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INVARIANTS AND FLOW GEOMETRY

BY

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Abstract. We continue the study of Riemannian manifolds (M,g) equipped with an isometric flow $\mathfrak{F}_\xi$ generated by a unit Killing vector field ξ. We derive some new results for normal and contact flows and use invariants with respect to the group of ξ-preserving isometries to charaterize special $(M,g,\mathfrak{F}_\xi)$, in particular Einstein, η-Einstein, η-parallel and locally Killing-transversally symmetric spaces. Furthermore, we introduce curvature homogeneous flows and flow model spaces and derive an algebraic characterization of Killing-transversally symmetric spaces by using the curvature tensor of special flow model spaces. All these results extend the corresponding theory in Sasakian geometry to flow geometry.

1. Introduction. Scalar curvature invariants have been used at many places to characterize special Riemannian manifolds. Characterizations of Einstein spaces and real, complex and quaternionic space forms by using quadratic scalar curvature invariants are well-known. They may be derived by means of the decomposition of spaces of (curvature) tensors into irreducible factors under the action of appropriate groups. We refer to [8], [14], [24], [25] where examples, applications and more references are given. Similar results are also known in Sasakian and almost contact metric geometry. See, for example, [5], [15], [18], [19]. Furthermore, scalar curvature invariants have also been used to derive the interesting result that a Riemannian manifold having the same curvature tensor as that of an irreducible symmetric space is itself locally symmetric, and hence locally isometric to that model space [26]. A similar theory has been developed for almost contact metric spaces, and in particular, for Sasakian manifolds [5], [6], [7]. These results belong to the study of curvature homogeneous spaces [20] and of manifolds having a homogeneous model (see [4] for a survey).

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The main purpose of this paper is to treat similar problems in flow geometry. This is the study of the geometry of Riemannian manifolds (M, g) equipped with an isometric flow $\mathfrak{F}_\xi$ generated by a unit Killing vector field ξ. It generalizes Sasakian geometry and has been developed in several papers. See Section 2 for some references. We introduce several scalar invariants with respect to the group of ξ-preserving isometries. In Section 2, we begin by collecting some basic material. In Section 3, we derive some useful new results for the special case of contact and normal flows, and which are mainly related to the problem of the local irreducibility or reducibility of the spaces equipped with such flows and of the base spaces of the local submersions determined by the isometric flows. In Section 4, we derive several inequalities for the introduced invariants and use them to prove characterizations for classes of special manifolds such as Einstein, η-Einstein and η-parallel spaces, and in particular, locally Killing-transversally symmetric spaces. This last class of manifolds consists of those spaces such that the reflections with respect to the flow lines of ξ are local isometries. They generalize the locally φ-symmetric spaces introduced in Sasakian geometry by T. Takahashi [21]. There they play a role which is very similar to that of the Hermitian symmetric spaces in complex geometry. Finally, in Section 5, we first introduce the notions of curvature homogeneous flows and flow model spaces and then prove a result concerning the Killing-transversally symmetric spaces by comparing the curvature tensor with that of an appropriate flow model space and by using the introduced invariants.

2. Preliminaries. Let (M, g) be an n-dimensional, connected, smooth Riemannian manifold with $n \geq 2$. Furthermore, let ∇ denote its Levi-Civita connection and R the corresponding Riemannian curvature tensor defined by $R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$ for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. Further, ϱ and τ denote the Ricci tensor and the scalar curvature, respectively.

A tangentially oriented foliation of dimension one on (M, g) is called a *flow*. The leaves of this foliation are the integral curves of a non-singular vector field on M and hence, after normalization, a flow is also given by a unit vector field. In particular, a non-singular Killing vector field defines a *Riemannian flow* also called an *isometric flow* [22].

In this paper, we consider and denote by $\mathfrak{F}_\xi$ an isometric flow generated by a *unit* Killing vector field ξ. This flow determines locally a Riemannian submersion. For each $m \in (M, g)$, let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the mapping $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$ is a submersion. Furthermore, let $\widetilde{g}$ denote the induced metric on $\widetilde{\mathcal{U}}$ given by

$$(\widetilde{g}(\widetilde{X}, \widetilde{Y}))^* = g(\widetilde{X}^*, \widetilde{Y}^*)$$

for $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$, where $\widetilde{X}^*$, $\widetilde{Y}^*$ denote the horizontal lifts of $\widetilde{X}$, $\widetilde{Y}$ with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then $\pi : (\mathcal{U}, g) \to (\widetilde{\mathcal{U}}, \widetilde{g})$ is a Riemannian submersion. Using O'Neill's integrability tensor A ([17]; see also [1]), we have

$$A_U\xi = \nabla_U\xi, \quad A_\xi U = 0,$$
$$A_X Y = (\nabla_X Y)^{\mathcal{V}} = -A_Y X, \quad g(A_X Y, \xi) = -g(A_X\xi, Y)$$

for $U \in \mathfrak{X}(M)$ and for horizontal vector fields X, Y (that is, $\eta(X) = \eta(Y) = 0$). Here, $\mathcal{V}$ denotes the vertical component.

Next, we define the operator H by

$$HU = -A_U\xi \tag{1}$$

and its $(0,2)$-version h by $h(U,V) = g(HU,V)$, $U,V \in \mathfrak{X}(M)$. Since ξ is a Killing vector field, h is skew-symmetric and $h = -d\eta$. Note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable. In this case, since the flow lines are geodesics, (M,g) is locally a product of an $(n-1)$-dimensional manifold and a curve.

The Levi-Civita connection $\widetilde{\nabla}$ on $(\widetilde{\mathcal{U}}, \widetilde{g})$ is determined by

$$\nabla_{\widetilde{X}^*}\widetilde{Y}^* = (\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y})^* + h(\widetilde{X}^*, \widetilde{Y}^*)\xi \tag{2}$$

for $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\widetilde{\mathcal{U}})$ and the curvature tensor R of (M,g) satisfies

$$R(X,\xi,Y,\xi) = g(HX,HY) = -g(H^2X,Y) \tag{3}$$

for all horizontal X,Y. Here, we use the notation $R(X,Y,Z,W) = g(R_{XY}Z,W)$. This shows that the ξ-sectional curvature $K(X,\xi)$ of the two-plane spanned by X and ξ is non-negative for all horizontal X. Since $H\xi = 0$, $K(X,\xi) = 0$ holds for all horizontal X if and only if $h = 0$. Further, $K(X,\xi)$ is strictly positive for all X if and only if H is of maximal rank $n-1$, in which case n is necessarily odd. Then η is a contact form and the flow $\mathfrak{F}_\xi$ is called a *contact flow*.

