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ON QUASITILTED ALGEBRAS WHICH ARE ONE-POINT EXTENSIONS OF HEREDITARY ALGEBRAS

BY

DIETER HAPPEL (CHEMNITZ) AND
INGER HEIDI SLUNGÅRD (TRONDHEIM)

Abstract. Quasitilted algebras have been introduced as a proper generalization of tilted algebras. In an earlier article we determined necessary conditions for one-point extensions of decomposable finite-dimensional hereditary algebras to be quasitilted and not tilted. In this article we study algebras satisfying these necessary conditions in order to investigate to what extent the conditions are sufficient.

1. Introduction. Let k be an algebraically closed field. By an *algebra* we mean a finite-dimensional k-algebra. If Λ is such an algebra, we denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. If X is a Λ-module, we denote by $\operatorname{pd}_\Lambda X$ (resp. $\operatorname{id}_\Lambda X$) the projective (resp. injective) dimension of X.

Definition 1. A finite-dimensional algebra Λ is called *quasitilted* if there exist a hereditary abelian category $\mathcal{H}$ which is locally finite, that is, has finite-dimensional Hom and Ext spaces, and a tilting object $T\in\mathcal{H}$ such that $\Lambda=\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$.

Quasitilted algebras give a proper generalization of tilted algebras. For example the canonical algebras (compare [Ri]), which are non-domestic, are quasitilted but not tilted. If Λ is a quasitilted algebra arising from the hereditary category $\mathcal{H}$, then the categories $\mathcal{H}$ and $\operatorname{mod}\Lambda$ are derived equivalent. Only two types of quasitilted algebras are known, those derived equivalent to the module category of a hereditary algebra (for example the tilted algebras) and those derived equivalent to the module category of a canonical algebra. The latter will be called quasitilted algebras of *canonical type* (see [LS] for a classification of this class of algebras). It has been conjectured that these are the only quasitilted algebras. The conjecture has been proved

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in some cases (see [HR1], [HR2], [Sk]), and we will use the fact that it is true if $\mathcal{H}$ has objects of finite length to get our main result in Section 3.

Since the quiver of a quasitilted algebra Λ has no oriented cycles, we can always view Λ as a one-point extension $\Lambda = \left(\begin{smallmatrix} k & 0 \\ M & A \end{smallmatrix}\right)$ of a quasitilted algebra A by the A-k-bimodule M. In many classes of examples, like for instance the wild canonical algebras, the algebra A can be chosen to be hereditary.

In [HRS] the case of an indecomposable hereditary algebra was studied. In the present article we investigate the case of one-point extensions of decomposable finite-dimensional hereditary algebras. Necessary conditions for such a one-point extension to be quasitilted and not tilted were determined in [HS]. In the present article we investigate these conditions, and we will see that they are not sufficient.

For further representation-theoretic terminology used here we refer to [ARS] or [Ri], in particular for the classical tilting theory which is described in [Ri].

2. One-point extensions of hereditary algebras. In this section we recall some definitions and results from [HRS] and [HS].

Let H be a finite-dimensional hereditary algebra, and let M be an H-module. The *one-point extension* $H[M]$ of H by M is then defined as the triangular matrix ring

$$H[M] = \begin{bmatrix} k & 0 \\ {}_H M_k & H \end{bmatrix}$$

with the obvious multiplication.

Before we consider the case when the hereditary algebra H is decomposable, we recall some known results for indecomposable algebras. For indecomposable finite-dimensional hereditary algebras a precise description of when $H[M]$ is quasitilted is given in [HRS, Theorem III.2.13]. We will not need the full information obtained there. We will only use the following partial results from [HRS].

LEMMA 2. *Let H be an indecomposable finite-dimensional hereditary algebra and let M be an H-module. Let $M = M_1 \amalg \ldots \amalg M_t$, where all M_i are indecomposable. Then we have the following*:

(a) *Suppose that all M_i are directing. Then $H[M]$ is quasitilted if and only if $M_1, \ldots, M_t$ lie on a complete slice. Moreover*, *$H[M]$ is tilted in this case.*

(b) *If some M_i is non-directing and $H[M]$ is quasitilted*, *then M is quasisimple regular.*

Another useful result when working with quasitilted algebras is the following result from [HRS].

