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ON BOUNDED UNIVALENT FUNCTIONS THAT OMIT TWO GIVEN VALUES

BY

DIMITRIOS BETSAKOS (HELSINKI)

Abstract. Let $a, b \in \{z : 0 < |z| < 1\}$ and let $S(a, b)$ be the class of all univalent functions f that map the unit disk $\mathbb{D}$ into $\mathbb{D} \setminus \{a, b\}$ with $f(0) = 0$. We study the problem of maximizing $|f'(0)|$ among all $f \in S(a, b)$. Using the method of extremal metric we show that there exists a unique extremal function which maps $\mathbb{D}$ onto a simply connnected domain D_0 bounded by the union of the closures of the critical trajectories of a certain quadratic differential. If $a < 0 < b$, we show that $D_0 = \mathbb{D} \setminus [-1, a] \setminus [b, 1]$.

1. Introduction. Let a, b be two distinct points in the unit disk $\mathbb{D}$, and assume that $a \neq 0 \neq b$. Let $S(a, b)$ be the class of univalent functions f that map $\mathbb{D}$ into $\mathbb{D} \setminus \{a, b\}$ and satisfy $f(0) = 0$. We study the following problem.

PROBLEM 1.1. *Find* $\max\{|f'(0)| : f \in S(a, b)\}$ *and determine the functions in* $S(a, b)$ *for which the maximum is attained.*

The existence of extremal functions is an easy consequence of a standard normal family argument. An equivalent formulation of the problem involves the conformal (inner) radius $R(0, D)$ at 0 of a domain D that contains 0 and possesses Green's function. $R(0, D)$ is defined as follows (see [2], p. 123): Let $g(z, 0, D)$ be Green's function of D with pole at 0 and let $c = \lim_{z\to 0}[g(z, 0, D) + \log |z|]$. Then $R(0, D) = e^c$. It is easy to see that if D is simply connected then $R(0, D) = |f'(0)|$, where f is a conformal mapping of $\mathbb{D}$ onto D with $f(0) = 0$. Thus Problem 1.1 is equivalent to the following:

Let $F_1(a, b)$ *be the class of all simply connected domains* $D \subset \mathbb{D} \setminus \{a, b\}$ *with* $0 \in D$. *Find* $\max\{R(0, D) : D \in F_1(a, b)\}$ *and determine the extremal domains.*

A reflection in the unit circle gives a third equivalent formulation of Problem 1.1:

Let $F_2(a, b)$ *be the class of all continua* $K \subset \mathbb{C}$ *with* $\mathbb{D} \cup \partial\mathbb{D} \cup \{1/\overline{a}, 1/\overline{b}\} \subset K$. *Find* $\min\{\operatorname{cap} K : K \in F_2(a, b)\}$ *and determine the extremal continua.*

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For the definition of the (logarithmic) capacity $\operatorname{cap} K$ of a continuum $K \subset \mathbb{C}$ we refer to [1], p. 14. In this third formulation the problem is similar to the Chebotarev problem in which one tries to find a continuum that contains given points and has minimal capacity (see [7], Ch. 1).

If we had *one* omitted value $a \in (0,1)$, the corresponding problem: *Find* $\max\{|f'(0)| : f(0) = 0,\ f : \mathbb{D} \to \mathbb{D} \setminus \{a\}\}$, is easy: The extremal function is unique and maps $\mathbb{D}$ onto $\mathbb{D} \setminus [a,1]$. This follows from the domain monotonicity of conformal radius (Schwarz's lemma) and the following symmetrization result which is due to Pólya, Szegő, Hayman and Jenkins (see [2], p. 126; [4], p. 136): *Let D be a simply connected domain that contains* 0. *Let D^* be the circular symmetrization of D with respect to the negative semiaxis. Then $R(0,D) \le R(0,D^*)$ with equality if and only if $D = e^{i\alpha} D^*$ for some* $\alpha \in \mathbb{R}$. A simple computation shows that

$$R(0, \mathbb{D} \setminus [a,1]) = \frac{4a}{(1+a)^2}. \tag{1.1}$$

The above argument implies that if $0 < a < b < 1$ in the context of Problem 1.1, then $\max\{R(0,D) : D \in F_1(a,b)\} = R(0, \mathbb{D} \setminus [a,1])$.

In Section 2 we use the method of extremal metric (see [4]; [7], Ch. 0) to give a qualitative solution of Problem 1.1: The extremal domain is unique and is bounded by curves that lie on the closure of the union of the critical trajectories of a certain quadratic differential. The formulae that describe this quadratic differential contain unknown constants.