Normal flows also appear naturally in this framework. $\mathfrak{F}_\xi$ is said to be a *normal flow* [9] if, for horizontal X,Y, the curvature transformations R_{XY} leave the horizontal subspaces invariant, or equivalently, $R(X,Y,X,\xi) = 0$. Note that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is equal to 1 (see [2] for more details). Moreover, if the ξ-sectional curvature is a non-vanishing constant $k = c^2$, then $H^2X = -kX$ for horizontal vectors X and $(M, c^2g, c^{-1}\xi, c\eta, c^{-1}H)$ is a Sasakian manifold. Nevertheless, there exist Riemannian manifolds equipped with a normal contact flow which cannot have any Sasakian structure. This is the case for the compact nilmanifolds $M(1,r)$ formed by the right cosets $\Gamma(1,r)\backslash H(1,r)$, where r is even or $r = 4s + 1$, $s \geq 1$, and $\Gamma(1, r)$ is the subgroup of the matrices in the generalized Heisenberg group $H(1, r)$ with integer entries [9].

The flow $\mathfrak{F}_\xi$ is normal if and only if

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2U \tag{4}$$

for all $U, V \in \mathfrak{X}(M)$. In this case, we have [9]

$$\begin{aligned} R_{UV}\xi &= \eta(V)H^2U - \eta(U)H^2V, \\ R_{U\xi}V &= g(HU, HV)\xi + \eta(V)H^2U \end{aligned} \tag{5}$$

for all $U, V \in \mathfrak{X}(M)$. Using also (2), it follows that the curvature tensors of ∇ and $\widetilde{\nabla}$ are related by

$$\begin{aligned} (\widetilde{R}_{\widetilde{X}\widetilde{Y}}\widetilde{Z})^* = R_{\widetilde{X}^*\widetilde{Y}^*}\widetilde{Z}^* &- g(H\widetilde{Y}^*, \widetilde{Z}^*)H\widetilde{X}^* \\ &+ g(H\widetilde{X}^*, \widetilde{Z}^*)H\widetilde{Y}^* + 2g(H\widetilde{X}^*, \widetilde{Y}^*)H\widetilde{Z}^* \end{aligned} \tag{6}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(\widetilde{\mathcal{U}})$. This yields

$$(\widetilde{\varrho}(\widetilde{X}, \widetilde{Y}))^* = \varrho(\widetilde{X}^*, \widetilde{Y}^*) + 2g(H\widetilde{X}^*, H\widetilde{Y}^*), \tag{7}$$

$$\widetilde{\tau}^* = \tau - \operatorname{tr} H^2. \tag{8}$$

Moreover, $\varrho(U, \xi) = -\eta(U) \operatorname{tr} H^2$. Furthermore, using (7), we get

$$((\widetilde{\nabla}_{\widetilde{X}}\widetilde{\varrho})(\widetilde{Y}, \widetilde{Z}))^* = (\nabla_{\widetilde{X}^*}\varrho)(\widetilde{Y}^*, \widetilde{Z}^*). \tag{9}$$

ϱ is said to be *η-parallel*, or $(M, g, \mathfrak{F}_\xi)$ is called *η-parallel*, if it satisfies $(\nabla_X \varrho)(Y, Z) = 0$ for all horizontal X, Y, Z. It follows from (9) that ϱ is η-parallel if and only if $\widetilde{\varrho}$ is parallel for each base space $(\widetilde{\mathcal{U}}, \widetilde{g})$. Similarly, we say that $(M, g, \mathfrak{F}_\xi)$ is *η-Einsteinian* if each $(\widetilde{\mathcal{U}}, \widetilde{g})$ is Einsteinian. From (7), (8) it follows that this is the case if and only if

$$\varrho(X, Y) = \lambda g(X, Y) - 2g(HX, HY)$$

for all horizontal X, Y, where $\lambda = (\tau - \operatorname{tr} H^2)(n - 1)^{-1}$. In particular, a Sasakian manifold M^{2n+1} is η-Einsteinian if and only if

$$\varrho(U, V) = ag(U, V) + b\eta(U)\eta(V)$$

where a, b are constants. In this case, $a + b = 2n$ and $a = (\tau - 2n)(2n)^{-1}$.

Next, from [9] we have the following formulas:

$$\begin{aligned} R_{HUV}W + R_{UHV}W = g(HV, W)H^2U &- g(HU, W)H^2V \\ &- g(H^2U, W)HV + g(H^2V, W)HU \\ &+ \eta(V)R_{HU\xi}W - \eta(U)R_{HV\xi}W \end{aligned} \tag{10}$$

for all $U, V, W \in \mathfrak{X}(M)$, and

$$(\nabla_\xi R)_{XY} Z = 0 \tag{11}$$

for all horizontal X, Y, Z. Hence, applying (5) and (10), we obtain

$$\begin{aligned}(\nabla_W R)_{UV}\xi = {} & g(W, H^2U)HV - g(W, H^2V)HU \\ & + \eta(W)\{\eta(U)H^3V - \eta(V)H^3U\} + HR_{UV}W\end{aligned} \tag{12}$$

for all tangent vectors U, V, W to M.

Next, let $\widetilde{H}$ be the $(1,1)$-tensor field on $\widetilde{\mathcal{U}}$ defined by $\widetilde{H}\widetilde{X} = \pi_* H\widetilde{X}^*$. Then $\mathfrak{F}_\xi$ is normal if and only if $\widetilde{\nabla}\widetilde{H} = 0$. Furthermore, in that case, (6) and (10) yield the useful relation

$$\widetilde{R}_{\widetilde{H}\widetilde{X}\widetilde{Y}} + \widetilde{R}_{\widetilde{X}\widetilde{H}\widetilde{Y}} = 0 \tag{13}$$

(see also [12]).

Finally, $(M, g, \mathfrak{F}_\xi)$ is called a *locally Killing-transversally symmetric space* (briefly, a *locally KTS-space*) if the reflections with respect to the flow lines are local isometries. Then the flow $\mathfrak{F}_\xi$ is automotically normal and $(M, g, \mathfrak{F}_\xi)$ is called a *globally Killing-transversally symmetric space* (briefly, a *KTS-space*) if ξ is complete and the local reflections can be extended to global isometries. Complete, simply connected locally KTS-spaces are KTS-spaces and then $(M, g, \mathfrak{F}_\xi)$ is a naturally reductive space. See [9], [10] for more details and further information.