LEMMA 3 [HRS, Proposition II. 1.15]. *Let Λ be a quasitilted algebra and let P be a finitely generated projective Λ-module. Then* $\mathrm{End}_\Lambda(P)^{\mathrm{op}}$ *is a quasitilted algebra.*

For $n \geq 1$, let $\mathbb{A}_n$ be the graph

$$\underset{1}{\bullet} \text{---} \underset{2}{\bullet} \cdots\cdots \underset{n-1}{\bullet} \text{---} \underset{n}{\bullet}$$

Let $\vec{\mathbb{A}}_n$ be a quiver with $\mathbb{A}_n$ as underlying graph. We can then form the hereditary path algebra $k\vec{\mathbb{A}}_n$. For each vertex i of the quiver $\vec{\mathbb{A}}_n$, there is a simple $k\vec{\mathbb{A}}_n$-module S_i concentrated at vertex i. Let P_i be the projective cover of S_i. We say that P_i *corresponds* to vertex i.

We now recall the necessary conditions we obtained in [HS] for the one-point extension $S[N]$ of a decomposable hereditary algebra S to be quasitilted but not tilted.

THEOREM 4 [HS, Theorem 3.4]. *Let S be a decomposable hereditary algebra, and let N be a non-zero S-module. Assume that $S[N]$ is an indecomposable algebra. If $S[N]$ is quasitilted but not tilted, then $S[N] \simeq (H \times k\vec{\mathbb{A}}_n)[M \amalg P]$, where H is an indecomposable hereditary algebra of infinite representation type, M is a quasisimple regular H-module and P is an indecomposable projective $k\vec{\mathbb{A}}_n$-module isomorphic either to P_1 or to P_n.*

In the following section we study one-point extension algebras of the form $(H \times k\vec{\mathbb{A}}_n)[M \amalg P]$.

3. Algebras of the form $\Lambda_n = (H \times k\vec{\mathbb{A}}_n)[M \amalg P]$.

Let H be an indecomposable finite-dimensional hereditary k-algebra and let M be an indecomposable H-module. For $n \geq 1$, let $k\vec{\mathbb{A}}_n$ be the path algebra of the quiver $\vec{\mathbb{A}}_n$. Let P be either the indecomposable projective $k\vec{\mathbb{A}}_n$-module P_1 corresponding to vertex 1 or the indecomposable projective $k\vec{\mathbb{A}}_n$-module P_n corresponding to vertex n. We will often, without loss of generality, assume that $P = P_1$.

We can now form the one-point extension $\Lambda_n = (H \times k\vec{\mathbb{A}}_n)[M \amalg P]$ of the hereditary algebra $H \times k\vec{\mathbb{A}}_n$ by $M \amalg P$. We denote $H[M]$ by Λ_0.

Note that Λ_n, $n \geq 1$, can be formed by rooting the quiver

$$\vec{\Delta}: \quad \underset{\omega}{\bullet} \longrightarrow \underset{1}{\bullet} \text{---} \underset{2}{\bullet} \cdots\cdots \underset{n-1}{\bullet} \text{---} \underset{n}{\bullet}$$

in the extension vertex ω of $H[M]$, where the orientation of $\vec{\Delta}$ is determined by the subquiver $\vec{\mathbb{A}}_n$. Since the quiver $\vec{\Delta}$ is a branch (see [Ri] for definition), Λ_n is a branch extension of H by M. See [LM] for more information about this notion.

We first prove that for given n, H and M, the orientation of the quiver of type $\mathbb{A}_n$ does not matter when determining whether Λ_n is quasitilted or not.

