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INVERSE LIMITS ON INTERVALS USING UNIMODAL BONDING MAPS HAVING ONLY PERIODIC POINTS WHOSE PERIODS ARE ALL THE POWERS OF TWO

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Abstract. We derive several properties of unimodal maps having only periodic points whose period is a power of 2. We then consider inverse limits on intervals using a single strongly unimodal bonding map having periodic points whose only periods are all the powers of 2. One such mapping is the logistic map, $f_\lambda(x) = 4\lambda x(1-x)$ on $[f(\lambda), \lambda]$, at the Feigenbaum limit, $\lambda \approx 0.89249$. It is known that this map produces an hereditarily decomposable inverse limit with only three topologically different subcontinua. Other examples of such maps are given and it is shown that any two strongly unimodal maps with periodic point whose only periods are all the powers of 2 produce homeomorphic inverse limits whenever each map has the additional property that the critical point lies in the closure of the orbit of the right endpoint of the interval.

0. Introduction. In [1], Barge and Ingram investigated inverse limits on $[0,1]$ using a single bonding map chosen from the family of logistic mappings. Theorem 7 of that paper yielded that at the Feigenbaum limit, the inverse limit is hereditarily decomposable. In the present paper, that theorem is reproved, but with a different argument. The process involves deducing several elementary properties of unimodal maps of the interval which have only periodic points whose period is a power of 2. Since the logistic map at the Feigenbaum limit has periodic points of periods all powers of 2 and no others, these results apply to this map. Most of these properties are well known for the logistic family (many can be calculated directly), but the authors know of no reference for these theorems in this more general setting. We then apply these properties along with a theorem of Collet and Eckmann [2, Theorem II.5.4, p. 116] to achieve our alternate proof that the inverse limit at the Feigenbaum limit is hereditarily decomposable.

The main advantage of this approach is that the theorem of Collet and Eckmann is more accessible (and its proof is easier to understand) than the theorem cited in the proof of Theorem 7 given in [1].

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In Section 3, we give an additional example of a unimodal map on $[0,1]$ which, like the logistic map at the Feigenbaum limit, has periodic points whose periods are all the powers of 2 and it has periodic points of no other periods. Then we prove the main theorem of this paper showing that, under certain conditions, any two strongly unimodal maps of intervals having periodic points whose periods are all the powers of 2 and having periodic points of no other period produce homeomorphic inverse limits.

The paper includes some combinatorial observations in Section 2 which the authors cannot locate in print in the form presented. These results are useful in deciding the nature of the kneading sequence for the logistic map at the Feigenbaum limit. Although the kneading sequence for the logistic map at the Feigenbaum limit is well understood, the approach using the results of this paper is simple to follow.

The *logistic family* of maps is given by $f_\lambda(x) = 4\lambda x(1-x)$ where $0 \le x \le 1$ and $0 \le \lambda \le 1$. For $\lambda > 0$, each of these maps has a critical point at $1/2$ while for $\lambda > 1/4$ each one has a non-zero fixed point at $1-1/(4\lambda)$. It is easy to see that, if f_λ is any member of this family, then $f_\lambda(x) \le \lambda$ for each x in $[0,1]$. Moreover, for $\lambda \ge (1+\sqrt{5})/4$, the interval $[f_\lambda(\lambda), \lambda]$ is mapped onto itself by f_λ. Let $\lambda_0 = 1/4$ and let $\lambda_1 = 3/4, \lambda_2, \lambda_3, \ldots$ denote the sequence of parameter values where the logistic family undergoes its first sequence of period-doubling bifurcations, and let λ_{c} (herein called the *Feigenbaum limit*, $\lambda_{\mathrm{c}} \approx 0.892486$) denote the limit point of this increasing sequence of positive numbers. Thus, if i is a positive integer and $\lambda_{i-1} \le \lambda < \lambda_i$, then f_λ has periodic points of period $1, 2, 4, \ldots, 2^{i-1}$.

