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KILLING TENSORS AND EINSTEIN–WEYL GEOMETRY

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Abstract. We give a description of compact Einstein–Weyl manifolds in terms of Killing tensors.

0. Introduction. In this paper we investigate compact Einstein–Weyl structures $(M,[g],D)$. In the first part we consider the Killing tensors on a Riemannian manifold (M,g). We prove that if a Killing tensor S has two eigenfunctions λ,μ such that $\dim\ker(S-\lambda I)=1$ and μ is constant then any section ξ of the bundle $D_\lambda=\ker(S-\lambda I)$ such that $g(\xi,\xi)=|\lambda-\mu|$ is a Killing vector field on (M,g). We prove that if (M,g) is compact and simply connected then every Killing tensor field with at most two eigenvalues λ,μ at each point of M such that μ is constant and $\dim D_\lambda\le 1$ admits a Killing eigenfield $\xi\in\mathfrak{iso}(M)$ $(S\xi=\lambda\xi)$. We also show that if the Ricci tensor of an $\mathcal{A}$-manifold has at most two eigenvalues at each point then these eigenvalues have to be constant on the whole of M.

In the second part we apply our results concerning Killing tensors and give a detailed description of compact Einstein–Weyl manifolds as a special kind of $\mathcal{A}\oplus\mathcal{C}^\perp$-manifolds first defined by A. Gray ([6]) (see also [1]). We show that the Ricci tensor of the standard Riemannian structure (M,g_0) of an Einstein–Weyl manifold $(M,[g],D)$ can be represented as $S+\Lambda\,\mathrm{Id}_{TM}$ where S is a Killing tensor and Λ is a smooth function on M. We prove that for compact simply connected manifolds there is a 1-1 correspondence between $\mathcal{A}\oplus\mathcal{C}^\perp$-Riemannian structures whose Ricci tensor has at most two eigenvalues at each point satisfying certain additional conditions and Einstein–Weyl structures. We also prove that if $(M,[g],D)$ is a compact Einstein–Weyl manifold with $\dim M\ge 4$ which is not conformally Einstein then the conformal scalar curvature s^D of $(M,[g],D)$ is nonnegative and that the center of the Lie algebra of the isometry group of the standard Riemannian structure (M,g_0) of $(M,[g],D)$ is nontrivial. Our results rely on some results of P. Gauduchon [3] and H. Pedersen and A. Swann ([9], [10]).

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1. Preliminaries. Let (M,g) be a smooth connected Riemannian manifold. Abusing the notation we sometimes write $\langle X,Y\rangle = g(X,Y)$. We denote by ∇ the Levi-Civita connection of (M,g). For a tensor $T(X_1,\ldots,X_k)$ we define another tensor $\nabla T(X_0,X_1,\ldots,X_k)$ by $\nabla T(X_0,X_1,\ldots,X_k) = \nabla_{X_0}T(X_1,\ldots,X_k)$. By a *Killing tensor* on M (we also call such tensors *$\mathcal{A}$-tensors*) we mean an endomorphism $S\in\operatorname{End}(TM)$ satisfying the following conditions:

$$\langle SX,Y\rangle = \langle X,SY\rangle \quad \text{for all } X,Y\in TM, \tag{1.1}$$

$$\langle \nabla S(X,X),X\rangle = 0 \quad \text{for all } X\in TM. \tag{1.2}$$

We also write $S\in\mathcal{A}$ if S is a Killing tensor. We call S a *proper $\mathcal{A}$-tensor* if $\nabla S\neq 0$. We denote by Φ the tensor defined by $\Phi(X,Y) = \langle SX,Y\rangle$.

We start with:

PROPOSITION 1.1. *For an endomorphism $S\in\operatorname{End}(TM)$, the following conditions are equivalent*:

(a) *the tensor S is an $\mathcal{A}$-tensor on (M,g).*

(b) *for every geodesic γ on (M,g), the function $\Phi(\gamma'(t),\gamma'(t))$ is constant on* $\operatorname{dom}\gamma$;

(c) *the condition*

$$\nabla_X\Phi(Y,Z)+\nabla_Y\Phi(Z,X)+\nabla_Z\Phi(X,Y) = 0 \tag{A}$$

is satisfied for all $X,Y,Z\in\mathfrak{X}(M)$.

Proof. By using polarization it is easy to see that (a) is equivalent to (c). Let now $X\in T_{x_0}M$ be any vector from TM and γ be a geodesic satisfying the initial condition $\gamma'(0) = X$. Then

$$\frac{d}{dt}\Phi(\gamma'(t),\gamma'(t)) = \nabla_{\gamma'(t)}\Phi(\gamma'(t),\gamma'(t)). \tag{1.3}$$

Hence $\frac{d}{dt}\Phi(\gamma'(t),\gamma'(t))_{t=0} = \nabla\Phi(X,X,X)$. The equivalence (a)$\Leftrightarrow$(b) follows immediately from the above relations. ∎

As in [2] define the integer-valued function $E_S(x) =$ (the number of distinct eigenvalues of S_x) and set $M_S = \{x\in M : E_S$ is constant in a neighbourhood of $x\}$. The set M_S is open and dense in M and the eigenvalues λ_i of S are distinct and smooth in each component U of M_S. The eigenspaces $D_\lambda = \ker(S-\lambda I)$ form smooth distributions in each component U of M_S. By ∇f we denote the gradient of a function f (i.e. $\langle\nabla f,X\rangle = df(X)$) and by $\Gamma(D_\lambda)$ (resp. $\mathfrak{X}(U)$) the set of all local sections of the bundle D_λ (resp. all local vector fields on U). Note that if $\lambda\neq\mu$ are eigenvalues of S then D_λ is orthogonal to D_μ.