LEMMA 5. *Let H be an indecomposable finite-dimensional hereditary k-algebra and let M be an indecomposable H-module. For $n \geq 1$, let $k\vec{\mathbb{A}}_n$ be the path algebra of the quiver $\vec{\mathbb{A}}_n$. Let P be either the indecomposable projective $k\vec{\mathbb{A}}_n$-module P_1 corresponding to vertex 1 or the indecomposable projective $k\vec{\mathbb{A}}_n$-module P_n corresponding to vertex n. If $\Lambda_n = (H \times k\vec{\mathbb{A}}_n)[M \amalg P]$ is quasitilted for one orientation of $\mathbb{A}_n$, then Λ_n is quasitilted for any orientation of $\mathbb{A}_n$.*

Proof. We index the vertices in $\mathbb{A}_n$ the following way:

$$\underset{1}{\bullet} \text{———} \underset{2}{\bullet} \cdots\cdots \underset{n-1}{\bullet} \text{———} \underset{n}{\bullet}$$

where $n \geq 1$. We can without loss of generality assume that $P = P_1$.

If $\vec{\mathbb{A}}_n$ contains a sink s different from 1, then the module P_s is simple projective and not injective. Thus we have an almost split sequence $0 \to P_s \to Q \to \operatorname{Tr}\operatorname{D} P_s \to 0$ with Q projective. This implies that $\operatorname{id}_{\Lambda_n} \operatorname{Tr}\operatorname{D} P_s = 1$. By assumption, Λ_n is quasitilted, so all projective Λ_n-modules are in $\mathcal{L}_{\Lambda_n} = \{X \in \operatorname{ind} \Lambda_n \mid \operatorname{pd}_{\Lambda_n} Y \leq 1$ for all Y with $Y \rightsquigarrow X\}$ (see [HRS, Theorem II 1.14]). Let $X \in \operatorname{ind} \Lambda_n$ and assume we have $0 \neq f \in \operatorname{Hom}(X, \operatorname{Tr}\operatorname{D} P_s)$. If X is not isomorphic to $\operatorname{Tr}\operatorname{D} P_s$ then f factors through Q, thus X is a predecessor of an indecomposable projective module in $\mathcal{L}_{\Lambda_n}$ and hence $\operatorname{pd}_{\Lambda_n} X \leq 1$. So all predecessors of $\operatorname{Tr}\operatorname{D} P_s$ are either a predeccesor of a module in $\mathcal{L}_{\Lambda_n}$ or isomorphic to $\operatorname{Tr}\operatorname{D} P_s$. This shows that $\operatorname{Tr}\operatorname{D} P_s \in \mathcal{L}_{\Lambda_n}$.

Let P_H be the direct sum of one module from each isomorphism class of indecomposable projective Λ_n-modules that comes from the algebra H, and let

$$T = \Big(\bigoplus_{\substack{j=1 \\ j \neq s}}^{n} P_j\Big) \oplus \operatorname{Tr}\operatorname{D} P_s \oplus P_\omega \oplus P_H.$$

Then T is an APR-tilting module in $\mathcal{L}_{\Lambda_n}$, hence $\Lambda_n' = \operatorname{End}_{\Lambda_n}(T)^{\mathrm{op}}$ is quasitilted [HRS, Proposition II 2.4]. Let $\vec{\mathbb{A}}_n'$ be the orientation of $\mathbb{A}_n$ corresponding to Λ_n'. In $\vec{\mathbb{A}}_n$ the vertex s was a sink, but it will be a source in $\vec{\mathbb{A}}_n'$.

Dually, assume that $\vec{\mathbb{A}}_n$ contains a source r different from 1. Then the simple module I_r is injective. Let I_H be the direct sum of one module from each isomorphism class of indecomposable injective Λ_n-modules that comes from the algebra H, and let

$$T = \Big(\bigoplus_{\substack{j=1 \\ j \neq s}}^{n} I_j\Big) \oplus \operatorname{D}\operatorname{Tr} I_r \oplus I_\omega \oplus I_H.$$

Then T is an APR-cotilting module in $\mathcal{R}_{\Lambda_n} = \{X \in \operatorname{ind} \Lambda_n \mid \operatorname{id}_{\Lambda_n} Y \leq 1$ for all Y with $X \rightsquigarrow Y\}$, so again $\Lambda'_n = \operatorname{End}_{\Lambda_n}(T)^{\mathrm{op}}$ is quasitilted. Let $\vec{\mathbb{A}}'_n$ be the orientation of $\mathbb{A}_n$ corresponding to Λ'_n. In $\vec{\mathbb{A}}_n$ the vertex r was a source, but it will be a sink in $\vec{\mathbb{A}}'_n$.