By a *continuum* we mean a compact, connected subset of a metric space. A *mapping* is a continuous function. If $X_1, X_2, X_3, \ldots$ is a sequence of metric spaces and $f_1, f_2, f_3, \ldots$ is a sequence of mappings (called *bonding maps*) such that $f_i : X_{i+1} \to X_i$ for each positive integer i, then by the *inverse limit* of the inverse limit sequence $\{X_i, f_i\}$ is meant the subset of the product space, $\prod_{i>0} X_i$, to which the point $(x_1, x_2, x_3, \ldots)$ belongs if and only if $f_i(x_{i+1}) = x_i$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is denoted by $\varprojlim\{X_i, f_i\}$. Here, the product space is metrizable with the metric $d(x,y) = \sum_{i>0} d_i(x_i, y_i)/2^i$ where, for each i, d_i is a metric for X_i bounded by one. It is well known that, when the spaces X_i are continua and the bonding mappings are continuous, the inverse limit exists and is a continuum. In the case when, for each i, $X_i = X$ and $f_i = f$, we denote the inverse limit by $\varprojlim\{X, f\}$.

The *shift homeomorphism* of $\varprojlim\{X, f\}$, denoted by $\widehat{f}$, is defined as $\widehat{f}(x) = (f(x_1), x_1, x_2, \ldots)$. A point x is said to be a *periodic point* for a mapping f provided there is a positive integer n such that $f^n(x) = x$. If n is the least positive integer k such that $f^k(x) = x$ then we say that x is periodic of *period* n. A mapping of a continuum is *monotone* provided each point-inverse is a continuum. A mapping f of an interval $[a,b]$ onto itself is called *unimodal on* $[a,b]$ provided f is not monotone and there is a point c of (a,b) such that $f(c)$ belongs to the set $\{a,b\}$ and f is monotone on $[a,c]$ and $[c,b]$. We call a map *strongly unimodal* provided it is unimodal and is a homeomorphism on $[a,c]$ and $[c,b]$. (Strongly unimodal maps are often called unimodal, see e.g. [2, p. 63].) Throughout this paper, if f is a unimodal map of the interval $[a,b]$ onto itself with $f(a)=a$, we denote the fixed point for f in $[c,b]$ by p.

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1. Properties of unimodal maps

THEOREM 1. *If f is a unimodal mapping of the interval $[a,b]$, $f(a)=a$ and f has a periodic point of period* 4, *then $f(b)<c$.*

Proof. It is clear that if f has a periodic point which is not a fixed point, then the orbit of that point must intersect $[c,b]$ since f is non-decreasing on $[a,c]$. Suppose $f(b)\ge c$. Then $f[c,b]$ is a subset of $[c,b]$ and $f^2|[c,b]$ is non-decreasing. Suppose x is a point of $[c,b]$ such that $f^2(x)\ne x$. If $f^2(x)>x$ then $f^4(x)\ge f^2(x)>x$ so $f^4(x)\ne x$. Similarly, if $f^2(x)<x$ then $f^4(x)\ne x$. Thus, f does not have a periodic point of period 4.

COROLLARY. *For $\lambda>\lambda_2$, $f_\lambda(\lambda)<1/2$. In particular, for $\lambda=\lambda_{\rm c}$, $f_\lambda(\lambda)<1/2$.*

Proof. For $\lambda>\lambda_2$, $f_\lambda:[0,\lambda]\twoheadrightarrow[0,\lambda]$ is unimodal and has a periodic point of period 4.

THEOREM 2 [4, Theorem 6, p. 1911]. *If $f:[a,b]\twoheadrightarrow[a,b]$ is a unimodal map and q is the first fixed point for f^2 between c and b, then f has a periodic point of odd period if and only if $f^2(b)<q$.*

COROLLARY. *For $\lambda_2<\lambda\le\lambda_{\rm c}$, $f_\lambda^2(\lambda)>1-1/(4\lambda)>1/2$.*

Proof. Note that for $\lambda_2<\lambda\le\lambda_{\rm c}$, f_λ has no period 2 points between $1/2$ and $1-1/(4\lambda)$ (the non-zero fixed point for f_λ) and no periodic points other than those whose period is a power of 2. Thus, the point q of Theorem 2 is the fixed point $1-1/(4\lambda)$. If $f_\lambda^2(\lambda)=1-1/(4\lambda)$ then f_λ^2 has periodic points of every period so f_λ has a periodic point whose period is not a power of 2.