THEOREM 1.2. *Let S be an $\mathcal{A}$-tensor on M and U be a component of M_S and $\lambda_1,\ldots,\lambda_k\in C^\infty(U)$ be eigenfunctions of S. Then for all $X\in D_{\lambda_i}$*

we have

$$\nabla S(X,X) = -\frac{1}{2}(\nabla\lambda_i)\|X\|^2 \tag{1.4}$$

and $D_{\lambda_i} \subset \ker d\lambda_i$. *If* $i \neq j$ *and* $X \in \Gamma(D_{\lambda_i})$, $Y \in \Gamma(D_{\lambda_j})$ *then*

$$\langle \nabla_X X, Y\rangle = \frac{1}{2}\frac{Y\lambda_i}{\lambda_j - \lambda_i}\|X\|^2. \tag{1.5}$$

Proof. Let $X \in \Gamma(D_{\lambda_i})$ and $Y \in \mathfrak{X}(U)$. Then $SX = \lambda_i X$ and

$$\nabla S(Y,X) + (S - \lambda_i I)(\nabla_Y X) = (Y\lambda_i)X \tag{1.6}$$

and consequently,

$$\langle \nabla S(Y,X), X\rangle = (Y\lambda_i)\|X\|^2. \tag{1.7}$$

Taking $Y = X$ in (1.7) we obtain $0 = X\lambda_i\|X\|^2$ by (1.2). Hence $D_{\lambda_i} \subset \ker d\lambda_i$. Thus from (1.6) it follows that $\nabla S(X,X) = (\lambda_i I - S)(\nabla_X X)$. Condition (A) implies $\langle \nabla S(X,Y), Z\rangle + \langle \nabla S(Z,X), Y\rangle + \langle \nabla S(Y,Z), X\rangle = 0$, hence

$$2\langle \nabla S(X,X), Y\rangle + \langle \nabla S(Y,X), X\rangle = 0. \tag{1.8}$$

Thus, (1.8) yields $Y\lambda_i\|X\|^2 + 2\langle \nabla S(X,X), Y\rangle = 0$. Consequently, $\nabla S(X,X) = -\frac{1}{2}(\nabla\lambda_i)\|X\|^2$. Let now $Y \in \Gamma(D_{\lambda_j})$. Then

$$\nabla S(X,Y) + (S - \lambda_j I)(\nabla_X Y) = (X\lambda_j)Y. \tag{1.9}$$

It is also clear that $\langle \nabla S(X,X), Y\rangle = \langle \nabla S(X,Y), X\rangle = (\lambda_j - \lambda_i)\langle \nabla_X Y, X\rangle$. Thus,

$$Y\lambda_i\|X\|^2 = -2(\lambda_j - \lambda_i)\langle \nabla_X Y, X\rangle = 2(\lambda_j - \lambda_i)\langle Y, \nabla_X X\rangle$$

and (1.5) holds. ∎

Corollary 1.3. *Let* $S, U, \lambda_1, \ldots, \lambda_k$ *be as above and* $i \in \{1, \ldots, k\}$. *Then the following conditions are equivalent*:

(a) *For all* $X \in \Gamma(D_{\lambda_i})$, $\nabla_X X \in D_{\lambda_i}$.
(b) *For all* $X, Y \in \Gamma(D_{\lambda_i})$, $\nabla_X Y + \nabla_Y X \in D_{\lambda_i}$.
(c) *For all* $X \in \Gamma(D_{\lambda_i})$, $\nabla S(X,X) = 0$.
(d) *For all* $X, Y \in \Gamma(D_{\lambda_i})$, $\nabla S(X,Y) + \nabla S(Y,X) = 0$.
(e) λ_i *is a constant eigenvalue of* S.

Note that if $X, Y \in \Gamma(D_{\lambda_i})$ then

$$\nabla S(X,Y) - \nabla S(Y,X) = (\lambda_i I - S)([X,Y]) \tag{1.10}$$

since from Theorem 1.2 it follows that $X\lambda_i = Y\lambda_i = 0$. Hence the distribution D_{λ_i} is integrable if and only if $\nabla S(X,Y) = \nabla S(Y,X)$ for all $X, Y \in \Gamma(D_{\lambda_i})$. Consequently, we obtain

Corollary 1.4. *Let* $\lambda_i \in C^\infty(U)$ *be an eigenvalue of an* $\mathcal{A}$*-tensor* S. *Then on* U *the following conditions are equivalent*:

(a) D_{λ_i} *is integrable and* λ_i *is constant.*
(b) *For all* $X, Y \in \Gamma(D_{\lambda_i})$, $\nabla S(X,Y) = 0$.
(c) D_{λ_i} *is autoparallel.*

Proof. This follows from (1.4), (1.10), Corollary 1.3 and the relation $\nabla_X Y = \nabla_Y X + [X,Y]$. ■

A Riemannian manifold (M,g) is called an $\mathcal{A}$-*manifold* (see [6]) if the Ricci tensor ϱ of (M,g) satisfies the condition

$$\nabla_X \varrho(X,X) = 0 \tag{A1}$$

for all $X \in TM$, i.e. if ϱ is a Killing tensor. By an $\mathcal{A}\oplus\mathcal{C}^\perp$-*manifold* we mean a Riemannian manifold (M,g) whose Ricci tensor satisfies the condition

$$\nabla_X \varrho(X,X) = \frac{2}{n+2} X\tau g(X,X) \tag{A2}$$

for all $X \in TM$ where $n = \dim M$ and τ denotes the scalar curvature of (M,g). We have

LEMMA 1.5. *Let* (M,g) *be a Riemannian manifold. Then* $(M,g) \in \mathcal{A}\oplus\mathcal{C}^\perp$ *if and only if there exists a function* $s \in C^\infty(M)$ *such that*

$$\nabla_X \varrho(X,X) = Xsg(X,X). \tag{1.11}$$

If (1.11) *holds then* $d\big(s - \frac{2}{n+2}\tau\big) = 0$.

Proof. From (1.11) we get

$$\mathfrak{C}_{X,Y,Z} \nabla_X \varrho(Y,Z) = \mathfrak{C}_{X,Y,Z} Xsg(Y,Z)$$

where $\mathfrak{C}$ denotes the cyclic sum. Hence

$$2\nabla_X \varrho(X,Y) + \nabla_Y \varrho(X,X) = 2Xsg(X,Y) + Ysg(X,X). \tag{1.12}$$

Define $\delta\varrho(Y) = \operatorname{tr}_g \nabla_{\cdot}\varrho(\cdot, Y)$. Then $\delta\varrho = \frac{1}{2}d\tau$ (see for example [1]). On the other hand, taking account of (1.12) we have

$$2\delta\varrho(Y) + \operatorname{tr}\nabla_Y \varrho(\cdot,\cdot) = 2g(\nabla s, Y) + nYs. \tag{1.13}$$

Since $\operatorname{tr}\nabla_Y \varrho(\cdot,\cdot) = Y\tau$ we finally obtain $2d\tau = (n+2)ds$. ■

2. $\mathcal{A}$-tensors with two eigenvalues. In this section we characterize certain $\mathcal{A}$-tensors with two eigenvalues. We start with:

THEOREM 2.1. *Let* S *be an* $\mathcal{A}$-*tensor on* (M,g) *with exactly two eigenvalues* λ, μ *and a constant trace. Then* λ, μ *are constant on* M. *The distributions* D_λ, D_μ *are both integrable if and only if* $\nabla S = 0$.