The following two propositions provide useful characterizations for locally KTS-spaces.

Proposition 2.1 [9]. *$(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X, Y, X, Y) = 0$$

for all horizontal X, Y.

Proposition 2.2 [9]. *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then the space is a locally KTS-space if and only if each base space $(\widetilde{\mathcal{U}}, \widetilde{g})$ of a local Riemannian submersion $\pi : \mathcal{U} \to \widetilde{\mathcal{U}} = \mathcal{U}/\xi$ is a locally symmetric space.*

So, according to the terminology used in [23], $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a *normal transversally symmetric foliation.* As already mentioned in the introduction, in Sasakian geometry the locally KTS-spaces coincide with the locally φ-symmetric spaces introduced in [21]. See also [3], [5] for more details and further references.

3. Normal and contact flows. In this section we give some more information about normal and contact flows and derive some new results. We begin with

LEMMA 3.1 [13]. *For all* $k \geq 1$, $\operatorname{tr} H^{2k}$ *is constant on a Riemannian manifold* (M, g) *equipped with a normal flow* $\mathfrak{F}_\xi$.

This implies

PROPOSITION 3.2. *The eigenvalues of* H^2 *are constant on a Riemannian manifold equipped with a normal flow.*

COROLLARY 3.3. *Let* (M, g) *be a Riemannian manifold and* $\mathfrak{F}_\xi$ *a normal flow on it. If the* ξ*-sectional curvature is pointwise constant, then it is globally constant.*

In what follows we denote by $-c_i^2$, $i = 1, \ldots, r$, the different eigenvalues of $H^2_{|\mathcal{H}}$ restricted to the horizontal distribution $\mathcal{H}$. The corresponding eigenspaces at $m \in M$ are denoted by $\mathcal{H}_{c_i^2}(m)$, $i = 1, \ldots, r$. Then, using (3), we have

$$\mathcal{H}_{c_i^2}(m) = \{X \in \mathcal{H}(m) \mid K(X, \xi) = c_i^2\}.$$

Now, we prove

PROPOSITION 3.4. *Let* $\mathfrak{F}_\xi$ *be a normal flow on* (M, g). *Then:*

(i) $\mathcal{H}_{c_i^2}$, $i = 1, \ldots, r$, *determines a differentiable distribution on* M *and for each* $m \in M$, $\mathcal{H}(m) = \mathcal{H}_{c_1^2}(m) \oplus \ldots \oplus \mathcal{H}_{c_r^2}(m)$ *is an* H*-invariant orthogonal decomposition of the horizontal subspace* $\mathcal{H}(m)$;

(ii) *the distribution* $\mathcal{D}_{c_i^2} : m \to \mathcal{H}_{c_i^2}(m) \oplus \xi(m)$ *is differentiable and involutive. Moreover, its integral manifolds are totally geodesic submanifolds of* M.

Proof. Since H^2 is symmetric, the differentiability of $\mathcal{H}_{c_i^2}$ and $\mathcal{D}_{c_i^2}$ follows at once. Next, we prove that $\mathcal{D}_{c_i^2}$ is involutive. In fact, we will prove more: if $U, V \in \mathcal{D}_{c_i^2}$, then also $\nabla_U V \in \mathcal{D}_{c_i^2}$. Hence, its integral manifolds are totally geodesic.

First, note that $U \in \mathcal{D}_{c_i^2}$ if and only if $H^2 U = c_i^2(\eta(U)\xi - U)$. Then, using (4), it follows that

$$H\nabla_U V = \nabla_U(HV) + c_i^2(\eta(V)U - g(U, V)\xi)$$

and so, from (1) we get

$$H^2\nabla_U V = c_i^2(\eta(\nabla_U V)\xi - \nabla_U V),$$

which proves the required result. ■

Furthermore, we also obtain distributions $\widetilde{\mathcal{H}}_{c_i^2}$, $i = 1, \ldots, r$, on each base space $\widetilde{\mathcal{U}}$ of a local submersion which assign to each point $\widetilde{m} = \pi(m)$ of $\widetilde{\mathcal{U}}$ the subspace $\widetilde{\mathcal{H}}_{c_i^2}(\widetilde{m}) = \pi_{*m}\mathcal{H}_{c_i^2}(m)$ of $T_{\widetilde{m}}\widetilde{\mathcal{U}}$. Note that $\widetilde{\mathcal{H}}_{c_i^2}(\widetilde{m})$ is well-defined because $\mathcal{H}_{c_i^2}$ is ξ-invariant, or equivalently, $\mathcal{H}_{c_i^2}$ is obtained locally as the horizontal lift of $\widetilde{\mathcal{H}}_{c_i^2}$. Here, we have

PROPOSITION 3.5. *For normal flows* $\mathfrak{F}_\xi$ *on* (M,g) *we have*:

(i) *the distributions* $\widetilde{\mathcal{H}}_{c_i^2}$, $i = 1,\dots,r$, *are differentiable and involutive*;

(ii) *let* $\widetilde{\mathcal{U}}_i$ *be the maximal integral manifold of* $\widetilde{\mathcal{H}}_{c_i^2}$ *through a point of* $\widetilde{\mathcal{U}}$.

Then $\widetilde{\mathcal{U}}_i$ *is a totally geodesic submanifold of* $\widetilde{\mathcal{U}}$ *and* $\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_1 \times \dots \times \widetilde{\mathcal{U}}_r$.

Proof. The differentiability is clear. Furthermore, taking into account that $\widetilde{\mathcal{H}}_{c_i^2}(\widetilde{m})$ is the eigenspace of $\widetilde{H}(\widetilde{m})$ on $T_{\widetilde{m}}(\widetilde{\mathcal{U}})$ corresponding to the eigenvalue $-c_i^2$ and that $\widetilde{\nabla}\widetilde{H} = 0$, we get $\widetilde{H}^2\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = -c_i^2\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}$ for all $\widetilde{X},\widetilde{Y} \in \widetilde{\mathcal{H}}_{c_i^2}$. This again implies the required result. The last part of (ii) follows from the uniqueness property of totally geodesic submanifolds. ■

Now, we turn to the consideration of normal contact flows. In this case $\dim M$ is odd and we put $\dim M = 2n+1$. First, we recall

PROPOSITION 3.6 [29]. *A Riemannian manifold equipped with a normal contact flow is locally irreducible with the group* $\mathrm{SO}(2n+1)$ *of all rotations as homogeneous holonomy group.*

Next, we prove

PROPOSITION 3.7. *Let* (M,g) *be a Riemannian manifold equipped with a normal flow* $\mathfrak{F}_\xi$. *If* $\operatorname{rank} H = 2k < 2n$, *then* (M,g) *is locally a product of a* $(2k+1)$*-dimensional Riemannian manifold*, *with* $\mathfrak{F}_\xi$ *as a contact flow on it*, *and a Riemannian manifold.*

Proof. In this case, $H^2_{|\mathcal{H}}$ has a zero eigenvalue and the corresponding eigenspace $\mathcal{H}_0(m)$ coincides, for each m, with $\operatorname{Ker} H_{|\mathcal{H}}$ at m since h is skew-symmetric. Hence, using (4), we have $\nabla_X Y \in \mathcal{H}_0$ for all $X, Y \in \mathcal{H}_0$. So, the distribution $\mathcal{H}_0$ is involutive and its integral manifolds are totally geodesic.