This shows that we can pass from one orientation of $\mathbb{A}_n$ to any other orientation by a sequence of APR-tilting and APR-cotilting modules. So if the algebra Λ_n is quasitilted for one orientation of $\mathbb{A}_n$, then Λ_n is quasitilted for any orientation of $\mathbb{A}_n$. ∎

Because of Theorem 4 we will mainly be interested in knowing when Λ_n is quasitilted if H is of infinite representation type and M is a quasisimple regular H-module. It is, however, not difficult to answer this question in other cases as well.

It is well known what happens when M is either preprojective or preinjective.

PROPOSITION 6. *Let M be an indecomposable preprojective* (*resp. preinjective*) *H-module. Then Λ_n is tilted for all $n \geq 0$.*

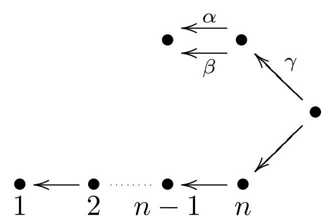
So Λ_n is tilted whenever H is of finite representation type. Hence from now on we assume that the hereditary algebra H is of infinite representation type and that M is an indecomposable regular H-module. It follows from Lemma 3 that Λ_0 is quasitilted whenever Λ_n, $n \geq 1$, are quasitilted. So by Lemma 2, we see that M must be a quasisimple regular H-module if Λ_n is quasitilted.

Let us first investigate Λ_n when H of tame representation type. From [LM, Proposition 3.6] we have the following easy consequence.

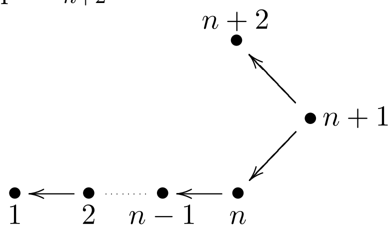
PROPOSITION 7. *Let H be of tame representation type and let M be a quasisimple regular H-module. Then Λ_n is quasitilted for all $n \geq 1$.*

Proof. As we have seen, Λ_n can be viewed as a branch extension of H by M with the branch $\vec{\Delta}$. Since H is tame and M is simple regular, we deduce from the dual result of [LM, Proposition 3.6] that Λ_n is quasitilted. ∎

EXAMPLE 8. Let H be the Kronecker algebra and let M be a simple regular H-module with dimension vector $\underline{\dim}\, M = (1\ 1)$. Let $\vec{\mathbb{A}}_n$ have linear orientation and let $P = P_n$ be the indecomposable projective module of length n. Hence Λ_n has the following quiver:



with the relation $\alpha\gamma = \beta\gamma$. In this case Λ_n is not only quasitilted, but also tilted. This can be seen by viewing Λ_n as a one-point coextension of the following algebra Γ of type $\mathbb{A}_{n+2}$:



We see that $\Lambda_n = [I_{n+2} \amalg S_{n+2}]\Gamma$. Since I_{n+2} and S_{n+2} are both directing and they lie on a complete slice, we conclude by Lemma 2 that Λ_n is tilted.

The only situation left to investigate is when H is of wild representation type and M is indecomposable regular. In this situation we have not been able to determine exactly when Λ_n is quasitilted, but we have a partial result.

We will prove the result by arguments on a locally finite hereditary abelian category with a tilting object. Before we state the result, we recall some definitions and basic results about such categories.

Let $\mathcal{H}$ be a locally finite hereditary abelian category with a tilting object T. Let $\mathcal{H}_0$ be the subcategory of $\mathcal{H}$ consisting of the objects of finite length and $\mathcal{H}_\infty$ be the additive subcategory of $\mathcal{H}$ whose indecomposable objects are of infinite length.

Given an object X in $\mathcal{H}$, we can form the perpendicular category $X^\perp = \{Y \in \mathcal{H} \mid \mathrm{Hom}_{\mathcal{H}}(X,Y) = 0 = \mathrm{Ext}^1_{\mathcal{H}}(X,Y)\}$. One can dually define ${}^\perp X$.