Proof. Note first that $p = \dim\ker(S - \lambda I)$, $q = \dim\ker(S - \mu I)$ are constant on M as $M_S = M$. We also have $p\lambda + q\mu = \operatorname{tr} S$ and $\operatorname{tr} S$ is constant on M. Hence

$$p\nabla\lambda + q\nabla\mu = 0 \tag{2.1}$$

on M. Note that $\nabla\lambda \in \Gamma(D_\mu)$, $\nabla\mu \in \Gamma(D_\lambda)$ (see Th. 1.2) thus $\nabla\lambda = \nabla\mu = 0$ since $TM = D_\lambda \oplus D_\mu$. Now suppose that D_λ is integrable. We show that $\nabla S(X,Y) = 0$ and $\nabla_X Y \in D_\mu$ if $X \in D_\lambda$ and $Y \in D_\mu$. We have $\nabla S(X,Y) = (\mu I - S)(\nabla_X Y) \in D_\lambda$ as D_λ is orthogonal to D_μ. Let $Z \in \Gamma(D_\lambda)$; then for any $X \in \Gamma(D_\lambda)$, $Y \in \mathfrak{X}(M)$ we have

$$\langle \nabla S(X,Y), Z\rangle = \langle Y, \nabla S(X,Z)\rangle = 0$$

since $\nabla S(X,Z) = 0$ (see Corollary 1.4). Hence $\nabla S(X,Y) = 0$ and $\nabla_X Y \in D_\mu$ if $X \in D_\lambda$ and $Y \in D_\mu$. If D_μ is also integrable then in view of Corollary 1.4, $\nabla S = 0$. ∎

We have also proved in passing:

COROLLARY 2.2. *Let S be an $\mathcal{A}$-tensor on (M,g) with two constant eigenvalues λ,μ. If D_λ is integrable then $\nabla S(X,Y) = 0$ for all $X \in \Gamma(D_\lambda)$, $Y \in \Gamma(D_\mu)$.*

COROLLARY 2.3. *Let (M,g) be an $\mathcal{A}$-manifold whose Ricci tensor S has exactly two eigenvalues λ,μ. Then λ,μ are constant.*

Proof. It is well known that if (M,g) is an $\mathcal{A}$-manifold then S has constant trace $\operatorname{tr} S = \tau$ (see [6] or Lemma 1.5). ∎

From now on we investigate $\mathcal{A}$-tensors with two eigenvalues λ,μ satisfying additional conditions: μ is constant and $\dim D_\lambda = 1$. It follows that D_λ is integrable. We also assume that D_λ is orientable (this happens for example if $\pi_1(M)$ has no subgroups of index 2). Otherwise we may consider a manifold $(\overline{M},\overline{g})$ and an $\mathcal{A}$-tensor $\overline{S}$ on $\overline{M}$ such that there exists a two-fold Riemannian covering $p : \overline{M} \to M$ for which $dp \circ \overline{S} = S \circ dp$ and $\overline{D}_\lambda = \ker(\overline{S} - \lambda I)$ is orientable. Let $\xi \in \Gamma(D_\lambda)$ be a global section of D_λ such that $\langle \xi,\xi\rangle = 1$. Then we have:

LEMMA 2.4. *Let (M,g) be a Riemannian manifold and $S \in \mathcal{A}$. Assume that S has exactly two eigenfunctions λ,μ such that μ is constant and $\lambda \in C^\infty(M)$. Let $\xi \in \Gamma(D_\lambda)$ be a unit vector field. Then the section $\sqrt{|\lambda-\mu|}\xi$ is a Killing vector field on (M,g). On the other hand, if a Riemannian manifold (M,g) admits a Killing vector field ξ then it admits an $\mathcal{A}$-tensor S such that ξ is an eigenfield of S.*

Proof. Denote by T the endomorphism of TM defined by $TX = \nabla_X \xi$. If $\Phi(X,Y) = \langle SX,Y\rangle$ then $\Phi(\xi,X) = \lambda\langle \xi,X\rangle$. Hence

$$\nabla\Phi(Y,\xi,X) + \Phi(TY,X) = \lambda\langle TY,X\rangle + Y\lambda\langle \xi,X\rangle. \tag{2.2}$$

Take $X = Y \in D_\mu$ in (2.2). Since $\nabla S(X,X) = 0$ (μ is constant) we obtain $\Phi(TX,X) = \lambda\langle TX,X\rangle$. On the other hand, $SX = \mu X$. Consequently, $\Phi(TX,X) = \mu\langle TX,X\rangle$. Hence

$$\langle TX,X\rangle = 0, \quad X \in D_\mu. \tag{2.3}$$

We also have

$$\langle \nabla_\xi \xi, X\rangle = \frac{1}{2}\frac{X\lambda}{\mu-\lambda} = -\frac{1}{2}X(\ln|\mu-\lambda|).$$

We now show that the field $\eta = \sqrt{|\mu-\lambda|}\xi$ is Killing. From (2.3) it follows that $\langle \nabla_X \eta, X\rangle = 0$ for $X \in D_\mu$. Notice that

$$\langle \nabla_\eta \eta, X\rangle = \frac{1}{2}\frac{X\lambda}{\mu-\lambda}\langle \eta,\eta\rangle = \frac{1}{2}X\lambda\varepsilon$$

where $\varepsilon = \operatorname{sgn}(\mu-\lambda)$. Since $\langle \eta,\eta\rangle = |\mu-\lambda|$ we get $2\langle \nabla_X\eta,\eta\rangle = -X\lambda\varepsilon$. Consequently, for $X \in \Gamma(D_\mu)$,

$$\langle \nabla_\eta \eta, X\rangle + \langle \nabla_X \eta, \eta\rangle = 0.$$

Note that $\xi\sqrt{|\mu-\lambda|} = 0$ (since $D_\lambda \subset \ker\lambda$). Thus it is clear that $\langle \nabla_\eta\eta,\eta\rangle = 0$. It follows that η is a Killing vector field and $\eta \in \mathfrak{iso}(M)$.