Furthermore, let $\mathcal{H}_0^\perp$ denote the distribution which assigns to each $m \in M$ the orthogonal complement $\mathcal{H}_0^\perp(m)$ of $\mathcal{H}_0(m)$ in T_mM. Then $\mathcal{H}_0^\perp$ is differentiable, and moreover, $\mathcal{H}_0^\perp(m) = \operatorname{Im} H(m) \oplus \mathbb{R}\xi(m)$. From (4) we get, for all $X \in \mathcal{H}$, $U \in \mathcal{H}_0^\perp$ and $A \in \mathcal{H}_0$,

$$g(\nabla_U HX, A) = g(H\nabla_U X, A) = -g(\nabla_U X, HA) = 0,$$
$$g(\nabla_U \xi, A) = -g(HU, A) = g(U, HA) = 0$$

and so, $\mathcal{H}_0^\perp$ is involutive with totally geodesic integral manifolds. Then, for each $m \in M$, there exists a small open neighborhood $\mathcal{U}$ of m such that $\mathcal{U} = \mathcal{U}_0 \times \mathcal{U}_0^\perp$ where $\mathcal{U}_0$ and $\mathcal{U}_0^\perp$ are integral manifolds through m of $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$, respectively. $\mathcal{U}_0^\perp$ is equipped with a contact normal flow generated by ξ. ■

From Proposition 3.6 we obtain at once

COROLLARY 3.8. *A Riemannian manifold equipped with a normal contact flow and with parallel Ricci tensor is an Einstein space, and moreover,* $\tau = -(2n+1)\,\mathrm{tr}\, H^2$.

Finally, we have

PROPOSITION 3.9. *Let* $\mathfrak{F}_\xi$ *be a normal contact flow on a Riemannian manifold* (M, g). *Then*:

(i) $\mathfrak{F}_\xi$ *fibers locally over a Kähler manifold*;

(ii) *if* $\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}_1 \times \ldots \times \widetilde{\mathcal{U}}_r$ *is a base space of a local submersion, then each integral manifold* $\mathcal{U}_i$ *of* $\mathcal{D}_{c_i^2}$, $i = 1, \ldots, r$, *is homothetic to a Sasakian manifold fibering over the Kähler submanifold* $\widetilde{\mathcal{U}}_i$ *of* $\widetilde{\mathcal{U}}$.

Proof. Since $\mathfrak{F}_\xi$ is a contact flow, all c_i^2 are non-zero. Then

$$J = c_1^{-1}\widetilde{H}_1 \times \ldots \times c_r^{-1}\widetilde{H}_r,$$

where $\widetilde{H}_i = \widetilde{H} \circ p_i$ (p_i being the projection $p_i : \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}_i$) is an almost complex structure on $\widetilde{\mathcal{U}}$ and $\widetilde{g}$ is a Hermitian metric. Moreover, $(\widetilde{\mathcal{U}}, \widetilde{g}, J)$ is a Kähler manifold because $\widetilde{\nabla}\widetilde{H} = 0$.

The second part of the proposition follows at once. ∎

We note that this result implies that for a contact locally KTS-space, each $\widetilde{\mathcal{U}}$ is a Riemannian product of Hermitian symmetric spaces $\widetilde{\mathcal{U}}_i$, $i = 1, \ldots, r$.

In what follows for normal contact flows, we put $\dim \mathcal{H}_{c_i^2}(m) = 2n_i$, $i = 1, \ldots, r$. Then $n = \sum_{i=1}^r n_i$. We denote such a manifold by $M^{2n+1}(c_1^2, \ldots, c_r^2; n_1, \ldots, n_r)$, $1 \le r \le n$, when necessary. Then a Sasakian manifold may be denoted by $M^{2n+1}(1; n)$.

4. Invariants and inequalities. In this section we use some classical scalar curvature invariants together with some new invariants to characterize some special Riemannian manifolds. We first introduce these new invariants. Let (M, g) be a Riemannian manifold equipped with an isometric flow $\mathfrak{F}_\xi$ and consider the following real-valued functions on M:

$$\|R_H\|^2 = \sum_{i,j,k,l} R^2_{ijkHl}, \qquad \|\varrho_H\|^2 = \sum_{i,j} \varrho^2_{iHj},$$
$$\langle h^2, \varrho\rangle = \sum_i \varrho_{H^2 e_i e_i}, \qquad \langle h^4, \varrho\rangle = \sum_i \varrho_{H^4 e_i e_i},$$
$$\mathrm{tr}\, H^2, \qquad \mathrm{tr}\, H^4$$

where $\{e_i \mid i = 1, \ldots, n\}$ is an arbitrary orthonormal basis for each $T_m M$, $m \in M$, and where we adopt the usual notational convention. From (1) it is easily seen that these functions are invariant for any ξ-preserving isometry of $(M, g, \mathfrak{F}_\xi)$.

Now, we derive some inequalities and characterizations. In [9], the following result is proved:

LEMMA 4.1. *A Riemannian manifold (M, g) equipped with a normal flow $\mathfrak{F}_\xi$ is a locally KTS-space if and only if for all vector fields X, Y, Z, V, W we have*

$$\begin{aligned}(\nabla_V R)_{XYZW} \\ = \eta(W)\{&-g(H^2X, V)g(HY, Z) + g(H^2Y, V)g(HX, Z) + R_{XYVHZ}\} \\ &+ \eta(X)\{g(HV, Z)g(H^2Y, W) - g(H^2Y, Z)g(HV, W) - R_{YHVZW}\} \\ &+ \eta(Y)\{g(V, HZ)g(H^2X, W) + g(H^2X, Z)g(HV, W) + R_{XHVZW}\} \\ &+ \eta(Z)\{g(V, H^2X)g(HY, W) - g(V, H^2Y)g(HX, W) - R_{XYVHW}\} \\ &+ \eta(V)\eta(W)\{\eta(Y)g(H^3X, Z) - \eta(X)g(H^3Y, Z)\} \\ &+ \eta(V)\eta(Z)\{\eta(X)g(H^3Y, W) - \eta(Y)g(H^3X, W)\}.\end{aligned}$$

From this we get

THEOREM 4.2. *Let (M, g) be a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$. Then*

$$\|\nabla R\|^2 \geq 4\{\|R_H\|^2 - 2\operatorname{tr} H^4 \operatorname{tr} H^2 - 4\langle h^4, \varrho\rangle\} \tag{14}$$

and equality holds if and only if $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space.