THEOREM 3. *Suppose f is a unimodal map of $[a,b]$ onto itself such that $f(a)=a$, $f(b)<c$ and q is the first fixed point for f^2 between c and b. If f has only periodic points whose period is a power of* 2, *then $f^2(b)>f(q)$.*

Proof. Theorem 2 implies that $f^2(b)\ge q$. If $f^2(b)=b$, the conclusion follows. Thus, since f is monotone on $[q,b]$ and $f(b)<c$, f^2 is unimodal on

$[q, b]$. Denote by r the fixed point between $f(q)$ and b. If $f^2(b) \leq f(q)$ then $f^4(b) \leq f^3(q) = f(q) < r$ so by Theorem 2, f^2 has a periodic point of odd period. This contradicts the assumption that the only periodic points of f are those whose period is a power of 2.

THEOREM 4. *Suppose f is a unimodal map of $[a, b]$ onto itself such that $f(a) = a$ and f has a periodic point of period 4. If f has only periodic points whose period is a power of 2, then $f(b) < f^3(b) < f^2(b) < b$.*

Proof. Denote by q the first fixed point for f^2 between c and b. By Theorem 3, $f^2(b) > f(q)$. Since $c < q \leq f(q) < f^2(b) < b$ and f is non-increasing on $[c, b]$, we have $f(b) \leq f^3(b) \leq q$. If $f^3(b) = f(b)$, then, since $f^2(b)$ is in $[c, b]$, f is constant on $[f^2(b), b]$. Thus, f^2 is constant on $[f^2(b), b]$ so $f^2(b)$ is a fixed point for f^2. Since f^2 is unimodal on $[p, b]$ and $f^2(b)$ is the last fixed point for f^2 in $[p, b]$, f^2 has no period 2 point. This involves a contradiction since f has a periodic point of period 4.

COROLLARY. *For $\lambda_2 < \lambda \leq \lambda_{\mathrm{c}}$, $f_\lambda(\lambda) < f^3_\lambda(\lambda) < f^2_\lambda(\lambda) < \lambda$.*

Proof. For $\lambda_2 < \lambda \leq \lambda_{\mathrm{c}}$, f_λ is unimodal, has a periodic point of period 4 and has only periodic points whose period is a power of 2.

THEOREM 5. *Suppose f is a unimodal map of $[a, b]$ onto itself such that $f(a) = a$ and n is a non-negative integer such that f has a periodic point of period 2^{n+2}. If f has only periodic points whose period is a power of 2, then $f^{2^n}(b) < f^{3\cdot 2^n}(b) < f^{2^{n+1}}(b) < b$.*

Proof. We proceed by induction. Observe that the case of $n = 0$ is Theorem 4. Assume the theorem is true for $n = k$ and that f has a periodic point of period 2^{k+3}. By Sharkovskiĭ's Theorem [3, Theorem 10.2, p. 62], f has a periodic point of period 4. Thus, $f(b) < c$. Since f has only periodic points whose period is a power of 2, by Theorem 3 we have $f^2(b) > f(q)$, therefore $f^2(b) > p$ where p is the fixed point of f between c and b. Then f^2 is a unimodal mapping of $[p, b]$ onto itself, $f^2|[p, b]$ has a periodic point of period 2^{k+2} and f^2 has only periodic points whose period is a power of 2. The conclusion follows by applying the inductive hypothesis to f^2 on $[p, b]$.

COROLLARY. *For $\lambda_{n+2} < \lambda \leq \lambda_{\mathrm{c}}$, $f^{2^n}_\lambda(\lambda) < f^{3\cdot 2^n}_\lambda(\lambda) < f^{2^{n+1}}_\lambda(\lambda) < \lambda$.*

Proof. For $\lambda_{n+2} < \lambda \leq \lambda_{\mathrm{c}}$, f_λ is unimodal, $f_\lambda(0) = 0$, f_λ has a periodic point of period 2^{n+2} and f_λ has only periodic points whose period is a power of 2.

The following is also an immediate consequence of Theorem 5.

