ON SYSTEMS OF NULL SETS

BY

K. P. S. BHASKARA RAO (BANGALORE) AND
R. M. SHORTT (MIDDLETOWN, CT)

Abstract. The collection of all sets of measure zero for a finitely additive, group-valued measure is studied and characterised from a combinatorial viewpoint.

Let \( X \) be a non-empty set and let \( A \) be a class of subsets of \( X \). Then \( A \) is a field if \( X \in A \) and \( A \) is closed under the operations of (finite) union and complementation, i.e. \( A \) is a Boolean algebra of subsets of \( X \). If \( A \) is any class of subsets of \( X \), then \( a(A) \) denotes the smallest field containing \( A \). A collection \( U \) of subsets of \( X \) is a u-system if \( \emptyset \in U \) and \( U \) is closed under the operation of proper difference: \( U_1 \setminus U_2 \in U \) whenever \( U_1 \supseteq U_2 \) for \( U_1, U_2 \in U \). It is easy to show that if \( U \) is a u-system such that \( X \in U \), and \( U_1, U_2 \in U \) with \( U_1 \cap U_2 = \emptyset \), then \( U_1 \cup U_2 \in U \): a u-system containing \( X \) is closed under formation of disjoint unions (and also complements).

Let \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_n \) be finite sequences of not necessarily distinct subsets of a set \( X \). For any \( k \geq 1 \), we define

\[
A(k) = \bigcup A_{i_1} \cap \ldots \cap A_{i_k}, \quad B(k) = \bigcup B_{i_1} \cap \ldots \cap B_{i_k},
\]

in each case intending the union of all \( k \)-fold intersections: the \((i_1, \ldots, i_k)\) are \( k \)-tuples of distinct indices \( i_j \). Then we have

\[
A(1) = A_1 \cup \ldots \cup A_m, \quad A(m) = A_1 \cap \ldots \cap A_m, \\
B(1) = B_1 \cup \ldots \cup B_n, \quad B(n) = B_1 \cap \ldots \cap B_n,
\]

and by convention, we put \( A(k) = \emptyset \) for \( k > m \) and \( B(k) = \emptyset \) for \( k > n \).

A collection \( M \) of subsets of \( X \) is an m-system if \( \emptyset \in M \) and whenever \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_n \) are sets in \( M \) such that

\((*)\)

\[
A(k + 1) \subseteq B(k) \subseteq A(k) \quad \text{for all } k \geq 1,
\]

then

\((**)\)

\[
\bigcup_{k=1}^{N} [A(k) \setminus B(k)] \in M, \quad \text{where } N \geq m, n.
\]

1991 Mathematics Subject Classification: 28A05, 28B10.
Clearly, every field is an \( m \)-system, and every \( m \)-system is a \( u \)-system. The converse implications do not hold, as is shown in an example given later. If \( A \) is a class of subsets of \( X \), then \( u(A) \) and \( m(A) \) denote, respectively, the smallest \( u \)-system and \( m \)-system containing \( A \). Then \( u(A) \subseteq m(A) \).

Given a non-empty set \( X \), let \( Z^X \) be the additive group of all functions from \( X \) to the integers \( Z \). If \( A \subseteq X \), then the \emph{indicator} of \( A \) is the function \( 1_A : X \to Z \) such that \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \). Given a collection \( A \) of subsets of \( X \), we define \( S(A) \) as the subgroup of \( Z^X \) generated by all the indicators \( 1_A \) for \( A \in A \).

**Lemma 1.** If \( A \) and \( B \) are collections of subsets of \( X \), then \( S(A \cup B) = S(A) + S(B) \).

**Lemma 2.** Let \( A \) be a collection of subsets of \( X \). For any \( E \subseteq X \), we have \( E \in m(A) \) if and only if \( 1_E \in S(A) \).

\begin{proof}
Suppose that \( 1_E \in S(A) \). Then there are sets \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_n \) in \( A \) such that \( 1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n} \). We see that the sets \( A_i \) and \( B_j \) satisfy condition \((*)\) in the definition of an \( m \)-system, so that \( E \), which is the set in \((**),\) must belong to \( m(A) \).

Now let \( M \) be the collection of all sets \( F \subseteq X \) such that \( 1_F \in S(A) \). It is easy to verify that \( M \) is an \( m \)-system containing \( A \), so that \( u(A) \subseteq M \). \( \blacksquare \)

The proof gives indication of a useful alternative definition of \( m \)-system: if \( A_i \) and \( B_j \) are sets in \( M \), and \( 1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n} \), then \( E \in M \).