Assume now that on a manifold (M,g) there exists a Killing vector field ξ and let $\alpha = \langle \xi,\xi\rangle$. Let μ be any real number and define a function $\lambda \in C^\infty(M)$ by

$$\lambda = \mu + \varepsilon\alpha$$

where $\varepsilon \in \{-1,1\}$. Then $|\lambda-\mu| = \langle \xi,\xi\rangle$. Define a $(1,1)$-tensor S on M as follows:

(a) $S\xi = \lambda\xi$,
(b) $SX = \mu X$ if $\langle X,\xi\rangle = 0$.

Then $S \in \mathcal{A}$. Note that the distribution $D = \{X : \langle X,\xi\rangle = 0\}$ is geodesic, i.e. if $X \in \Gamma(D)$ then $\nabla_X X \in \Gamma(D)$. It follows that $\nabla S(X,X) = 0$ if $X \in \Gamma(D)$.

Note also that

$$\nabla S(\xi,\xi) = -\tfrac{1}{2}\nabla\alpha\langle \xi,\xi\rangle.$$

Indeed, since $\xi\alpha = 0$, we have

$$\nabla S(\xi,\xi) + (S - (\mu+\varepsilon\alpha)\,\mathrm{Id})(\nabla_\xi\xi) = 0.$$

Since $\nabla_\xi\xi = -\frac{1}{2}\nabla\alpha$ and $\xi\alpha = 0$ we have $\nabla_\xi\xi \in \Gamma(D)$. Hence $\nabla S(\xi,\xi) - \varepsilon\alpha\nabla_\xi\xi = 0$ and consequently $\nabla S(\xi,\xi) = -\frac{1}{2}\varepsilon\alpha\nabla\alpha$.

It is clear that S is self-adjoint. Note that $\nabla S(X,\xi) + (S-\lambda\,\mathrm{Id})(\nabla_X\xi) = \varepsilon X\alpha\xi$. Thus

$$2\langle \nabla S(\xi,\xi),\xi\rangle + \langle \nabla S(X,\xi),\xi\rangle = -\varepsilon\alpha X\alpha + \varepsilon\alpha X\alpha = 0.$$

If $X,Y \in \Gamma(D)$ then $\nabla S(X,Y) + (S-\mu\,\mathrm{Id})(\nabla_X Y) = 0$. Hence

$$\begin{aligned}\langle \nabla S(X,Y),\xi\rangle &+ \langle \nabla S(\xi,X),Y\rangle + \langle \nabla S(Y,\xi),X\rangle \\ &= \varepsilon\alpha\langle \nabla_X Y,\xi\rangle + \varepsilon\alpha\langle \nabla_Y X,\xi\rangle = \varepsilon\alpha\langle \nabla_X Y + \nabla_Y X,\xi\rangle = 0.\end{aligned}$$

It is also clear that $\langle \nabla S(\xi,\xi),\xi\rangle = 0$ and $\mathfrak{C}_{X,Y,Z}\nabla S(X,Y,Z) = 0$ for $X,Y,Z \in \Gamma(D)$. Hence S is a Killing tensor. ■

In the sequel we need several facts concerning Killing vector fields. The first is well known.

LEMMA 2.5. *Let* $X \in \mathfrak{iso}(M)$. *If* c *is a geodesic in* (M,g) *then* $g(X,\dot{c})$ *is constant on* $\operatorname{dom} c$.

COROLLARY 2.6. *Let* $X \in \mathfrak{iso}(M)$ *and let* c *be a geodesic in* (M,g) *such that* $\lim_{t\to t_0} X\circ c(t) = 0$ *for a certain* $t_0 \in \operatorname{dom} c$. *Then* $g(X_{c(t)},\dot{c}(t)) = 0$ *for all* $t \in \operatorname{dom} c$.

LEMMA 2.7. *Let* (M,g) *be complete and* $X \in \mathfrak{X}(M)$, $X \in \mathfrak{iso}(M-\overline{U})$ *where* U *is an open subset of* M. *If* $X|_{\partial U} = 0$ *then* $X|_{M-\overline{U}} = 0$.

Proof. Let $x_0 \in U$ and let $x_1 \in M-\overline{U}$ be such that $X_{x_1} \neq 0$. Consider a geodesic $c(t)$ such that $c(0) = x_1$ and $c(1) = x_0$. From Corollary 2.6 it follows that $g(\dot{c}(0), X_{x_1}) = 0$. Let $V \subset U$ be a neighbourhood of x_0. Since $\exp_{x_1}\dot{c}(0) = x_0$ it follows that there exists a neighbourhood W of $\dot{c}(0)$ in $T_{x_0}M$ such that $\exp_{x_1}(W) \subset V$. Take a vector $Y \in W$ such that

$$g(Y,X_{x_1}) \neq 0. \tag{2.4}$$

The geodesic $d(t) = \exp tY$ intersects ∂U, hence $g(\dot{d}(t), X_{d(t)}) = 0$ if $d(t) \in M-\overline{U}$, a contradiction with (2.4). ■

Next we prove:

THEOREM 2.8. *Let* (M,g) *be a compact Riemannian manifold*, $U \subset M$ *be an open*, *nonempty subset of* M *and* $X \in \mathfrak{iso}(U)$ *be a Killing vector field on* U. *Assume also that there exists a function* $\phi \in C^\infty(M)$ *such that* $\phi|_U = g(X,X)$ *and* $N := M - U = \{x : \phi(x) = 0\}$. *Then* $\operatorname{int} N = \emptyset$ *and* X *extends to a Killing vector field* $\overline{X} \in \mathfrak{X}(M)$ *such that* $g(\overline{X},\overline{X}) = \phi$.

Proof. If $V = \operatorname{int} N \neq \emptyset$ then $X \in \mathfrak{iso}(M-\overline{V})$ and $X|_{\partial V} = 0$. From Lemma 2.7 it then follows that $X|_{M-\overline{V}} = 0$. Hence $\operatorname{int} N = \emptyset$.