THEOREM 6. *If f is a mapping of $[a, b]$ onto itself and f has periodic points whose periods are all the powers of 2 and no others, then $f(b) < f^2(b) < f^4(b) < f^8(b) < \ldots < b$.*

2. Properties of f_{λ_c}. The key notion external to this paper but used in this section is that of a homterval [2, p. 107]. A *homterval* for a mapping f is an open interval J such that $f^n|J$ is a homeomorphism of J onto its image for all $n \geq 1$. As noted in the introduction, the key theorem we use is Theorem II.5.4 of [2, p. 116]:

Let f be S-unimodal and suppose f has no stable periodic orbits. Then f has no homterval.

A map is *S-unimodal* on $[a,b]$ provided f is strictly increasing on $[a,c]$ and strictly decreasing on $[c,b]$, $f'(x) \neq 0$ if $x \neq c$, $f''(c) < 0$, f''' is continuous, f maps $[f(b),b]$ into itself and f has negative Schwarzian derivative. At $\lambda = \lambda_c$, f_λ satisfies the hypothesis of the quoted theorem and, consequently, has no homterval.

Before we state and prove our next theorems, we make some observations based on the results we have demonstrated thus far. Suppose f is a unimodal map of $[a,b]$ onto itself with critical point c, $a < c < b$, such that $f(c) = b$ and $f(a) = a$, which has only periodic points whose periods are powers of 2. Such maps include members f_λ of the logistic family for $\lambda \leq \lambda_c$. There are, in addition, piecewise linear maps with this property as well. In Section 3 we will indicate one way such maps can be constructed.

We first make some combinatorial observations which allow one to write the order (in the reals) of the orbit of b under f for as many iterations as one wishes (but, in general, only up to 2^n iterations in case f has a periodic point of period 2^{n+1} but none of period 2^{n+2}). The notation $\langle\langle n_1\ n_2\ \ldots\ n_k\rangle\rangle$ will be used in this section as an abbreviation for $f^{n_1}(b) < f^{n_2}(b) < \ldots < f^{n_k}(b)$. In this context, the Corollary to Theorem 4 would read that if $\lambda_2 < \lambda \leq \lambda_c$ and $f = f_\lambda$ then $\langle\langle 1\ 3\ 2\ 0\rangle\rangle$ (where f^0 denotes the identity) and Theorem 6 would read that for $\lambda = \lambda_c$ and $f = f_\lambda$, $\langle\langle 1\ 2\ 4\ 8\ \ldots\ 0\rangle\rangle$.

If f has a periodic point of period 8 (recall that we assume all of its periodic points have periods which are powers of 2), using Theorem 5 with $n = 1$ we observe that $\langle\langle 2\ 6\ 4\ 0\rangle\rangle$. Since f decreases on $[c,b]$ and $f^2(b) > c$, applying f to $\langle\langle 2\,6\,4\,0\rangle\rangle$ yields $\langle\langle 1\,5\,7\,3\rangle\rangle$. Recalling that $f^3(b) < f^2(b)$ allows us to conclude that $\langle\langle 1\ 5\ 7\ 3\ 2\ 6\ 4\ 0\rangle\rangle$ is true. Reasoning similarly, if f has a periodic point of period 16, then we can conclude that $\langle\langle 1\ 5\ 7\ 3\ 2\ 6\ 4\ 0\rangle\rangle$ must be true for f^2, i.e. $\langle\langle 2\,10\,14\,6\,4\,12\,8\,0\rangle\rangle$. Because f decreases on $[c,b]$ it follows that $\langle\langle 1\ 9\ 13\ 5\ 7\ 15\ 11\ 3\rangle\rangle$ and finally $\langle\langle 1\ 9\ 13\ 5\ 7\ 15\ 11\ 3\ 2\ 10\ 14\ 6\ 4\ 12\ 8\ 0\rangle\rangle$.