For objects X in $\mathcal{H}_\infty$ which are torsion and exceptional, i.e. indecomposable with $\mathrm{Ext}^1_{\mathcal{H}}(X,X) = 0$ and $\mathrm{Ext}^1_{\mathcal{H}}(T,X) = 0$, we have the following result.

THEOREM 9 [HR2, Theorem 4.14]. *Let $\mathcal{H}$ be a locally finite hereditary abelian category with a tilting object T. If X is a torsion exceptional object in $\mathcal{H}_\infty$, then $X^\perp$ is equivalent to* mod H' *for some hereditary algebra H'.*

As mentioned in the Introduction, it has been conjectured that the tilted algebras and the quasitilted algebras of canonical type are the only quasitilted algebras. This has been proved by the first author and I. Reiten in the following case.

THEOREM 10 [HR2, Theorem 6.1]. *Let $\mathcal{H}$ be a connected locally finite hereditary abelian category with a tilting object. If $\mathcal{H}_0 \neq \emptyset$, then $\mathcal{H}$ is derived equivalent to* mod A, *where A is either a hereditary algebra or a canonical algebra.*

These two results are used in the proof of the following.

THEOREM 11. *Let H be of wild representation type and let M be an indecomposable regular H-module. For* $n \geq 0$, *assume that Λ_n is quasitilted but not tilted. Then Λ_{n+1} is not quasitilted.*

Proof. We may by Lemma 5 assume that $\vec{\mathbb{A}}_{n+1}$ has the following orientation:

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \cdots\cdots \underset{n}{\bullet} \longrightarrow \underset{n+1}{\bullet}$$

We also assume that $P = P_1$. If $\Lambda_{n+1} = (H \times k\vec{\mathbb{A}}_{n+1})[M \amalg P]$ is quasitilted, then there exist a locally finite hereditary abelian category $\mathcal{H}$ and a tilting object $T \in \mathcal{H}$ such that $\Lambda_{n+1} = \operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}}$. Let ω be the extension vertex of Λ_{n+1}. We can write T as $T = T' \amalg T_\omega \amalg T_1 \amalg \ldots \amalg T_{n+1}$ where $\operatorname{End}_{\mathcal{H}}(T')^{\mathrm{op}} \simeq H$, $\operatorname{End}_{\mathcal{H}}(T' \amalg T_\omega)^{\mathrm{op}} \simeq H[M]$ and $\operatorname{End}_{\mathcal{H}}(T' \amalg T_\omega \amalg T_1 \amalg \ldots \amalg T_i)^{\mathrm{op}} \simeq \Lambda_i$ for all $1 \leq i \leq n+1$.

If $T_{n+1} \in \mathcal{H}_\infty$, we find by Theorem 9 that ${}^\perp T_{n+1}$ is equivalent to $\operatorname{mod} H'$ for some hereditary algebra H'. The module $T' \amalg T_\omega \amalg T_1 \amalg \ldots \amalg T_n$ is a tilting object in ${}^\perp T_{n+1}$ (see [HR2, Theorem 2.5]). This implies that $\operatorname{End}_{\mathcal{H}}(T' \amalg T_\omega \amalg T_1 \amalg \ldots \amalg T_n)^{\mathrm{op}} \simeq \Lambda_n$ is tilted. But Λ_n was assumed not to be tilted, so T_{n+1} has to be of finite length in $\mathcal{H}$. Hence $\mathcal{H}_0 \neq \emptyset$, so by Theorem 10 we see that Λ_{n+1} is either tilted or quasitilted of canonical type. Since Λ_n is not tilted, Λ_{n+1} cannot be tilted either. Hence Λ_{n+1} has to be quasitilted of canonical type.

