v + D_v - F_u + 2DC - 2BE = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\varrho = D_v - F_u + CD - BE = A_v - C_u + BE - CD \neq 0,\\ &\mu = E_u - C_v + E(A - D) + C(F - C) \neq 0. \end{aligned} \tag{4.3}$$

Further, we have the following formulas for $\nabla\,\mathrm{Ric}$:

$$(4.4)\qquad \begin{aligned} &(\nabla_U \mathrm{Ric})(U,V) = \varrho_u - (A+D)\varrho - B\mu = M,\\ &(\nabla_U \mathrm{Ric})(V,U) = -M - 2B\mu,\\ &(\nabla_V \mathrm{Ric})(U,V) = \varrho_v - (C+F)\varrho - D\mu = \overline{M},\\ &(\nabla_V \mathrm{Ric})(V,U) = -\overline{M} - 2D\mu,\\ &(\nabla_X \mathrm{Ric})(U,U) = 0 \quad \text{for all } X,\\ &(\nabla_U \mathrm{Ric})(V,V) = \mu_u - 2D\mu = N,\\ &(\nabla_V \mathrm{Ric})(V,V) = \mu_v - 2F\mu = \overline{N}. \end{aligned}$$

Here we have introduced new auxiliary functions $M, \overline{M}, N, \overline{N}$.

A family Φ of linear isomorphisms given by (3.14) preserves Ric if and only if

$$(4.5)\qquad Q = 0, \quad \text{i.e.,} \quad \Phi(U) = PU_p, \quad \Phi(V) = SU_p + TV_p$$

and

$$(4.6)\qquad \mu = T^2\mu(p), \quad \varrho = PT\varrho(p), \quad PT \neq 0.$$

Further, Φ preserves $\nabla\,\mathrm{Ric}$ if and only if

$$(4.7)\qquad \begin{aligned} &M = P^2Tm, \quad \overline{M} = PT(Sm + T\overline{m}),\\ &N = PT^2n - 2PSTB(p)\mu(p),\\ &\overline{N} = T^3\overline{n} + ST^2(n - 2D(p)\mu(p)) - 2S^2TB(p)\mu(p) \end{aligned}$$

and

$$(4.8)\qquad B\mu = P^2TB(p)\mu(p), \quad D\mu = PT^2D(p)\mu(p),$$

where $M, \overline{M}, N$ and $\overline{N}$ are defined by (4.4) and

$$(4.9)\qquad m = M(p), \quad \overline{m} = \overline{M}(p), \quad n = N(p), \quad \overline{n} = \overline{N}(p).$$

From (4.6) and (4.7) we see that Φ_p preserves Ric_p and $(\nabla\,\mathrm{Ric})_p$ if and only if

$$(4.10)\qquad P(p) = T(p) = 1$$

and for $s = S(p)$ we have

$$(4.11)\qquad ms = 0, \quad B(p)s = 0, \quad (n - 2D(p)\mu(p))s = 0.$$

We have two cases:

A. One of the numbers $m, B(p)$ and $n-2D(p)\mu(p)$ is nonzero. Then $s=0$ and the joint isotropy subgroup of Ric and $\nabla\,\mathrm{Ric}$ at p reduces to $\{\mathrm{Id}\}$. This means that $\mathfrak{g}(p;1) = (0)$. Now, if ∇ is analytic and curvature homogeneous up to order 2 then, according to Theorem 2.1, it is locally homogeneous.

B. Suppose now that

$$m = 0, \quad B(p) = 0, \quad n - 2D(p)\mu(p) = 0 \tag{4.12}$$

and that ∇ is curvature homogeneous up to order 1. Then according to (4.7) and (4.8) we have, in a neighborhood $\mathcal{V}$ of p,

$$M = 0, \quad B = 0, \quad N - 2D\mu = 0. \tag{4.13}$$

In particular, the 1-dimensional distribution $\mathcal{D}$ determined by U is totally geodesic. Taking a new local coordinate system (u, v) as in case B of Section 3 we can assume that $A = 0$ in a neighborhood of p.