In the special case where a and b lie on a diameter of $\mathbb{D}$ we give a complete solution of Problem 1.1:

THEOREM 1.2. *Let $-1 < a < 0 < b < 1$. Then*

$$\max\{|f'(0)| : f \in S(a,b)\} = 4(b + b^{-1} - a - a^{-1})^{-1}. \tag{1.2}$$

The maximum is attained only for the function $f \in S(a,b)$ that maps $\mathbb{D}$ onto $\mathbb{D} \setminus [-1,a] \setminus [b,1]$.

A rescaling and a convergence argument show that Theorem 1.2 implies a classical result due to Lavrent'ev [8]:

COROLLARY 1.3. *Let $b_1 < 0 < b_2$ and $f : \mathbb{D} \to \mathbb{C} \setminus \{b_1, b_2\}$ be a univalent function with $f(0) = 0$. Then*

$$|f'(0)| \le 4 b_1 b_2 (b_1 - b_2)^{-1}. \tag{1.3}$$

Equality holds if f maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, b_1] \setminus [b_2, +\infty)$.

Kuz'mina [6] generalized Lavrent'ev's result to arbitrary $b_1, b_2 \in \mathbb{C}$.

The following generalizes Theorem 1.2.

THEOREM 1.4. *Let n be a positive integer*, $r \in (0,1)$, $s \in (0,1)$ *and* $\xi_j = e^{2\pi i j/(2n)}$, $j = 0, 1, \ldots, 2n-1$. *Let $a_j = r\xi_j$ for $j = 0, 2, \ldots, 2n-2$*

and $a_j = s\xi_j$ *for* $j = 1, 3, \ldots, 2n-1$. *Consider the family* F *of all simply connected domains in* $\mathbb{D} \setminus \{a_0, a_1, \ldots, a_{2n-1}\}$ *that contain* 0. *Then*

$$\forall D \in F, \qquad R(0, D) \le R(0, D_*), \tag{1.4}$$

where $D_* = \mathbb{D} \setminus \bigcup_{j=0}^{2n-1} [a_j, \xi_j]$. *Equality holds if and only if* $D = D_*$.

We also prove the following:

THEOREM 1.5. *Let* $-1 < a < 0 < b < 1$. *Let* $F_3(a, b)$ *be the class of all domains* D *in* $\mathbb{D} \setminus \{a, b\}$ *such that* $0 \in D$ *and* $\mathbb{C} \setminus D$ *has a component that contains both* a *and* b. *Then*

$$\forall D \in F_3(a, b), \qquad R(0, D) \le R(0, G), \tag{1.5}$$

where $G = \mathbb{D} \setminus [-1, a] \setminus [b, 1]$. *If equality holds in* (1.5) *for some* $D \in F_3(a, b)$ *which is regular for the Dirichlet problem*, *then* $D = G$.

Since $F_1(a, b) \subset F_2(a, b)$, Theorem 1.5 also generalizes Theorem 1.2.

The proofs of the theorems are in Section 2.

2. Application of the method of extremal metric. It is well known that there exists a relation between conformal radius and extremal length (see [7], p. 30). Precisely,

$$\lim_{r \to 0} \left[\lambda(D_r, \partial D, D) + \frac{\log r}{2\pi} \right] = \frac{1}{2\pi} \log R(0, D), \tag{2.1}$$

where $D_r = \{|z| < r\}$, D is a domain with Green function and $\lambda(D_r, \partial D, D)$ is the extremal distance between ∂D and D_r with respect to the domain $D \setminus \operatorname{clos} D_r$. Thus Problem 1.1 is the problem for the extremal metric for the family of the closed Jordan curves in $\mathbb{D}$ that separate 0 from a, b and from the boundary of $\mathbb{D}$. Therefore we may apply the theory of Jenkins (see [3]; [7], Ch. 0, especially Th. 0.2). This theory implies that the maximal conformal radius $R(0, D)$ for $D \in F_1(a, b)$ is attained *uniquely* for an admissible domain with respect to a quadratic differential which has the following form:

$$Q(z)dz^2 = \frac{P(z)}{z^2(z - a)(1 - \overline{a}z)(z - b)(1 - \overline{b}z)} dz^2, \tag{2.2}$$

where $P(z)$ is a polynomial which can only have zeros of even multiplicity on $\partial\mathbb{D}$ (see Th. 0.2 in [7]).