Proof. Denote by A the right-hand side of the expression for ∇R given in Lemma 4.1 and define the $(0,5)$-tensor T by

$$T(V, X, Y, Z, W) = (\nabla_V R)(X, Y, Z, W) - A(V, X, Y, Z, W).$$

Then by using (5) and (12), we obtain

$$\begin{aligned}\|T\|^2 = \|\nabla R\|^2 - 4\|R_H\|^2 + 8\operatorname{tr} H^4 \operatorname{tr} H^2 - 16\operatorname{tr} H^6 \\ + 8\sum_{i,j}(R_{H^2ijiH^2j} + R_{iHjHjH^2i}).\end{aligned}$$

Furthermore, applying (10), we get

$$\sum_{i,j} R_{H^2ijiH^2j} = \sum_{i,j} R_{iHjHjH^2i} = \langle h^4, \varrho\rangle + \operatorname{tr} H^6.$$

The required result then follows easily. ∎

Next, from (12) we obtain

$$(\nabla_W \varrho)(U, \xi) = \varrho(U, HW) + g(U, HW)\operatorname{tr} H^2$$

and hence, by using (11), we get

LEMMA 4.3. *A Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ is η-parallel if and only if for all vector fields U, V, W we have*

$$(\nabla_W \varrho)(U,V) = \eta(U)\{\varrho(V,HW) + g(V,HW)\operatorname{tr} H^2\} + \eta(V)\{\varrho(U,HW) + g(U,HW)\operatorname{tr} H^2\}. \tag{15}$$

Proceeding now as in the proof of Theorem 4.2, we obtain

THEOREM 4.4. *On a Riemannian manifold (M,g) equipped with a normal flow $\mathfrak{F}_\xi$ we have*

$$\|\nabla\varrho\|^2 \ge 2\{\|\varrho_H\|^2 - (\operatorname{tr} H^2)^3 - 2\langle h^2, \varrho\rangle \operatorname{tr} H^2\}$$

and equality holds if and only if $(M,g,\mathfrak{F}_\xi)$ is η-parallel.

Now, we derive a similar characterization for η-Einstein manifolds. We have

THEOREM 4.5. *On a Riemannian manifold (M,g) equipped with a normal flow $\mathfrak{F}_\xi$ we have*

$$\|\varrho\|^2 \ge \frac{1}{n-1}(\tau - \operatorname{tr} H^2)^2 + (\operatorname{tr} H^2)^2 + 4\langle h^2, \varrho\rangle - 4\operatorname{tr} H^4$$

and equality holds if and only if $(M,g,\mathfrak{F}_\xi)$ is η-Einsteinian.

Proof. As is well-known, for an m-dimensional Riemannian manifold we always have $m\|\varrho\|^2 \ge \tau^2$ and equality holds if and only if the manifold is an Einstein space. Now,

$$\|\widetilde{\varrho}\|^2 = \|\varrho\|^2 - (\operatorname{tr} H^2)^2 + 4\operatorname{tr} H^4 - 4\langle h^2, \varrho\rangle.$$

Then the result follows from $(n-1)\|\widetilde{\varrho}\|^2 \ge \widetilde{\tau}^2$, (7), (8) and the definition of an η-Einstein space. ■

Furthermore, it is easy to check the following

LEMMA 4.6. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with a normal contact flow. Then (M,g) is an Einstein space if and only if*

$$\varrho_H = -(\operatorname{tr} H^2)h$$

where ϱ_H is given by $\varrho_H(U,V) = \varrho(HU,V)$ for all tangent vectors U,V.

A similar procedure as above then yields, by taking $T = \varrho_H + (\operatorname{tr} H^2)h$,

THEOREM 4.7. *Let (M^{2n+1}, g) be a Riemannian manifold equipped with a normal contact flow. Then*

$$\|\varrho_H\|^2 \ge (\operatorname{tr} H^2)^3 + 2\langle h^2, \varrho\rangle \operatorname{tr} H^2$$

where equality holds if and only if (M,g) is an Einstein space.

When $(M^{2n+1}, g, \mathfrak{F}_\xi)$ is of type $M(1;n)$ (that is, the manifold has a Sasakian structure (ξ, η, φ, g)), we have, putting $H = \varphi$,

$$\|R_\varphi\|^2 = \|R\|^2 - 4n, \qquad \|\varrho_\varphi\|^2 = \|\varrho\|^2 - 4n^2, \qquad \langle h^2, \varrho\rangle = -\langle h^4, \varrho\rangle = 2n - \tau.$$

So, as a corollary, we obtain the following already known results (see [18], [19] and [5] for more details and applications):

Corollary 4.8. *Let $(M^{2n+1}, \xi, \eta, \varphi, g)$ be a Sasakian manifold. Then:*

(i) $\|\nabla R\|^2 \geq 4(\|R\|^2 - 4\tau + 8n^2 + 4n)$ *and equality holds if and only if the manifold is locally φ-symmetric*;

(ii) $\|\nabla \varrho\|^2 \geq 2(\|\varrho\|^2 - 4n\tau + 4n^2(2n+1))$ *and equality holds if and only if the manifold is η-parallel*;

(iii) $\|\varrho\|^2 \geq (1/(2n))(\tau - 2n)^2 + 4n^2$ *and, for $n > 1$, equality holds if and only if the manifold is an η-Einstein space*;

(iv) $\|\varrho\|^2 \geq 4n(\tau - n(2n+1))$ *and equality holds if and only if the manifold is an Einstein space.*

We finish this section with a first application of Theorem 4.2 by providing an alternative proof of a result given in [9].