From $(4.2)_1$ and $(4.4)_1$ we get

$$D_u + D^2 = 0, \quad \varrho_u - D\varrho = 0. \tag{4.14}$$

Suppose first that $D \equiv 0$. Then $\varrho_u = 0$ and from (4.13) we have $N = 0$. Hence (4.4) gives $\mu_u = 0$ and both ϱ and μ depend on v only. From (4.6) we see that P and T depend on v only, and (4.7) shows that $\overline{M} = PT^2\overline{m}$ and $\overline{N} = T^3\overline{n}$ depend on v only. From $(4.4)_7$ we see that then F depends on v only, and $(4.3)_1$ shows that $\varrho = 0$, which is a contradiction. Hence $D \not\equiv 0$.

Then (4.14) implies

$$D(u, v) = 1/(u + f(v)), \quad \varrho(u, v) = \varphi(v)(u + f(v)), \tag{4.15}$$

where $f(v)$ and $\varphi(v)$ are arbitrary functions. The same coordinate transformation as in (3.35) guarantees that $\varphi(v) = 1$ and $f(v) = 0$. The second equation of (4.3) gives $\varrho = -C_u - CD$ and hence

$$C(u, v) = -\tfrac{1}{3}u^2 - \psi(v)/u, \tag{4.16}$$

where $\psi(v)$ is an arbitrary function. From $(4.2)_2$ we get by integration

$$F(u, v) = \psi(v)/u - \tfrac{2}{3}u^2 + \chi(v), \tag{4.17}$$

where $\chi(v)$ is a new arbitrary function.

From $(4.4)_6$ and (4.13) we obtain $\mu_u - 4D\mu = 0$ and hence

$$\mu = u^4\kappa, \quad N = 2u^3\kappa, \tag{4.18}$$

where κ is an arbitrary function of v, $\kappa \neq 0$. Next, from $(4.4)_3$ we get

$$\overline{M} = (1 - \kappa)u^3 - u\chi \tag{4.19}$$

and from (4.7) we infer $\overline{M} = (\overline{m}/n)N$. Substituting $\overline{M}$ and N from $(4.18)_2$ and (4.19) we get $\chi = 0$ and $\kappa = \text{const}$.

From $(4.4)_7$ we obtain

$$\overline{N} = \tfrac{4}{3}u^6 - 2\psi u^3. \tag{4.20}$$

From (4.7) we get, by (4.12), $\overline{N} = T^3\overline{n}$ and from (4.6) it follows that $\overline{N}^2 = \mu^3(\overline{n}^2/\mu(p)^3) = (\overline{n}^2/\mu(p)^3)\kappa^3u^{12}$. Hence $\psi = 0$. We conclude that

$$C = -\tfrac{1}{3}u^2, \quad D = 1/u, \quad F = -\tfrac{2}{3}u^2 \tag{4.21}$$

and from $(4.3)_2$ we calculate

$$E = -\tfrac{2}{9}u^5 + ue(v), \tag{4.22}$$

where $e(v)$ is an arbitrary function.

We see that we have the same formulas for D, C, F, E as in Section 3 (cf. (3.41)–(3.43)) except the first coefficient in (4.22).

The calculation of affine Killing vector fields is exactly the same as in the previous section (see (3.51) and (3.52)) and hence ∇ is locally homogeneous in the given neighborhood of p.

We conclude:

PROPOSITION 4.1. *Let both $\mathcal{M}$ and ∇ be analytic. If the symmetric part of* Ric *has rank one, then curvature homogeneity up to order* 2 *implies local homogeneity.*