Now we investigate the hereditary category $T_\omega^\perp$. The module $T' \amalg T_1 \amalg \ldots \amalg T_{n+1}$ is a tilting object in $T_\omega^\perp$ (see [HR2, Theorem 2.5]). Since $\operatorname{End}_{\mathcal{H}}(T' \amalg T_1 \amalg \ldots \amalg T_{n+1})^{\mathrm{op}} \simeq H \times k\vec{\mathbb{A}}_{n+1}$, we deduce that $T_\omega^\perp$ is derived equivalent to the module category of a wild hereditary algebra. Hence $T_\omega^\perp$ does not have tubes. Since T_{n+1} is a direct summand of a tilting object and is in $\mathcal{H}_0$, $\mathcal{H}$ must have tubes of rank greater than 1. If $T_\omega \in \mathcal{H}_0$, this would imply that $T_\omega^\perp$ had tubes. Since this is not the case, we have $T_\omega \in \mathcal{H}_\infty$.

Let P_{n+1} be the indecomposable projective Λ_{n+1}-module corresponding to vertex $n+1$. Then the following exact sequence is the almost split sequence starting at the simple projective Λ_{n+1}-module $S_{n+1} = P_{n+1}$:

$$0 \to P_{n+1} \to P_n \to \operatorname{Tr} \mathrm{D} P_{n+1} \to 0.$$

This shows in particular that $\operatorname{pd}_{\Lambda_{n+1}} \operatorname{Tr} \mathrm{D}\, P_{n+1} = 1$, and it is easily seen that $\operatorname{id}_{\Lambda_{n+1}} P_{n+1} = 1$. Then by [Ha, 4.7],

$$P_{n+1} \to P_n \to \operatorname{Tr} \mathrm{D}\, P_{n+1} \to P_{n+1}[1]$$

is an AR-triangle in $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{n+1})$. Since $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{n+1})$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{H})$ are derived equivalent, this means that we have an AR-triangle

$$T_{n+1} \to T_n \to Z_{n+1} \to T_{n+1}[1]$$

in $\mathcal{D}^{\mathrm{b}}(\mathcal{H})$. So we have an irreducible map $T_{n+1} \to T_n$ in $\mathcal{H}$, and hence T_{n+1} and T_n are in the same component of $\mathcal{H}$.

The almost split sequence in $\operatorname{mod} \Lambda_{n+1}$ starting with P_i, $2 \le i \le n$, is the following:

$$0 \to P_i \to \operatorname{Tr} \mathrm{D} P_{i+1} \amalg P_{i-1} \to \operatorname{Tr} \mathrm{D} P_i \to 0.$$

It is easily seen that $\operatorname{pd}_{\Lambda_{n+1}} \operatorname{Tr} \mathrm{D} P_i = 1$ and $\operatorname{id}_{\Lambda_{n+1}} P_i = 1$. Just as above this will give us an AR-triangle

$$T_i \to Z_{i+1} \amalg T_{i-1} \to Z_i \to T_i[1]$$

in $\mathcal{D}^{\mathrm{b}}(\mathcal{H})$. Hence we have an irreducible map $T_i \to T_{i-1}$ in $\mathcal{H}$, and therefore T_i and T_{i-1} are in the same component of $\mathcal{H}$ for $2 \le i \le n$.

The almost split sequence in $\operatorname{mod} \Lambda_{n+1}$ starting with P_1 is the following:

$$0 \to P_1 \to \operatorname{Tr} \mathrm{D} P_2 \amalg P_\omega \to \operatorname{Tr} \mathrm{D} P_1 \to 0.$$

As in the cases above, this gives us an irreducible map $T_1 \to T_\omega$ in $\mathcal{H}$, and therefore T_1 and T_ω are in the same component of $\mathcal{H}$.

This means that $T_1, \ldots, T_{n+1}$ and T_ω all are in the same component of $\mathcal{H}$. But T_{n+1} is of finite length and T_ω is of infinite length, so this is not possible. Hence Λ_{n+1} is not quasitilted. ∎

From this result we see that Λ_n can only be quasitilted and not tilted for at most one n. Hence we have the following consequence of Theorem 11.

COROLLARY 12. *Let H be hereditary of wild representation type and let M be an indecomposable regular H-module. If Λ_n is quasitilted but not tilted for $n \ge 1$, then Λ_i is tilted for all $0 \le i \le n-1$.*

4. Extensions of quasitilted algebras and examples. Before we illustrate Theorem 11 by some examples, we prove a result which gives an easy indication of whether an algebra is quasitilted or not.