The set $M-N$ is connected. If $M-N = U_1 \cup U_2$ where $U_1 \cap U_2 = \emptyset$ and U_i are open in $M-N$ hence in M then $\partial U_i \subset N$. If $U_i \neq \emptyset$ for $i = 1,2$ then $\partial U_i \neq \emptyset$ and we would have a contradiction with Lemma 2.7. Note that X extends to a continuous vector field $\overline{X}$ on M such that $\overline{X}|N = 0$.

Let ε be the radius of injectivity of (M,g). Assume that $\varepsilon' < \varepsilon$ and let $x_0 \in N_i$. Since $\operatorname{int} N = \emptyset$ there exists a point $x_1 \in M-N$ such that $d(x_1,x_0) < \varepsilon'$. Note that $\exp_{x_1} : V \to M$ where $V := \{v \in T_{x_1}M : \|v\| < \varepsilon\}$ is a diffeomorphism. Assume that $x_0 = \exp_{x_1} v$. Then $\|v\| < \varepsilon'$ and there exists $\eta > 0$ such that $V_1 := \{u \in T_{x_1}M : \|u-v\| < \eta\} \subset V$.

If $U_1 = \exp_{x_1} V_1$ then $(U_1, \exp_{x_1}^{-1})$ is a local chart on M. For $u \in V_1$, denote by $J_u(t)$ the Jacobi vector field along the geodesic $c_u(t) = \exp_{x_1} tu$ and satisfying the initial conditions

$$J_u(0) = X_{x_0}, \qquad J'_u(0) = (\nabla_u X)_{x_0}. \tag{2.5}$$

Define a vector field Y on U_1 by $Y(\exp u) = J_u(1)$. Since $J_u(1)$ depends smoothly on the parameter u (J_u is the solution of the differential equation $\nabla^2_{\dot{c}} J + R(J,\dot{c})\dot{c} = 0$ with initial conditions (2.5) depending smoothly on u) it follows that Y is a smooth vector field $Y \in \mathfrak{X}(U_1)$.

We show that $Y = \overline{X}|_{U_1}$. If a geodesic c_u does not intersect N it is clear that $Y \circ c_u = X \circ c_u$. In the other case, since $M - N$ is connected and $\operatorname{int} N = \emptyset$ we can approximate a geodesic intersecting N by geodesics c_u disjoint from N, which proves the result in general. Since x_0 was an arbitrary point from N it follows that $\overline{X}$ is a smooth extension of X. It is also clear that $\overline{X} \in \mathfrak{iso}(M)$. ■

LEMMA 2.9. *Assume that* (M,g) *is a compact, connected Riemannian manifold and* $\phi \in C^\infty(M)$ *is a function on* M *which is not identically* 0. *Let* $N = \{x : \phi(x) = 0\}$ *and let* D *be the* 1*-dimensional distribution over* $M - N$. *Assume also that for any unit local section* $\xi_V \in \Gamma(D|_V)$ *of* D *with* $\operatorname{dom} \xi_V = V$ *the field* $\eta_V = \sqrt{|\phi|}\xi_V$ *is Killing, i.e.* $\eta_V \in \mathfrak{iso}(V)$. *Let* $U_+ = \{x : \phi(x) > 0\}$ *and* $U_- = \{x : \phi(x) < 0\}$. *Then* $\operatorname{int} N = \emptyset$ *and either* $U_+ = \emptyset$ *or* $U_- = \emptyset$.

Proof. Assume for example that $U_+ \neq 0$. We show that $\operatorname{int} N \cup U_- = \emptyset$. Let $c : I = [a,b] \to M$ be a geodesic on M such that $\operatorname{im} c \subset M - N$. Then we can find open sets $\{U_1, \ldots, U_k\}$ such that $\operatorname{im} c \subset \bigcup U_i$ and $c([t_i, t_{i+1}]) \subset U_i$ where $a = t_1 < \ldots < t_k < t_{k+1} = b$ and there exist local sections $\xi_i = \xi_{U_i}$ of D such that $\|\xi_i\| = 1$. We can assume that $\xi_i = \xi_{i+1}$ on $U_i \cap U_{i+1}$. Define local Killing vector fields $\eta_i = \sqrt{|\phi|}\xi_i$. Note that ϕ has constant sign along c and $\eta_i|_{U_i \cap U_{i+1}} = \eta_{i+1}|_{U_i \cap U_{i+1}}$.

Define a vector field J along c by $J|_{c([t_i, t_{i+1}])} = \eta_i \circ c$. Then J is a well-defined Jacobi vector field along c. In particular, $g(J,\dot{c}) = \text{const}$ and $\|J\|^2 = |\phi|$. On the other hand, let $c : [a,b] \to M$ be a geodesic on M such that $c(a) \in M - N$ and $g(\dot{c}(a), \eta_V(a)) \neq 0$ where η_V is a local Killing vector field on $V \subset M - N$ constructed as above. Then $\operatorname{im} c \cap N = \emptyset$. Otherwise we would have an increasing sequence $\{t_i\}$ of real numbers such that $\lim_{i\to\infty} c(t_i) \in N$ and $c([a, t_i]) \subset M - N$. The Jacobi vector field J constructed as above would then satisfy two conditions:

(a) $g(J,\dot{c}) = g(J(a), \dot{c}(a)) \neq 0$, and
(b) $\|J(t_i)\|^2 = |\phi \circ c(t_i)| \to 0$,

which gives a contradiction.

Assume now that $U_+ \neq \emptyset$. Let $x_0 \in U_+$. Note that $\partial(N \cup U_-) \subset N$. Assume that $\operatorname{int} N \cup U_- \neq \emptyset$ and let $x_1 \in \operatorname{int} N \cup U_-$. Let $\eta = \eta_V$ be a local Killing vector field defined in the neighbourhood of the point x_0 and let $X = \eta_{x_0} \in T_{x_0}M$. Then, as in the proof of Lemma 2.7 we find a geodesic $d : I = [0,1] \to M$ and an open neighbourhood V_1 of the point x_1 such that

$g(\dot{d}(0), X) \neq 0$ and $d(1) \in V_1 \subset \operatorname{int} N \cup U_-$. In particular, $\operatorname{im} d \cap N \neq \emptyset$, which gives a contradiction with the above considerations. ∎

Our present aim is to prove :