Finally, we give one example of this type which is curvature homogeneous up to order 1 but not locally homogeneous. For this purpose we put, on $\mathbb{R}^2[u,v]$,

$$A = B = D = 0, \qquad C = F = e^v u, \qquad E = \tfrac{1}{2}e^v u^2 + h(u,v), \tag{4.23}$$

where

$$h(u,v) = -\tfrac{1}{6}e^{-v}(1-2ue^v)^3. \tag{4.24}$$

(Compare with (3.53).) Then (4.2) is satisfied identically and choosing $p = (0,0)$ as a basic point in $\mathbb{R}^2[u,v]$, we get, from (4.3) and (4.4),

$$\begin{aligned}
&\varrho = -C_u = -e^v, \qquad \varrho(p) = -1,\\
&M = 0, \qquad m = 0,\\
&\mu = h'_u = (1-2ue^v)^2, \qquad \mu(p) = 1,\\
&N = h''_{uu} = -4e^v(1-2ue^v), \qquad n = -4,\\
&\overline{M} = -e^v(1-2ue^v), \qquad \overline{m} = -1,\\
&\overline{N} = -2ue^v(1-2ue^v)(3-2ue^v), \qquad \overline{m} = 0.
\end{aligned} \tag{4.25}$$

Here P, Q, T and S are given by the same formulas as in (3.60) and (3.62). We easily check that all formulas (4.6) and (4.7) are satisfied and the connection ∇ is curvature homogeneous up to order 1 on the same domain of $\mathbb{R}^2[u,v]$ as described at the end of Section 3.

The calculation of the Killing vector fields is an easy modification of the procedure used in the example of Section 3. We can again express $a(u,v)$ and $b(u,v)$ in the form (3.55). From $(3.46)_3$ we again obtain (3.56) and hence we can express $r(v)$ and $s(v)$ through $q(v)$. From $(3.46)_5$ we get, after making substitutions for $C, E, F, r(v), s(v)$ and their derivatives, the pair of equations

$$2q''(v) + q'(v) = 0, \qquad 2q'''(v) - 2q^{(4)}(v) - 3q'(v) = 0, \tag{4.26}$$

from which we deduce $q'(v) = 0$. Hence $q(v) = -r(v)$ are constants and $s(v) = 0$. We get the 1-dimensional Lie algebra of affine Killing vector fields spanned by (3.59) and our affine manifold is not locally homogeneous.

5. The case where Sym(Ric) **is nondegenerate.** We are going to prove the following

PROPOSITION 5.1. *Let both $\mathcal{M}$ and ∇ be analytic, and denote by g the symmetric part of* Ric. *If g is nondegenerate, and if ∇ is curvature homogeneous up to order* 2, *then ∇ is locally homogeneous.*

Proof (an easy adaptation of the proof of Theorem 2.1 in [5]). Choose $p \in \mathcal{M}$. For a linear endomorphism A of $T_p\mathcal{M}$ and a fixed basis (X, Y) of $T_p\mathcal{M}$ we set $AX = \alpha X + \beta Y$, $AY = \gamma X + \delta Y$. One sees that $A \cdot \mathrm{Ric} = 0$ iff $A \cdot g = 0$ and $A \cdot \omega = 0$, where ω denotes the skew-symmetric part of Ric.

Let now (X, Y) be a g-orthonormal basis of $T_p\mathcal{M}$, i.e.,

$$g(X, X) = 1, \qquad g(X, Y) = 0, \qquad g(Y, Y) = \varepsilon = \pm 1. \tag{5.1}$$

We have

$$\{A \in \mathfrak{gl}(T_p\mathcal{M}) \mid A \cdot g = 0\} = \left\{ \begin{bmatrix} 0 & -\varepsilon\beta \\ \beta & 0 \end{bmatrix} \;\middle|\; \beta \in \mathbb{R} \right\}. \tag{5.2}$$

Now we have either $\omega = 0$ on $\mathcal{M}$ or ω is a volume element. In the second case the condition $A \cdot \omega = 0$ is equivalent to $\mathrm{tr}(A) = 0$. Hence we always obtain

$$\mathfrak{g}(p; 0) = \left\{ \begin{bmatrix} 0 & -\varepsilon\beta \\ \beta & 0 \end{bmatrix} \;\middle|\; \beta \in \mathbb{R} \right\}. \tag{5.3}$$

We now observe that $\mathfrak{g}(p; 0)$ is *reductive* in $\mathfrak{gl}(T_p\mathcal{M})$. We have two cases. Either $\mathfrak{g}(p; 1) = \mathfrak{g}(p; 0)$ and then Theorem 2.2 and curvature homogeneity up to order one imply local homogeneity. Or $\mathfrak{g}(p; 1) = \{0\}$ and then Theorem 2.1 and curvature homogeneity up to order two imply local homogeneity, as well. ∎