THEOREM 13. *Let Λ be a quasitilted algebra which is not tilted, and let $\Lambda[M]$ be the one-point extension of Λ by a Λ-module M. If $\Lambda[M]$ is quasitilted, then $\Lambda[M]$ is quasitilted of canonical type.*

Proof. If $\Lambda[M]$ is quasitilted, then there exist a locally finite hereditary abelian category $\mathcal{H}$ and a tilting object $T \in \mathcal{H}$ such that $\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{op}} = \Lambda[M]$. Let ω be the extension vertex in $\Lambda[M]$. We can write T as $T = T' \amalg T_\omega$ where $\operatorname{End}_{\mathcal{H}}(T')^{\mathrm{op}} \simeq \Lambda$. If $\mathcal{H}_0 = \emptyset$, then T_ω must be of infinite length. By using Theorem 9 we see that $T_\omega^{\perp}$ is equivalent to $\operatorname{mod} H'$ where H' is a hereditary algebra. Since T' is a tilting object in $T_\omega^{\perp}$, this would imply that $\operatorname{End}_{\mathcal{H}}(T')^{\mathrm{op}} \simeq \Lambda$ is tilted. Since Λ is not tilted, we must therefore have $\mathcal{H}_0 \ne \emptyset$. Also, $\Lambda[M]$ cannot be tilted, since Λ is not. So by Theorem 10, we find that $\Lambda[M]$ has to be quasitilted of canonical type. ∎

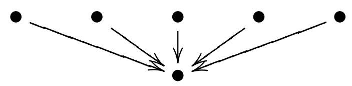
The *Coxeter polynomial* of an algebra A is the characteristic polynomial of the Coxeter transformation of A. It was shown in [LP, Proposition 4.2] that the Coxeter polynomial of a canonical algebra $C = C(p, \lambda)$ is of the following form:

$$\chi(x) = (x-1)^2 \prod_{i=1}^{t} \frac{x^{p_i}-1}{x-1}.$$

We say that a Coxeter polynomial of this form is of *canonical type*. Two algebras derived equivalent to each other have the same Coxeter polynomial, so all quasitilted algebras of canonical type have Coxeter polynomials of this type. From Theorem 13 we now have the following easy consequence.

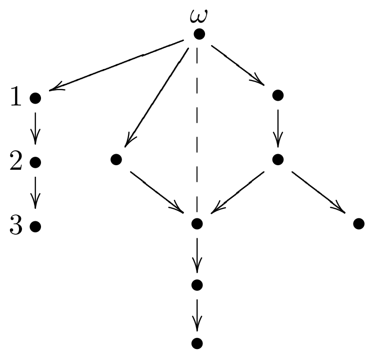
COROLLARY 14. *Let Λ be a quasitilted algebra which is not tilted, and let $\Lambda[M]$ be the one-point extension of Λ by a Λ-module M. If the Coxeter polynomial of $\Lambda[M]$ is not of canonical type, then $\Lambda[M]$ is not quasitilted.*

EXAMPLE 15. Let H be given by the quiver



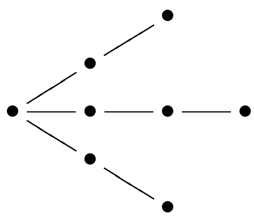
and let M be an indecomposable regular H-module with dimension vector $\underline{\dim}\, M = \binom{1\,1\,1\,1\,1}{2}$ such that $\Lambda_0 = H[M]$ is a wild canonical algebra. This of course means that Λ_0 is quasitilted and not tilted. So we find by using Proposition 11 that Λ_1 is not quasitilted. This can also be seen by using the dual of Corollary 14. The Coxeter polynomial of Λ_1 is $\chi(x) = (x^4 - x^3 + x^2 - x + 1)(x+1)^4$, which is not of canonical type.

EXAMPLE 16. In [Hü] it was shown that the algebra with quiver

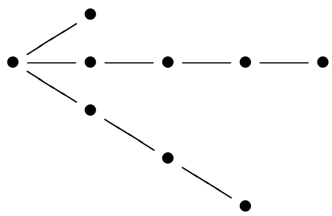


is quasitilted but not tilted. It is clearly of type Λ_3. Proposition 11 tells us that Λ_4 is not quasitilted, and that the algebras Λ_0, Λ_1 and Λ_2 are all tilted.