THEOREM 2.10. *Assume that (M, g) is a compact manifold and S is a Killing tensor on M with two eigenfunctions λ, μ such that $\mu \in \mathbb{R}$ and $\lambda \in C^\infty(M)$. Assume also that on the set $U = \{x \in M : \lambda(x) \neq \mu\}$ the distribution $D_\lambda = \ker(S - \lambda I)$ satisfies the condition $\dim D_\lambda | U = 1$. Then there exists a two-fold Riemannian covering $p : (M', g') \to (M, g)$ and a Killing vector field $X' \in \mathfrak{iso}(M')$ such that $S'X' = (\lambda \circ p)X'$ where S' is the lift of S to M'. If $D_\lambda|_U$ is orientable or if M is simply connected then there exists a Killing vector field $X \in \mathfrak{iso}(M)$ such that $X \in \Gamma(D_\lambda)$. Furthermore, the function $\phi = \lambda - \mu$ has constant sign on U and U is dense in M.*

Proof. Note that for every point $x_0 \in U$ there exists an open neighbourhood V of x_0 such that $D_\lambda|_V$ is spanned by a unit vector field ξ_V. From Lemma 2.4 it follows that $X_V = \sqrt{|\lambda - \mu|}\,\xi_V$ is a Killing vector field on V and $X_V \in \Gamma(D_\lambda|_V)$. Note that $-X_V$ is also a Killing vector field satisfying the last condition. If $x_0 \in N := M - U = \{x : \lambda(x) = \mu\}$ then we can define $X|_V$ on a neighbourhood V of x_0 as in the proof of Theorem 2.8: $X(\exp u) = J_u(1)$ where $\exp_{x_1} u = x_0$ and $V = \exp V_1$, since we need X to be defined only in an arbitrary small neighbourhood of the point x_1. We also obtain in this way two possible Killing vector fields $X_V, -X_V$ on V. Hence for every $x_0 \in M$ we have a neighbourhood V of x_0 and two Killing vector fields $X_V, -X_V$ defined on V such that $X_V|_U = \sqrt{|\lambda - \mu|}\,\xi|_{V \cap U}$ where $\xi \in \Gamma(D_\lambda)$ and $\|\xi\| = 1$.

Consider the set of germs $M' = \{[X_V]_x : x \in M\}$ of local Killing vector fields (V, X_V) with the usual topology. Then $p : M' \to M$ where $p([X]_x) = x$ is a two-fold topological covering. We lift the structure of Riemannian manifold on M' from M. Then p is a Riemannian submersion (and a local isometry) and $p : (M', g') \to (M, g)$ is a two-fold Riemannian covering. We define a field X' on M' by $X'_{[X_V]_x} = X^l_x$ where X^l_x denotes the lift of $X_V(x) \in T_xM$ to $T_{[X]_x}M'$. It is clear that $X' \in \mathfrak{iso}(M')$ and that $(S' - \lambda \circ p \operatorname{Id})X' = 0$. If $D_\lambda|U$ is orientable then we can take in the above construction the germs of fields which agree with the orientation and then $p : M' \to M$ is an isometry. If M is simply connected then M' is a union of two components each of them isometric to M, which concludes the proof. ∎

3. Einstein–Weyl geometry and Killing tensors. We start with some basic facts concerning Einstein–Weyl geometry. For more details see [10], [9], [4], [3].

Let M be an n-dimensional manifold with a conformal structure $[g]$ and a torsion-free affine connection D. This defines an *Einstein–Weyl* (E–W) *structure* if D preserves the conformal structure, i.e. there exists a 1-form ω on M such that

$$Dg = \omega \otimes g \tag{3.1}$$

and the Ricci tensor ϱ^D of D satisfies the condition

$$\varrho^D(X,Y) + \varrho^D(Y,X) = \overline{\Lambda} g(X,Y) \quad \text{for every } X,Y \in TM$$

for some function $\overline{\Lambda} \in C^\infty(M)$. P. Gauduchon proved ([5]) the fundamental theorem that if M is compact then there exists a Riemannian metric $g_0 \in [g]$ for which $\delta\omega_0 = 0$ and g_0 is unique up to homothety. We call g_0 the *standard metric* of the E–W structure $(M,[g],D)$. Let ϱ be the Ricci tensor of (M,g) and denote by S the Ricci endomorphism of (M,g), i.e. $\varrho(X,Y) = g(X,SY)$. We recall two important theorems (see [9]):

THEOREM 3.1. *A metric g and a 1-form ω define an E–W structure if and only if there exists a function $\Lambda \in C^\infty(M)$ such that*

$$\varrho^\nabla + \tfrac{1}{4}\mathcal{D}\omega = \Lambda g \tag{3.2}$$

where $\mathcal{D}\omega = (\nabla_X\omega)Y + (\nabla_Y\omega)X + \omega(X)\omega(Y)$. If (3.2) *holds then*

$$\overline{\Lambda} = 2\Lambda + \operatorname{div}\omega - \tfrac{1}{2}(n-2)\|\omega^\sharp\|^2. \tag{3.3}$$

THEOREM 3.2. *Let M be a compact E–W manifold and let g be the standard metric with the corresponding 1-form ω. Then $\omega^\sharp$ is a Killing vector field on M.*

The above theorems yield

THEOREM 3.3. *Let $(M,[g])$ be a compact E–W manifold and let g be the standard metric on M. Then (M,g) is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold. The manifold (M,g) is Einstein or the Ricci tensor ϱ^∇ of (M,g) has exactly two eigenfunctions $\lambda_0 \in C^\infty(M)$, $\lambda_1 = \Lambda$ satisfying the following conditions*:

(a) $(n-4)\lambda_1 + 2\lambda_0 = C_0 = \text{const}$,

(b) $\lambda_0 \le \lambda_1$ *on* M,

(c) $\dim\ker(S - \lambda_0 \operatorname{Id}) = 1$, $\dim\ker(S - \lambda_1 \operatorname{Id}) = n-1$ *on* $U = \{x : \lambda_0(x) \ne \lambda_1(x)\}$.

In addition, $\lambda_0 = (1/n)\mathrm{Scal}_g^D$ *where* $\mathrm{Scal}_g^D = \operatorname{tr}_g \varrho^D$ *denotes the conformal scalar curvature of* (M,g,D).