Corollary 4.9. *A locally symmetric space equipped with a normal contact flow is a space form.*

Proof. First, we prove that the ξ-sectional curvature is constant. From (12) we obtain, for horizontal vectors U, V,

$$\begin{aligned}(\nabla_{HV} R)(U, HV, \xi, HU) &+ (\nabla_V R)(U, \xi, HV, H^2U)\\ &= g(HU, HU)g(H^2V, H^2V) - g(HV, HV)g(H^2U, H^2U).\end{aligned}$$

Since $\nabla R = 0$, we get

$$\|HU\|^2\|H^2V\|^2 = \|HV\|^2\|H^2U\|^2.$$

So, let $U \in \mathcal{H}_{c_i^2}$ and $V \in \mathcal{H}_{c_j^2}$ for $i \neq j$. Then we get at once $c_i^2 = c_j^2$. This means that the ξ-sectional curvature is a pointwise, and hence global, constant. We denote this constant by c^2. Then we get

$$\|R_H\|^2 = c^2(\|R\|^2 - 4nc^4), \qquad \langle h^4, \varrho\rangle = c^4(\tau - 2nc^2)$$

and so, (14) reduces to

$$0 \geq \|R\|^2 + 4c^2(n(2n+1)c^2 - \tau).$$

Furthermore, on a $(2n+1)$-dimensional manifold we always have

$$\|R\|^2 \geq \tau^2\{n(2n+1)\}^{-1}$$

and equality holds if and only if the manifold is a space of constant curvature. From this, we get

$$0 \geq \|R\|^2 + 4c^2(n(2n+1)c^2 - \tau) \geq (\tau - 2n(2n+1)c^2)^2\{n(2n+1)\}^{-1} \geq 0.$$

This implies $\|R\|^2 = \tau^2\{n(2n+1)\}^{-1}$ and the result follows. ∎

It also follows from the proof that

$$\tau = 2n(2n+1)c^2$$

and this means that the sectional curvature equals c^2. Hence, for $c^2 = 1$, we obtain the well-known result that a locally symmetric Sasakian manifold is a space of constant curvature $+1$.

REMARK 4.10. Combining Theorems 4.4 and 4.7, we may also obtain an alternative proof of Corollary 3.8.

5. Curvature homogeneous flows, flow model spaces and locally KTS-spaces. A Riemannian manifold (M, g) is said to be *curvature homogeneous* [20] if and only if, for each pair of points p and q in M, there exists a linear isometry $F : T_pM \to T_qM$ such that $F^*R_q = R_p$. This is equivalent to the fact that, with respect to suitable orthonormal bases for the tangent spaces T_pM and T_qM, R_p and R_q must have the same components. Furthermore, given a homogeneous space $(M' = G/H, g')$ with G-invariant metric g' and curvature tensor R', (M, g) is said to have the same curvature tensor as (M', g') if for every point $p \in M$ and fixed point $o \in M'$, there exists a linear isometry $F : T_pM \to T_oM'$ such that $F^*R'_o = R_p$. In this case (M, g) is curvature homogeneous and (M', g') is called a *model space* of (M, g). We refer to [4] for a survey about curvature homogeneity and model spaces.

Now, let $(M, g, \mathfrak{F}_\xi)$ be a Riemannian manifold equipped with an isometric flow $\mathfrak{F}_\xi$ and let $A(M)$ denote the group of all isometries of (M, g) which leave ξ invariant. Then $\mathfrak{F}_\xi$ is called a *homogeneous flow* if $A(M)$ acts transitively on M. Note that the flow on a KTS-space is necessarily a homogeneous flow [10].

Furthermore, $\mathfrak{F}_\xi$ is said to be a *curvature homogeneous flow* if (M, g) is curvature homogeneous and if in addition the linear isometries F preserve ξ. So, a homogeneous flow is automatically a curvature homogeneous flow. Next, let (M', g') be equipped with a homogeneous flow $\mathfrak{F}_{\xi'}$. We call $(M', g', \mathfrak{F}_{\xi'})$ a *flow model space* of $(M, g, \mathfrak{F}_\xi)$ if for every point $p \in M$ and a fixed point $o \in M'$, there exists a linear isometry $F : T_pM \to T_oM'$ such that $F_*\xi = \xi'$ and $F^*R'_o = R_p$.

Now, let $(M', g', \mathfrak{F}_{\xi'})$ be a flow model space of $(M, g, \mathfrak{F}_\xi)$. Clearly, if $\mathfrak{F}_{\xi'}$ is normal, then so is $\mathfrak{F}_\xi$. Furthermore, since the ξ-sectional curvatures are preserved, it follows that $\operatorname{rank} H' = \operatorname{rank} H$ and so, $\mathfrak{F}_\xi$ is a contact flow if and only if $\mathfrak{F}_{\xi'}$ is a contact flow, or equivalently, by using Propositions 3.6 and 3.7, (M, g) is locally irreducible if and only if (M', g') is locally irreducible. Moreover, $(M', g', \mathfrak{F}_{\xi'})$ and $(M, g, \mathfrak{F}_\xi)$ have the same configuration $(c_1^2, \ldots, c_r^2; n_1, \ldots, n_r)$ with $F\mathcal{H}_{c_i^2} = \mathcal{H}'_{c_i^2}$, $i = 1, \ldots, r$. Note that we do not know if F preserves H (see also the particular cases at the end of this section).

Now, we state the main theorem of this section.

THEOREM 5.1. *Let $\mathfrak{F}_\xi$ be a normal flow on a Riemannian manifold (M, g) with η-parallel Ricci tensor. If $(M, g, \mathfrak{F}_\xi)$ has a KTS-space as a flow model space, then it is a locally KTS-space.*

Note that a locally KTS-space is necessarily η-parallel.

In the proof of this theorem we use the following lemma.