This process may be formalized by defining three operations on sequences. If $s = \langle\langle n_1\ n_2\ n_3\ \ldots\ n_k\rangle\rangle$, the *doubling* operation D is defined by $Ds = \langle\langle 2n_1\ 2n_2\ 2n_3\ \ldots\ 2n_k\rangle\rangle$, the *reverse and add one* operation R is defined by $Rs = \langle\langle n_k+1\ n_{k-1}+1\ \ldots\ n_1+1\rangle\rangle$ and, if $t = \langle\langle m_1\ m_2\ \ldots\ m_j\rangle\rangle$, the *merge two sequences* operation M is defined by $M(s,t) = \langle\langle n_1\ n_2\ \ldots n_k\ m_1\ m_2\ \ldots\ m_j\rangle\rangle$. Then defining the sequence $s_0, s_1, s_2, \ldots$ inductively by

$s_0 = \langle\langle 1\ 0 \rangle\rangle$ and $s_n = M(RDs_{n-1}, Ds_{n-1})$ for a mapping f having periodic points whose periods are all the powers of 2 completely determines the sequences. Under this scheme, the last sequence determined in the previous paragraph was s_3.

REMARK. We can also mark the location of c in these sequences, which is helpful in calculating the kneading sequence for a mapping f having periodic points whose periods are all the powers of 2 and which is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$. We know that $f(b) < c$. If $f^3(b) \le c$ then the fact that f increases on $[0, c]$ and $\langle\langle 1\ 5\ 7\ 3\ 2\ 6\ 4\ 0 \rangle\rangle$ (in particular, $f^7(b) < f^3(b)$) yield $f^8(b) < f^4(b)$, a contradiction to Theorem 6. If $c \le f^7(b)$ then $\langle\langle 1\ 9\ 13\ 5\ 7\ 15\ 11\ 3\ 2\ 10\ 14\ 6\ 4\ 12\ 8\ 0 \rangle\rangle$ (in particular, $f^7(b) < f^{15}(b)$) and the fact that f decreases on $[c, b]$ yield that $f^{16}(b) < f^8(b)$, again a contradiction to Theorem 6. Without proof we observe that it follows that $f^{2^n-1}(b) < c$ if n is odd and $f^{2^n-1}(b) > c$ if n is even.

The remaining theorems in this section are concerned with f_{λ_c}.

THEOREM 7. *The sequence* $\{f_{\lambda_c}^{2^n}(\lambda_c)\}_{n\ge 0}$ *converges to* λ_c.

Proof. Denote λ_c by λ. Since the sequence $f^{2^n}(\lambda)$ is increasing, if it does not converge to λ, then it must converge to some number $s < \lambda$. Let $J = (s, \lambda)$. We show that J is a homterval for f by showing that if n is a positive integer then $1/2$ is not in $f^n(J)$. To this end, note that there is a positive integer j such that $n < 2^j - 1$. Since J is a subset of $(f^{2^j}(\lambda), \lambda)$ and $f^i[(f^{2^j}(\lambda), \lambda)]$ does not contain $1/2$ for $1 \le i \le n$, it follows that $f^n(J)$ does not contain $1/2$. Thus, J is a homterval, contrary to [2, Theorem II.5.4].

REMARK. The techniques used in the proof of Theorem 7 could be slightly modified to argue that the sequence $\{f_{\lambda_c}^{2^n-1}(\lambda_c)\}_{n\ge 0}$ converges to $1/2$, by considering separately the cases where n is odd and n is even. We make use of this observation in the proof of our main theorem, Theorem 11.

THEOREM 8. $M_{\lambda_c} = \varprojlim\{[0,1], f_{\lambda_c}\}$ *is hereditarily decomposable.*

Proof. Denote λ_c by λ and f_λ by f. The map f has periodic points of period 2^n for $n = 0, 1, 2, \ldots$ and periodic points of no other period [2, p. 54]. We observe that, by Bennett's Theorem [4, Theorem 1], M_λ is the union of a ray R and a continuum K such that $K = \overline{R} - R$ and hence M_λ is decomposable. If H is a subcontinuum of M_λ and i is a positive integer, denote by H_i the projection of H onto the ith factor space. Suppose H is a subcontinuum of M_λ. We show that H is an arc, homeomorphic to M_λ or to the union of two copies of M_λ intersecting at a common endpoint of a ray.

Suppose that, for some positive integer i, $H_i = \pi_i H$ intersects $[0, f(\lambda))$. If $j > i$, then H_j intersects $[0, f(\lambda))$ because a point of $[0, f(\lambda))$ only has a point-inverse in $[0, f(\lambda))$; see Figure 1. If there is a positive integer N such that if $n \geq N$, then H_n is a subset of $[0, f(\lambda))$, then H is an arc. On the other hand, if for infinitely many integers j, H_j intersects $[f(\lambda), \lambda]$ then H_k contains $[f(\lambda), \lambda]$ for $k = 1, 2, 3, \ldots$ Thus, in this case, H is homeomorphic to M_λ.