Proof. Note that $\omega(X) = g(\xi,X)$ where $\xi \in \mathfrak{iso}(M)$ and

$$\varrho^\nabla + \tfrac{1}{4}(n-2)\omega\otimes\omega = \Lambda g \tag{3.4}$$

(see [10], p. 101 and [3]). It is also clear that $\nabla_X\omega(X) = g(\nabla_X\xi, X) = 0$. Thus $\nabla_X(\omega\otimes\omega)(X,X) = 0$. From (3.4) it follows that

$$\nabla_X\varrho(X,X) = X\Lambda g(X,X). \tag{3.5}$$

This means that $(M,g)\in\mathcal{A}\oplus\mathcal{C}^\perp$ and $d\big(\Lambda - \frac{2}{n+2}\tau\big) = 0$, where τ is the scalar curvature of (M,g). From (3.5) it follows that the tensor $T = S - \Lambda\,\mathrm{Id}$ is a Killing tensor. Denote by ξ the Killing vector field dual to ω. Note that $\varrho(\xi,\xi) = \big(\Lambda - \frac14(n-2)\|\xi\|^2\big)\|\xi\|^2$ and if $X\perp\xi$ then $SX = \Lambda X$. Hence the tensor S has two eigenfunctions $\lambda_0 = \Lambda - \frac14(n-2)\|\xi\|^2$ and $\lambda_1 = \Lambda$. This proves (b).

Note that

$$\tau = \lambda_0 + (n-1)\lambda_1 = n\Lambda - \tfrac14(n-2)\|\xi\|^2$$

and $2\tau - (n+2)\Lambda = C_0 = \mathrm{const}$. Thus $C_0 = (n-2)\Lambda - \frac12(n-2)\|\xi\|^2$. However, $(n-4)\lambda_1 + 2\lambda_0 = (n-2)\Lambda - \frac12(n-2)\|\xi\|^2$, which proves (a).

Note also that (see for example [10], p. 100 and [3], p. 8)

$$\frac1n s_g^D = \Lambda - \frac{n-2}{4}\|\xi\|^2 = \lambda_0, \tag{3.6}$$

which finishes the proof. ■

On the other hand, the following theorem holds.

THEOREM 3.4. *Let (M,g) be a compact $\mathcal{A}\oplus\mathcal{C}^\perp$-manifold. Assume that the Ricci tensor ϱ of (M,g) has exactly two eigenfunctions λ_0,λ_1 satisfying the conditions*:

(a) $(n-4)\lambda_1 + 2\lambda_0 = C_0 = \mathrm{const}$,

(b) $\lambda_0 \le \lambda_1$ *on* M,

(c) $\dim\ker(S-\lambda_0\,\mathrm{Id}) = 1$, $\dim\ker(S-\lambda_1\,\mathrm{Id}) = n-1$ *on* $U = \{x : \lambda_0(x)\ne\lambda_1(x)\}$.

Then there exists a two-fold Riemannian covering (M',g') of (M,g) and a Killing vector field $\xi\in\mathfrak{iso}(M')$ such that $(M',[g'])$ admits two different E–W structures with the standard metric g' and the corresponding 1-forms $\omega_\mp = \mp\xi^\sharp$. The condition (b) *may be replaced by the condition*

(b1) *there exists a point $x_0\in M$ such that $\lambda_0(x_0) < \lambda_1(x_0)$.*

Proof. Let τ be the scalar curvature of (M,g). Then $\tau = (n-1)\lambda_1 + \lambda_0$ and $C_0 = (n-4)\lambda_1 + 2\lambda_0$. It follows that

$$\lambda_1 = \frac{2\tau - C_0}{n+2},\qquad \lambda_0 = \frac{(n-1)C_0 - (n-4)\tau}{n+2}. \tag{3.7}$$

In particular, $\lambda_0,\lambda_1\in C^\infty(M)$. Let S be the Ricci endomorphism of (M,g) and define the tensor $T := S - \lambda_1\,\mathrm{Id}$. Since from (3.7) we have $d\lambda_1 = \frac{2}{n+2}d\tau$ it follows that T is a Killing tensor with two eigenfunctions: $\mu = 0$ and $\lambda = \lambda_0 - \lambda_1$. Note that on the set $U = \{x : \lambda\ne\mu\}$ we have $\dim D_\lambda|_U = 1$.

Thus we can apply Theorem 2.10. Hence there exists a two-fold Riemannian covering $p : (M', g') \to (M, g)$ and a Killing vector field $\xi \in \mathfrak{iso}(M')$ such that $S'\xi = (\lambda_0 \circ p)\xi$ where S' is the Ricci endomorphism of (M', g'). Note also that $\|\xi\|^2 = |\lambda - \mu| = |\lambda_0 - \lambda_1|$. Define the 1-form ω on M' by $\omega = c\xi^\sharp$ where

$$c = 2\sqrt{\frac{1}{n-2}}.$$

It is easy to check that with such a choice of ω equation (3.4) is satisfied and $\delta\omega = 0$. Thus (M', g', ω) defines an E–W structure and g' is the standard metric for $(M', [g'])$. Note that $(M, g', -\omega)$ gives another E–W structure corresponding to the field $-\xi$. ∎

COROLLARY 3.5. *Let (M, g) be a compact simply connected manifold satisfying the assumptions of Theorem* 3.4. *Then $(M, [g])$ admits two E–W structures with the standard metric g.*

Next we give a slight generalization of a result of K. P. Tod (see [9], Corollary 6.2).

COROLLARY 3.6. *Let $(M, [g], D)$ be a compact E–W manifold which is not conformally Einstein and let g be the standard metric on M. Then the center of the Lie algebra of the isometry group of (M, g) is at least one-dimensional. The component of identity of the isometry group of (M, g) coincides with the component of the identity G_e of the symmetry group G of $(M, [g], D)$.*

Proof. The field $\xi = \omega^\sharp$ is a Killing vector field and on the open and dense subset $U = \{x : \xi_x \neq 0\}$ of M the distribution $D_\lambda = \ker(S - \lambda\,\mathrm{Id})$ is spanned by ξ. We shall show that $\xi \in \mathfrak{z}(\mathfrak{iso}(M))$ where $\mathfrak{z}(\mathfrak{g})$ denotes the center of the Lie algebra $\mathfrak{g}$. Let $\eta \in \mathfrak{iso}(M)$. Since $\eta\tau = 0$ from (3.7) it follows that $\eta(\lambda_0 - \lambda_1) = 0$. Hence $\eta g(\xi, \xi) = 0$. It follows that

$$g([\xi, \eta], \xi) = 0. \tag{3.8}$$

Since $S\xi = \lambda_0\xi$ we get $S[\eta, \xi] = \lambda_0[\eta, \xi]$. Hence on the set U the field $[\eta, \xi]$ is parallel to ξ. From (3.8) we obtain $[\eta, \xi] = 0$ on U. Hence $[\eta, \xi] = 0$ on M and $\xi \in \mathfrak{z}(\mathfrak{iso}(M, g))$.