LEMMA 5.2. *Let $\mathfrak{F}_\xi$ be a normal flow on a Riemannian manifold (M, g) and let $(M', g', \mathfrak{F}_{\xi'})$ be a flow model space of $(M, g, \mathfrak{F}_\xi)$. Then*

$$\|R'_{H'}\|^2 = \|R_H\|^2, \qquad \|\varrho'_{H'}\|^2 = \|\varrho_H\|^2,$$
$$\langle h'^2, \varrho' \rangle = \langle h^2, \varrho \rangle, \qquad \langle h'^4, \varrho' \rangle = \langle h^4, \varrho \rangle.$$

Proof. Proposition 3.7 implies that it suffices to prove the lemma for contact flows. So, we suppose that $\mathfrak{F}_\xi$ and $\mathfrak{F}_{\xi'}$ are both contact flows. Then $(M, g, \mathfrak{F}_\xi)$ and $(M', g', \mathfrak{F}_{\xi'})$ have the same configuration $(c_1^2, \ldots, c_r^2; n_1, \ldots, n_r)$. Now, denote by $\widetilde{\tau}_i$, $\widetilde{\tau}'_i$, $\widetilde{\varrho}_i$ and $\widetilde{\varrho}'_i$, $i = 1, \ldots, r$, the corresponding scalar curvatures and Ricci tensors of $\widetilde{\mathcal{U}}_i$ and $\widetilde{\mathcal{U}}'_i$, respectively. Using (7), we then get

$$\widetilde{\tau}_i = 4n_i c_i^2 + \sum_{\alpha=1}^{2n_i} \varrho_{\alpha\alpha}.$$

Hence, it follows at once that $\widetilde{\tau}_i = \widetilde{\tau}'_i$ for $i = 1, \ldots, r$. Furthermore applying (7) again, we get

$$\langle h^2, \varrho \rangle = -\sum_{i=1}^{r} c_i^2 \widetilde{\tau}_i + 2 \operatorname{tr} H^4, \qquad \langle h^4, \varrho \rangle = \sum_{i=1}^{r} c_i^4 \widetilde{\tau}_i + 2 \operatorname{tr} H^6$$

and this implies $\langle h^2, \varrho \rangle = \langle h'^2, \varrho' \rangle$, $\langle h^4, \varrho \rangle = \langle h'^4, \varrho' \rangle$. Next, a similar computation yields

$$\|\varrho_H\|^2 = \sum_{i=1}^{r} c_i^2 (\|\widetilde{\varrho}_i\|^2 - 4c_i^2 \widetilde{\tau}_i) - 4 \operatorname{tr} H^6$$

and moreover, we have

$$\|\widetilde{\varrho}_i\|^2 = \sum_{\alpha,\beta=1}^{2n_i} \varrho_{\alpha\beta}^2 + 4c_i^2(\widetilde{\tau}_i - 2n_i c_i^2).$$

So, we get $\|\widetilde{\varrho}_i\|^2 = \|\widetilde{\varrho}'_i\|^2$, and hence $\|\varrho_H\|^2 = \|\varrho'_{H'}\|^2$.

Next, using (3), we obtain

$$\|R_H\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{2n} R^2_{\alpha\beta\gamma H\delta} - 2\operatorname{tr} H^6$$

and so, by using (6), we have

$$\|R_H\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{2n} \widetilde{R}^2_{\alpha\beta\gamma\widetilde{H}\delta} - 6\operatorname{tr} H^4 \operatorname{tr} H^2 - 8\operatorname{tr} H^6 + 4\sum_{\alpha,\beta=1}^{2n} (\widetilde{R}_{\alpha\widetilde{H}^2\beta\widetilde{H}\alpha\widetilde{H}\beta} + \widetilde{R}_{\alpha\widetilde{H}\alpha\widetilde{H}^2\beta\widetilde{H}\beta}).$$

Using (13) and the first Bianchi identity, this may be written in the form

$$\|R_H\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{2n} \widetilde{R}^2_{\alpha\beta\gamma\widetilde{H}\delta} - 12\sum_{i=1}^{r} c_i^4\widetilde{\tau}_i - 6\operatorname{tr} H^4 \operatorname{tr} H^2 - 8\operatorname{tr} H^6. \tag{16}$$

But

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{2n} \widetilde{R}^2_{\alpha\beta\gamma\widetilde{H}\delta} = \sum_{i=1}^{r} c_i^2\|\widetilde{R}_i\|^2 \tag{17}$$

and from (6) and (13) we get

$$\|\widetilde{R}_i\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{2n_i} R^2_{\alpha\beta\gamma\delta} + 12c_i^2\{\widetilde{\tau}_i - n_i(2n_i+1)c_i^2\}.$$

Hence, this yields $\|\widetilde{R}_i\|^2 = \|\widetilde{R}'_i\|^2$ and so, from this, (16) and (17) we obtain

$$\|R_H\|^2 = \sum_{i=1}^{r} c_i^2(\|\widetilde{R}_i\|^2 - 12c_i^2\widetilde{\tau}_i) - 6\operatorname{tr} H^4 \operatorname{tr} H^2 - 8\operatorname{tr} H^6$$

from which we have $\|R_H\|^2 = \|R'_{H'}\|^2$. This completes the proof of the lemma. ∎

Remark 5.3. It follows from the proof of this lemma that if $\mathfrak{F}_\xi$ is a curvature homogeneous normal flow, then the invariants $\|R_H\|^2$, $\|\varrho_H\|^2$, $\langle h^2, \varrho\rangle$ and $\langle h^4, \varrho\rangle$ are constant.

Now, we turn to the

Proof of Theorem 5.1. Since $(M', g', \mathfrak{F}_{\xi'})$ is a KTS-space, it follows from Theorem 4.2 that

$$\|\nabla' R'\|^2 = 4(\|R'_{H'}\|^2 - 2\operatorname{tr} H'^4 \operatorname{tr} H'^2 - 4\langle h'^4, \varrho'\rangle)$$

and hence, using Lemma 5.2, we get

$$\|\nabla' R'\|^2 = 4(\|R_H\|^2 - 2\operatorname{tr} H^4 \operatorname{tr} H^2 - 4\langle h^4, \varrho\rangle). \tag{18}$$

Next, we prove that $\|\nabla R\|^2 = \|\nabla' R'\|^2$. The required result then follows from (18) and Theorem 4.2. To prove this equality, we use the so-called Lichnerowicz formula, which, since $\|R\|^2 = \|R'\|^2$ is constant, may be written in the form (see [14], in particular (2.18), for more details)

$$\langle \nabla^2 \varrho, \bar{R} \rangle = \sum_{\alpha,\beta,\gamma,\delta} \nabla^2_{\alpha\beta} \varrho_{\gamma\delta} R_{\alpha\gamma\beta\delta} = -\frac{1}{4}\|\nabla R\|^2 - \frac{1}{2}\langle \varrho, \dot{R} \rangle + \check{\bar{R}} + \frac{1}{4}\check{R}$$

where

$$\begin{aligned} \check{R} &= \sum_{\alpha,\beta,\gamma,\delta,\lambda,\mu} R_{\alpha\beta\gamma\delta} R_{\gamma\delta\lambda\mu} R_{\lambda\mu\alpha\beta}, \\ \check{\bar{R}} &= \sum_{\alpha,\beta,\gamma,\delta,\lambda,\mu} R_{\alpha\gamma\beta\delta} R_{\gamma\lambda\delta\mu} R_{\lambda\alpha\mu\beta}, \\ \langle \varrho, \dot{R} \rangle &= \sum_{\alpha,\beta,\lambda,\mu,\nu} \varrho_{\alpha\beta} R_{\alpha\lambda\mu\nu} R_{\beta\lambda\mu\nu}. \end{aligned}$$