**Lemma 3.** Let \( A \) be a collection of subsets of \( X \). Then \( S(m(A)) = S(u(A)) = S(A) \).

\begin{proof}
Clearly, \( S(A) \subseteq S(u(A)) \subseteq S(m(A)) \). The inclusion \( S(m(A)) \subseteq S(A) \) follows from the preceding lemma. \( \blacksquare \)

**Example.** We show that the concepts of \( u \)-system and \( m \)-system are in general distinct. Put
\[
Y = \{0, 1\}^3, \quad X = \{(a_1, a_2, a_3) \in Y : a_1 + a_2 \geq a_3\}, \quad A_i = \{(a_1, a_2, a_3) \in X : a_i = 1\} \quad \text{for} \quad i = 1, 2, 3.
\]
Then the collection
\[
U = \{\emptyset, A_1, A_2, A_3, X \setminus A_1, X \setminus A_2, X \setminus A_3, X\}
\]
is a \( u \)-system, but \( m(U) \) contains the additional set
\[
E = \{(1, 1, 1), (1, 0, 0), (0, 1, 0)\};
\]
we have \( 1_E = 1_{A_1} + 1_{A_2} - 1_{A_3} \).

Let \( A \) be a field of subsets of a set \( X \) and let \( G \) be an Abelian group. A function \( \mu : A \to G \) is a \((G\text{-valued})\) charge if \( \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) \)
whenever $A_1$, $A_2$ are disjoint sets in $A$. Every $G$-valued charge $\mu$ induces a unique homomorphism $\varphi : S(A) \to G$ such that $\varphi(1_A) = \mu(A)$ for every $A \in A$; using the same equation, we see that each homomorphism $\varphi : S(A) \to G$ is induced by a charge $\mu : A \to G$. Zero sets of group-valued charges are characterised in the

**Theorem.** Let $M$ be a collection of subsets of a non-empty set $X$ and define $A = a(M)$. The following conditions are equivalent:

(i) there is an Abelian group $G$ and a charge $\mu : A \to G$ such that $M = \{ A \in A : \mu(A) = 0 \}$;
(ii) $M$ is an $m$-system.

**Proof.** (i)$\Rightarrow$(ii). Let $\varphi : S(A) \to G$ be the homomorphism induced by $\mu$. If $A_i$ and $B_j$ are sets in $A$ with $\mu(A_i) = \mu(B_j) = 0$ and $1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n}$, then $\mu(E) = \varphi(1_E) = 0$. The collection $M = \{ A \in A : \mu(A) = 0 \}$ is thus closed under the operation that defines $m$-systems.

(ii)$\Rightarrow$(i). Define $G = S(A)/S(M)$ and let $\varphi : S(A) \to G$ be the standard projection onto the quotient. Define $\mu : A \to G$ by $\mu(A) = \varphi(1_A)$. By Lemma 2, $M = \{ A \in A : \mu(A) = 0 \}$. $\blacksquare$

Quotient groups of the form $S(a(A\cup B))/[S(A)+S(B)]$, where $A$ and $B$ are fields, arise naturally in and have been studied for their connection with the problem of joint extensions of group-valued charges (see [1], [2]). With this application in mind, we now prove that the $u$-system and the $m$-system generated by the union of two fields coincide.

**Theorem.** Let $A$ and $B$ be fields of subsets of a set $X$. For $E \subseteq X$, we have $1_E \in S(A)+S(B)$ if and only if $E \in u(A\cup B)$. Then $u(A\cup B) = m(A\cup B)$.

**Proof.** From Lemma 2 and the inclusion $u(A\cup B) \subseteq m(A\cup B)$, we see that $1_E \in S(A\cup B) = S(A)+S(B)$ whenever $E \in u(A\cup B)$. Now suppose that $1_E \in S(A\cup B)$. Then $1_E = h+k$ for functions $h \in S(A)$ and $k \in S(B)$. Since constant functions in $\mathbb{Z}^X$ belong to $S(A) \cap S(B)$, it involves no loss of generality to assume that $h \geq 0$ and $k \leq 0$. Then we have

$$E = \bigcup_{i=0}^{\infty} \{ x : k(x) \geq -i \} \setminus \{ x : h(x) \leq i \},$$

a finite disjoint union of proper differences of sets of $B$ with sets of $A$. Thus $E \in u(A\cup B)$.

We have shown that $1_E \in S(A\cup B)$ if and only if $E \in u(A\cup B)$. Lemma 2 then implies that $u(A\cup B) = m(A\cup B)$. $\blacksquare$
REFERENCES


Indian Statistical Institute
Bangalore 560059, India
E-mail: kpsbrao@isibang.ernet.in

Wesleyan University
Middletown, CT 06457, U.S.A.
E-mail: rshortt@wesleyan.edu

Received 23 May 1994