Note that $D = \nabla - K$ where $2K(X, Y) = \omega(X)Y + \omega(Y)X - g(X, Y)\xi$. If $\eta \in \mathfrak{iso}(M, g)$ then $L_\eta\nabla = 0, L_\eta K = 0$, thus $L_\eta D = 0$. Consequently, $\mathrm{Iso}_e(M, g) \subset G_e$. The inclusion $G_e \subset \mathrm{Iso}_e(M, g)$ is proved in [8] (Lemma 2.2, p. 410). (Note that the Euclidean sphere is conformally Einstein.) ∎

COROLLARY 3.7. *Let (M, g) be a compact simply connected $\mathcal{A}$-manifold whose Ricci tensor ϱ has two constant eigenvalues λ, μ such that $\lambda \le \mu$ and* $\dim D_\lambda = 1$. *Then $(M, [g])$ admits two E–W structures with the standard metric g.*

Finally, we prove that the conformal scalar curvature of a compact E–W manifold which is not conformally Einstein is nonnegative. Hence Corollary 4.4 in [10] is not correct.

THEOREM 3.8. *Let* $(M, [g])$ *be a compact E–W manifold and* $\dim M \geq 4$. *If* $(M, [g])$ *is not conformally Einstein then* $s^D \geq 0$ *on* M.

Proof. For $\dim M = 4$ the result is known (see [10], p. 103). Let (M, g) be the standard Riemannian manifold for the E–W manifold $(M, [g])$ and assume that $\dim M > 4$. Set $s^D = s^D_g$. Note that (see [10], p. 101)

$$\Delta s^D = -\frac{n(n-4)}{4}\Delta\|\omega\|^2 = -\frac{n(n-4)}{4}\Delta\|\xi\|^2 \tag{3.9}$$

where $\xi = \omega^\sharp$ and $\Delta\phi = \mathrm{tr}_g \operatorname{Hess}\phi$. Since ξ is a Killing vector field we have

$$-\frac{1}{2}\Delta\|\xi\|^2 = \varrho(\xi,\xi) - \|\nabla\xi\|^2 = \frac{1}{n}s^D\|\xi\|^2 - \|\nabla\xi\|^2. \tag{3.10}$$

Consequently, we obtain

$$\Delta s^D = \frac{n(n-4)}{2}\left(\frac{1}{n}s^D\|\xi\|^2 - \|\nabla\xi\|^2\right). \tag{3.11}$$

Let a point $x_0 \in M$ satisfy the condition $s^D(x_0) = \inf\{s^D(x) : x \in M\}$. Then $\Delta s^D(x_0) \geq 0$. From (3.11) it follows that

$$\frac{1}{n}s^D(x_0)\|\xi_{x_0}\|^2 \geq \|(\nabla\xi)_{x_0}\|^2. \tag{3.12}$$

If $\xi_{x_0} = 0$ then from (3.12) it follows that $\nabla\xi_{x_0} = 0$ and consequently $\xi = 0$ on M. Thus in this case (M, g) is Einstein. If $\xi_{x_0} \neq 0$ then from (3.12) we obtain $s^D(x_0) \geq 0$. Hence if $(M, [g])$ is not conformally Einstein then $s^D \geq 0$. ■

COROLLARY 3.9. *Let* $(M, [g])$ *be a compact E–W manifold with* $\dim M \geq 4$ *which is not locally conformally Einstein. Then* $b_1(M) = 0$.

Proof. From Theorem 2.4 of [10] it follows that if $s^D \geq 0$ and s^D is not identically 0 then $b_1(M) = 0$. It is also well known that if $s^D = 0$ then $(M, [g])$ is locally conformally Einstein (see [3]). ■

COROLLARY 3.10. *Let* $(M, [g], D)$ *be a compact E–W manifold which is not locally conformally Einstein. Assume that* $\chi(M) \neq 0$. *Then the standard Riemannian structure* (M, g_0) *has nonconstant scalar Riemannian curvature* τ_0, *in particular cannot be locally homogeneous.*

Proof. Note that an $\mathcal{A}\oplus\mathcal{C}^\perp$-manifold (M, g_0) has constant scalar curvature if and only if is an $\mathcal{A}$-manifold. Note also that if the standard structure (M, g_0) is an $\mathcal{A}$-manifold which is not locally conformally Einstein then $\chi(M) = 0$ (since it admits a global one-dimensional distribution D_λ). This contradiction shows that τ_0 is nonconstant. ■

REMARK. Note that every four-dimensional compact E–W manifold which is not locally conformally Einstein has nonzero Euler characteristic, hence it does not admit a locally homogeneous standard metric.

COROLLARY 3.11. *A compact E–W manifold which is not conformally Einstein is locally conformally Einstein if and only if its standard Riemannian structure* (M,g) *is an* $\mathcal{A}$*-manifold with two* (*constant*) *eigenvalues* λ,μ *such that* $\lambda=0<\mu$ *and* $\dim\ker S=1$, *where* S *is the Ricci endomorphism of* (M,g). *If these conditions on* (M,g) *are satisfied then the Ricci tensor of* (M,g) *is parallel*, $\nabla S=0$ *and the universal covering* $(\widetilde{M},\widetilde{g})$ *of* (M,g) *is* $(\mathbb{R},dt)\times(M_1,g_1)$, *where* M_1 *is a compact*, *simply connected Einstein manifold with positive scalar curvature.*

Proof. It is clear that then $\nabla\xi=0$ and $\|\xi\|=\text{const}$. Hence the scalar curvature τ of (M,g) is constant. Thus $(M,g)\in\mathcal{A}$. Note that if M is compact then $\widetilde{M}$ is complete. Hence we can apply the results from [7] and the de Rham theorem. ∎

REMARK. This last result was proved by P. Gauduchon (see [3], Th. 3, p. 10). We wanted here to prove it using only properties of Killing tensors.

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