We are now going to construct an example of the same type which is curvature homogeneous up to order one but not locally homogeneous. For our example we require that (in the standard notation)

$$\mathrm{Ric}(U, U) = \mathrm{Ric}(V, V) = 0, \qquad \mathrm{Ric}(V, U) = \mathrm{Ric}(U, V) \neq 0. \tag{5.4}$$

We set again

$$\varrho = \mathrm{Ric}(U, V) \neq 0. \tag{5.5}$$

Then (2.4) implies the system of PDEs

$$
\begin{aligned}
&B_v - D_u + D(A-D) + B(F-C) = 0,\\
&E_u - C_v + E(A-D) + C(F-C) = 0,\\
&C_u - A_v + F_u - D_v = 0.
\end{aligned}
\tag{5.6}
$$

This system is, in particular, satisfied by the functions

$$
A = F = -\frac{2}{u+v}, \qquad B = (u+v)^2, \qquad C = D = E = 0 \tag{5.7}
$$

defined in the half-plane $\{(u,v) \in \mathbb{R}^2 \mid u+v>0\}$, which we denote by $\mathcal{H}$. We choose $p = (1/2, 1/2)$ as a fixed point. The formulas (5.5) and (2.4) imply

$$
\varrho = \frac{2}{(u+v)^2}. \tag{5.8}
$$

Next, we calculate $\nabla \operatorname{Ric}$. We easily see that

$$
\begin{aligned}
&(\nabla_U \operatorname{Ric})(U,U) = -2B\varrho = -4,\\
&(\nabla_X \operatorname{Ric})(Y,Z) = 0 \quad \text{if at least one of the arguments is } V.
\end{aligned}
\tag{5.9}
$$

Consider the field $\Phi = \{\Phi_x : T_x\mathcal{H} \to T_p\mathcal{H}\}$ of linear isomorphisms defined by

$$
\Phi(U) = U_p, \qquad \Phi(V) = \frac{1}{(u+v)^2}V_p. \tag{5.10}
$$

We see at once that, with Φ defined by (5.10), the space $(\mathcal{H}, \nabla)$ becomes curvature homogeneous up to order 1.

We now prove that $(\mathcal{H}, \nabla)$ is not locally homogeneous. The corresponding system of equations for the affine Killing vector fields $X = a(u,v)U + b(u,v)V$ can be written, in this case, in the form

$$
\begin{aligned}
&a_{uu} + Aa_u - Ba_v + A_u a + A_v b = 0,\\
&b_{uu} + 2Ba_u - Ab_u - Bb_v + B_u a + B_v b = 0,\\
&a_{uv} + Aa_v = 0,\\
&b_{uv} + Ba_v + Fb_u = 0,\\
&a_{vv} - Fa_v = 0,\\
&b_{vv} + Fb_v + F_u a + F_v b = 0.
\end{aligned}
\tag{5.11}
$$

From the integrability condition for $(5.11)_3$ and $(5.11)_5$ we easily see that $a_v = 0$ and hence

$$
a = q(u). \tag{5.12}
$$

Using $(5.11)_1$ we calculate

$$
b(u,v) = -\tfrac{1}{2}q''(u)(u+v)^2 + q'(u)(u+v) - q(u). \tag{5.13}
$$

The equations $(5.11)_4$ and $(5.11)_6$ are satisfied identically by (5.12), (5.13). The equation $(5.11)_5$ implies the single condition

$$(q^{(4)}(u) - 6q'(u))(u+v) + 4q'''(u) = 0. \tag{5.14}$$

Hence we get $q'(u) = 0$ and $q(u)$ is constant. All affine Killing vector fields are constant multiples of

$$X = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \tag{5.15}$$

and the space $(\mathcal{H}, \nabla)$ is not locally homogeneous.

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Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha, Czech Republic
E-mail: kowalski@karlin.mff.cuni.cz

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: opozda@im.uj.edu.pl

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