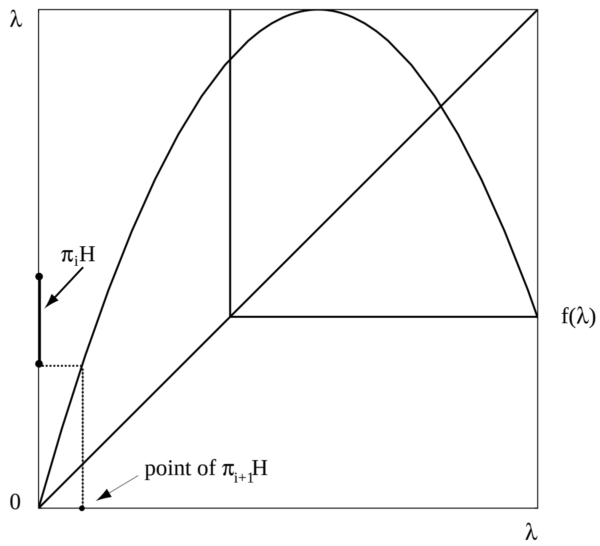

Fig. 1. Graph of f_{λ_c} on $[0, \lambda_c]$

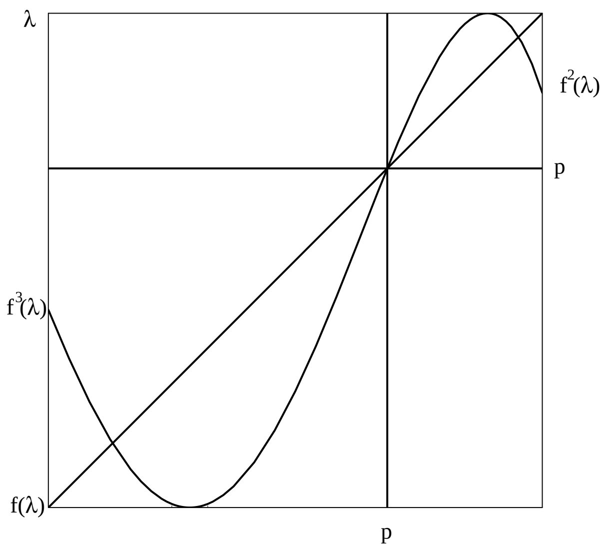

Fig. 2. Graph of $f^2_{\lambda_c}$ on $[f_{\lambda_c}(\lambda_c), \lambda_c]$

If, for each i, H_i lies in $[f(\lambda),\lambda]$, and for some i, H_i intersects $(f^3(\lambda), f^2(\lambda))$, then H is an arc, homeomorphic to M_λ or to the union of two copies of M_λ intersecting at a common endpoint of a ray. This can be seen by looking at f^2 on $[f(\lambda),\lambda]$ and applying the previous argument to the interval $[f(\lambda),p]$ or the interval $[p,\lambda]$ where p is the non-zero fixed point of f (see Figure 2). If, for each i, H_i lies in $[f(\lambda),f^3(\lambda)]$ then $\pi_i\widehat{f}\,[H]$ lies in $[f^2(\lambda),\lambda]$ so either $\pi_i H$ or $\pi_i\widehat{f}\,[H]$ lies in $[f^2(\lambda),\lambda]$. Since $\widehat{f}$ is a homeomorphism, we may assume H_i lies in $[f^2(\lambda),\lambda]$.

Proceeding inductively, suppose there is a positive integer n such that for each i, H_i (or $\pi_i\widehat{f}^{\,2^{n-1}j}[H]$ for some j) lies in $[f^{2^n}(\lambda),\lambda]$ and, for some i, H_i intersects $(f^{3\cdot 2^n}(\lambda), f^{2\cdot 2^n}(\lambda))$. Then H is an arc, homeomorphic to M_λ or to the union of two copies of M_λ. If for each n and each i, H_i (or the projection of an appropriate shift of H) lies in $[f^{2^n}(\lambda),\lambda]$ then H is degenerate.