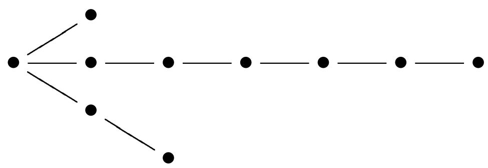
The Coxeter polynomial of Λ_4 is $\chi(x) = x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1$, so we can also see by the dual of Corollary 14 that Λ_4 is not quasitilted. For the algebras Λ_0, Λ_1 and Λ_2 one can find a complete slice in the preinjective component of the corresponding AR-quiver, so this proves that they are tilted. In fact, Λ_0 is tilted from an algebra of type



Λ_1 is tilted from an algebra of type

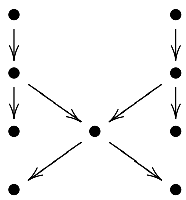


Λ_2 is tilted from an algebra of type



The next example is one where we cannot use Proposition 11 or Corollary 14 to decide if the algebra Λ_1 is quasitilted. It is possible in this example to decide whether Λ_1 is quasitilted in another way.

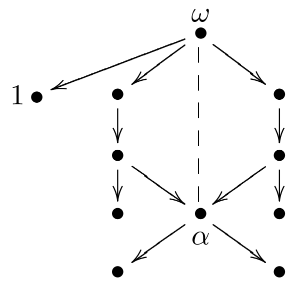
EXAMPLE 17. Let H be the hereditary algebra given by the quiver



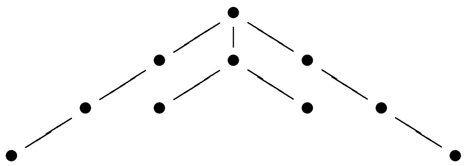
and let M be the indecomposable regular H-module with dimension vector

$$\underline{\dim}\, M = \begin{pmatrix} 1 & & 1 \\ 1 & & 1 \\ 1 & 1 & 1 \\ 1 & & 1 \end{pmatrix}.$$

We then look at the algebra $\Lambda_1 = (H \times k\vec{\mathbb{A}}_1)[M \amalg P]$. Hence Λ_1 is given by the following quiver with the indicated commutativity relation:



One can determine that Λ_0 is tilted from an algebra of type

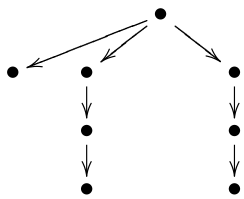


The Coxeter polynomial of Λ_1 is

$$\chi(x) = (x-1)^2 \frac{(x^8-1)}{(x-1)} \frac{(x^2-1)}{(x-1)} \frac{(x^2-1)}{(x-1)},$$

so Corollary 14 does not tell us if Λ_1 is quasitilted or not. By the following argument one can however see that Λ_1 is not quasitilted.

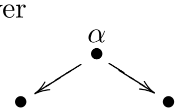
Inside Λ_1 we have the following subalgebra H' of type $\widetilde{\mathbb{E}}_7$:



H' is tame hereditary, and hence especially tame concealed. Let M' be the indecomposable H'-module with dimension vector

$$\underline{\dim}\, M' = \left(\begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & 1 \\ & 0 & 0 \end{smallmatrix} \right).$$

This is a simple regular H'-module in a tube of rank 3. Now Λ_1 is obtained by rooting the hereditary quiver



in the coextension vertex α of $[M']H'$. From the dual result of [Sk, Lemma 3.2] we then conclude that Λ_1 is not quasitilted.

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Fakultät für Mathematik
Technische Universität Chemnitz
Postfach 964
D-09107 Chemnitz, Germany
E-mail: happel@mathematik.tu-chemnitz.de

Institutt for matematiske fag
Fakultet for fysikk, informatikk og matematikk
Norges teknisk-naturvitenskaplige universitet
N-7491 Trondheim, Norway
E-mail: ingersl@math.ntnu.no

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