A detailed computation (long but straightforward, hence omitted here), using (3), (4) and (15), yields

$$\begin{aligned} \langle \nabla^2 \varrho, \bar{R} \rangle &= 3\langle h^4, \varrho \rangle + 3 \operatorname{tr} H^4 \operatorname{tr} H^2 \\ &\quad - \operatorname{tr} H^2 \sum_{\alpha,\beta} (R_{\alpha H\alpha\beta H\beta} + R_{\alpha H\beta\beta H\alpha}) \\ &\quad - \sum_{\alpha,\beta,\gamma} (R_{\alpha H\alpha\beta\gamma} \varrho_{\gamma H\beta} + R_{\alpha\gamma\beta H\alpha} \varrho_{\gamma H\beta}). \end{aligned} \tag{19}$$

Furthermore, from (10) and the first Bianchi identity, we obtain

$$\sum_{\alpha,\beta} (R_{\alpha H\alpha\beta H\beta} + R_{\alpha H\beta\beta H\alpha}) = 3\{2 \operatorname{tr} H^4 - \langle h^2, \varrho \rangle - (\operatorname{tr} H^2)^2\}$$

and from (6), (7) and (13) we get

$$\begin{aligned} \sum_{\alpha,\beta,\gamma} R_{\alpha\gamma\beta H\alpha} \varrho_{\gamma H\beta} &= \frac{1}{2} \sum_{\alpha,\beta,\gamma} R_{\alpha H\alpha\beta\gamma} \varrho_{\gamma H\beta} \\ &= \sum_{i=1}^{r} c_i^2 \|\widetilde{\varrho}_i\|^2 - 3\langle h^4, \varrho \rangle + 4 \operatorname{tr} H^6 - \operatorname{tr} H^2 \langle h^2, \varrho \rangle. \end{aligned}$$

Substituting these expressions in (19) yields

$$\begin{aligned} \langle \nabla^2 \varrho, \bar{R} \rangle = 3\Big\{ &-\sum_{i=1}^{r} c_i^2 \|\widetilde{\varrho}_i\|^2 + 4\langle h^4, \varrho \rangle + 2\langle h^2, \varrho \rangle \operatorname{tr} H^2 \\ &- 4 \operatorname{tr} H^6 - \operatorname{tr} H^4 \operatorname{tr} H^2 + (\operatorname{tr} H^2)^3 \Big\}. \end{aligned}$$

So, we get $\langle \nabla^2 \varrho, \bar{R} \rangle = \langle \nabla'^2 \varrho', \bar{R}' \rangle$ and then $\|\nabla R\|^2 = \|\nabla' R'\|^2$. This completes the proof of Theorem 5.1. ∎

In Theorem 5.1, we did not yet prove that $(M, g, \mathfrak{F}_\xi)$ is locally isometric to the model space. Note that for the result in [26] mentioned in the introduction, the local isometry followed immediately. Here, this seems to be more difficult. In this respect, we state the following result [11]:

PROPOSITION 5.4. *Let $(M_1, g_1, \mathfrak{F}_{\xi_1})$ and $(M_2, g_2, \mathfrak{F}_{\xi_2})$ be locally KTS-spaces and $o_1 \in M_1$, $o_2 \in M_2$. Further, let $L : T_{o_1}M_1 \to T_{o_2}M_2$ be a linear isometry satisfying*

(i) $L\xi_1 = \xi_2$,
(ii) $L \circ H_1 = H_2 \circ L$,
(iii) $LR_{1UV}W = R_{2LULV}LW$,

for all $U, V, W \in T_{o_1}M_1$. Then there exists an isometry f of a neighborhood $\mathcal{U}_1$ of o_1 onto a neighborhood $\mathcal{U}_2$ of o_2 such that $f(o_1) = o_2$, $f_*\xi_1 = \xi_2$ and $f_{*o_1} = L$.

From this we may conclude that if in Theorem 5.1 we also have $F \circ H = H' \circ F$, then $(M, g, \mathfrak{F}_\xi)$ and $(M', g', \mathfrak{F}_{\xi'})$ are locally isometric. This situation occurs in the following cases:

(i) $\mathfrak{F}_\xi$ is a normal contact flow and the configuration is of type $(c_1^2, \dots, c_r^2; 1, \dots, 1)$;
(ii) $\dim M = 3$;
(iii) $\dim M = 5$ and the ξ-sectional curvature is not pointwise constant, that is, the manifold is not homothetic to a Sasakian manifold;
(iv) $\operatorname{rank} H \leq 2$ (in particular, $\dim M = 4$).

Furthermore, using the fact that a 3-dimensional Sasakian manifold is a locally φ-symmetric space if and only if the scalar curvature is constant [3], [28], we immediately have

THEOREM 5.5. *Let $(M, g, \varphi, \xi, \eta)$ be a three-dimensional Sasakian manifold whose curvature tensor is the same as that of a φ-symmetric space $(M', \xi', \eta', \varphi', g')$. Then $(M, \xi, \eta, \varphi, g)$ is locally φ-symmetric. If, moreover, $(M', \xi', \eta', \varphi', g')$ is a flow model space of $(M, \xi, \eta, \varphi, g)$, then both manifolds are locally isometric.*

REMARK 5.6. In [6], Theorem 5.1 has also been proved for Sasakian manifolds and a φ-symmetric space (M', g') without imposing that (M', g') is a flow model space.

We finish with an application of Theorem 5.1 for two interesting classes of manifolds. First, we recall that a *D'Atri space* is a Riemannian manifold all of whose local geodesic symmetries are volume-preserving (up to sign). Furthermore, a Riemannian manifold is said to be a *C-space* if all Jacobi operators have constant eigenvalues along the corresponding geodesics. We refer to [16], [27] for more details and further references. In particular, for both kinds of spaces the Ricci tensor is a Killing tensor (that is, $\nabla\varrho$ is cyclic-parallel). In [13] we considered such spaces when they are equipped with a normal or a normal contact flow. In this last case, it turns out that D'Atri and C-spaces are η-parallel. Hence, we have

COROLLARY 5.7. *Any D'Atri or C-space which is equipped with a normal contact flow and which has a KTS-space as flow model space is a locally KTS-space.*

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