Thus, each non-degenerate subcontinuum of M_λ is decomposable and the proof is complete.

THEOREM 9. $M_{\lambda_c}=\varprojlim\{[0,1],f_{\lambda_c}\}$ *is the union of a ray R and a continuum C such that $\overline{R}-R=C$ and C is the union of two copies of M_{λ_c} intersecting at a common endpoint of the ray.*

Proof. Use Bennett's Theorem [4] and the techniques of the proof of Theorem 8.

THEOREM 10. $M_{\lambda_c}=\varprojlim\{[0,1],f_{\lambda_c}\}$ *has only three topologically different subcontinua: arcs, copies of M_{λ_c} and unions of two copies of M_{λ_c} intersecting at a common endpoint of a ray.*

REMARK. Although the theorems in this paper are stated for unimodal maps for which $f(a)=a$, they hold for unimodal maps for which $f(b)=a$ since such a map can be "embedded" in a unimodal map for which $f(a)=a$. Thus, by conjugacy, these theorems hold for all unimodal maps of intervals having only periodic points whose periods are powers of 2 (with an appropriate change of endpoint of the interval in question along with the sense of the inequalities).

3. Other examples and main theorem. Other examples of maps of intervals which exhibit periodic points whose periods are all the powers of 2 and no others include the map g_μ defined on $[-1,1]$ by $g_\mu(x)=1-\mu x^2$ for $\mu\approx 1.401155$ (see [2, p. 36]) and the map s_λ defined on $[0,1]$ by $s_\lambda(x)=\sin(\lambda\pi x)$ for $\lambda\approx 0.86526$ as well as at appropriate parameter values in any other full family such as the one-parameter family of quadratic maps x^2+c. In this section we construct another example of a unimodal map having periodic points whose periods are all the powers of 2 and no others. We discovered this map independently although we subsequently found it mentioned in a paper of Nitecki [5, p. 50]. He includes no proof that it is such an example so we provide the example and a proof for the sake of completeness.

Example. Suppose f is a mapping of $[0,1]$ into itself. Denote by $\mathcal{D}(f)$ the map of $[0,1]$ into itself defined by

$$\mathcal{D}(f)(x) = \begin{cases} \frac{2}{3} + \frac{1}{3} f(1-3x), & 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{3} - (1+f(0))\big(x - \frac{2}{3}\big), & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ 1-x, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

It is easy to show that if p is a periodic point of period n for f then $(1-p)/3$ is a periodic point of period $2n$ for $\mathcal{D}(f)$. Moreover, the only periodic points for $\mathcal{D}(f)$ in $[0,1/3]$ arise in this manner from periodic points for f.

Now, define ϕ_1 to be the identity on $[0,1]$ and, inductively, define ϕ_{n+1} to be $\mathcal{D}(\phi_n)$ for $n = 1,2,3,\ldots$ Then, for each j, ϕ_j has periodic points whose periods are $1,2,4,\ldots,2^{j-1}$ and no others. The sequence $\phi_1, \phi_2, \phi_3, \ldots$ converges uniformly to a function ϕ which is easily seen to have the property that $\mathcal{D}(\phi) = \phi$. From this it follows that ϕ has periodic points whose periods are $1,2,4,\ldots$ and no others.

Remark. We think it interesting to note that if one begins the process described above with *any* function ψ_1 from $[0,1]$ into $[0,1]$ (continuous or not), then the resulting sequence $\psi_1, \psi_2, \psi_3, \ldots$ will converge to ϕ.

Remark. The critical point for ϕ is $1/4$ while the orbit of 1 under ϕ is the set of endpoints of the deleted open intervals in the usual middle-thirds Cantor set. Thus, the critical point for ϕ is a limit point of the orbit of 1 under ϕ and so ϕ satisfies the hypothesis of Theorem 11 (below).

An example which fails. Among our attempts to construct other examples of these functions with periodic points whose periods are all the powers of 2 and no others, we hit upon an interesting scheme which appears to work but fails. Let f_1 be the mapping of $[0,1]$ onto itself defined by $f_1(x) = 1-x$. Starting from the point $(1,0)$, construct f_2 from two linear pieces one of which emanates from $(1,0)$ with slope -2 (i.e., twice the slope of f_1) and the other emanates from the point $(1/2,1)$ where the first piece strikes the top of $[0,1] \times [0,1]$ and has slope $1/2$ (i.e., half of the slope of f_1 with a change of sign). The mapping f_2 has periodic points of periods $1,2$ and 4. The endpoints of the interval lie in a period 4 orbit. Now, starting from the point $(0,3/4)$, construct f_3 from three linear pieces as follows: the first piece emanates from the point $(0,3/4)$ with slope 1 (i.e., twice the slope of the piece of f_2 eminating from $(0,3/4)$), the second piece emanates from the point $(1/4,1)$ where the first piece strikes the top of $[0,1] \times [0,1]$ and has slope $-1/4$ (i.e., half of the slope of the piece of f_2 which is being replaced along with a change of sign), the third piece consists of the portion of $2(1-x)$ emanating from the point of intersection of the second piece of f_3 and this piece of f_2. Then f_3 has periodic points of period $1, 2, 4, 8$ and no others and the endpoints of the interval lie in a period 8 orbit. Continuing this process of "pivoting" on points on alternating sides along with doubling and halving slopes continues to produce mappings f_j with appropriate periods and having the endpoints of the interval in a period 2^j orbit for $j = 4, 5$ but *fails* for $j = 6$.

We now proceed with the main theorem of this paper. We denote by $\mathcal{O}_f(x)$ the set $\{x, f(x), f^2(x), f^3(x), \ldots\}$ and we use $\mathrm{cl}(M)$ to denote the closure of the set M. As we have noted earlier, the core of the logistic map at the Feigenbaum limit and the mapping ϕ constructed in this section satisfy the hypothesis of Theorem 11. The cores of the mappings g_μ and s_λ mentioned at the beginning of this section also satisfy the hypothesis of Theorem 11.

THEOREM 11. *Suppose* $f : [a, b] \to [a, b]$ *and* $g : [a', b'] \to [a', b']$ *are strongly unimodal maps with critical points* c *and* c', *respectively*, *such that* $f(b) = a$, $g(b') = a'$ *and* f *and* g *have periodic points whose periods are all the powers of* 2 *and each has no other periodic points. If* c *belongs to* $\mathrm{cl}(\mathcal{O}_f(b))$ *and* c' *belongs to* $\mathrm{cl}(\mathcal{O}_g(b'))$, *then* $\varprojlim\{[a, b], f\}$ *and* $\varprojlim\{[a', b'], g\}$ *are homeomorphic.*

Proof. Using Theorem 5, the remark preceding Theorem 7 and the assumption that c belongs to $\mathrm{cl}(\mathcal{O}_f(b))$ and c' belongs to $\mathrm{cl}(\mathcal{O}_g(b'))$, it follows that $\mathrm{cl}(\mathcal{O}_f(b))$ and $\mathrm{cl}(\mathcal{O}_g(b'))$ are Cantor sets. Also, from the observations in Section 2 about the orbit of the right endpoint, we note that $f^n(b) < f^m(b)$ if and only if $g^n(b') < g^m(b')$. Thus, there is a homeomorphism h from $[a, b]$ onto $[a', b']$ such that $h(f^n(b)) = g^n(b)$ for each n. We define a sequence $\phi_1, \phi_2, \phi_3, \ldots$ of maps from $[a, b]$ onto $[a', b']$ which induces a homeomorphism between the inverse limits as follows. Let $\phi_1 = h$. Then let

$$\phi_{i+1} = \begin{cases} (g|[a', c'])^{-1}\phi_i f(x), & x \in [a, c], \\ (g|[c', b'])^{-1}\phi_i f(x), & x \in [c, b]. \end{cases}$$

It is easy to see that the induced mapping is a homeomorphism.

COROLLARY. $\varprojlim\{[0, 1], \phi\}$ *and* $\varprojlim\{[f_{\lambda_c}(\lambda_c), \lambda_c], f_{\lambda_c}|[f_{\lambda_c}(\lambda_c), \lambda_c]\}$ *are homeomorphic.*

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