Using the uniqueness of the extremal domain we can easily prove Theorem 1.2:

Proof of Theorem 1.2. Let G denote the extremal domain. By symmetry $\overline{G} = \{\overline{z} : z \in G\}$ is also an extremal domain. By uniqueness, $G = \overline{G}$. The domain monotonicity of conformal radius implies $G = \mathbb{D} \setminus [-1, a] \setminus [b, 1]$. An easy calculation shows that $R(0, G) = 4(b + b^{-1} - a - a^{-1})^{-1}$.

The critical trajectories of $Q(z)dz^2$ divide the extended complex plane $\mathbb{C}'$ into two simply connected domains D_0 and D_∞, symmetric with respect to $\partial\mathbb{D}$ and such that $D_0 \subset \mathbb{D}$ and $D_\infty \subset \mathbb{C}' \setminus \mathbb{D}$. The unit circle $\partial\mathbb{D}$ is a subset of the union Φ' of the closures of the critical trajectories of $Q(z)dz^2$.

D_0 is the unique extremal domain for Problem 1.1. Since it is simply connected there is a connected subset of Φ' that joins a and $\partial\mathbb{D} \subset \Phi'$. So $P(z)$ has at least a (double) zero on $\partial\mathbb{D}$. This also follows from Jenkins's Basic Structure Theorem ([4], Th. 3.5).

Because of symmetry, ∞ is a double pole of $Q(z)dz^2$. Hence the degree of $P(z)$ must be exactly 4. We conclude that $P(z)$ must have one of the following three forms:

Case 1: $P(z) = -B_1(z - e^{i\upsilon})^2(z - e^{i\phi})^2$, where $B_1 > 0$, $\upsilon \in [0, 2\pi)$, $\phi \in [0, 2\pi)$, $\phi \neq \upsilon$.

Case 2: $P(z) = -B_2(z - e^{i\chi})^2(z - re^{i\theta})(z - e^{i\theta}/r)$, where $B_2 > 0$, $\chi \in [0, 2\pi)$, $\theta \in [0, 2\pi)$, $0 < r < 1$.

Case 3: $P(z) = -B_3(z - e^{i\xi})^4$, where $B_3 > 0$ and $\xi \in [0, 2\pi)$.

Note that if $re^{i\theta} = a$ or if $re^{i\theta} = b$ in Case 2, then this case reduces to the case $a = b$ which has been studied in the introduction. In the sequel we assume that $re^{i\theta} \neq a$ and $re^{i\theta} \neq b$.

Next we use the results of Jenkins ([4], Ch. 3) on the structure of the trajectories of quadratic differentials:

In Case 1: $Q(z)dz^2$ has two double zeros $e^{i\upsilon}$, $e^{i\phi}$ on $\partial\mathbb{D}$. So four critical trajectories must meet at each of $e^{i\upsilon}$ and $e^{i\phi}$. Two of them are arcs of the unit circle, one lies in $\mathbb{D}$ and the last lies in $\mathbb{C} \setminus \mathbb{D}$. Since a, b are simple poles, a critical trajectory emanates from each one of them (see Th. 3.2 in [4]). Thus the boundary of D_0 consists of $\partial\mathbb{D}$ and two analytic arcs γ_1,γ_2. The arc γ_1 joins a to $e^{i\upsilon}$ and γ_2 joins b to $e^{i\phi}$. The arcs γ_1 and γ_2 meet the unit circle at right angles.

In Case 2: $Q(z)dz^2$ has a double zero at $e^{i\chi}$, a simple zero at $re^{i\theta}$ and simple poles at a, b. So four critical trajectories meet at $e^{i\chi}$, three meet at $re^{i\theta}$ and one emanates from each of a, b. Hence $\partial D_0 = \partial\mathbb{D} \cup \delta_1 \cup \delta_2 \cup \delta_3$, where $\delta_1, \delta_2, \delta_3$ are analytic arcs in $\mathbb{D}$ such that δ_1 joins a to $re^{i\theta}$, δ_2 joins b to $re^{i\theta}$ and δ_3 joins $re^{i\theta}$ to $e^{i\chi}$.

In Case 3: $Q(z)dz^2$ has a zero of order four at $e^{i\xi}$ and simple poles at a, b. So six critical trajectories must meet at $e^{i\xi}$ and one must emanate from each of a, b. Hence $\partial D_0 = \partial\mathbb{D} \cup \sigma_1 \cup \sigma_2$, where σ_1, σ_2 are analytic arcs in $\mathbb{D}$ such that σ_1 joins a to $e^{i\xi}$ and σ_2 joins b to $e^{i\xi}$.

The solution of Problem 1.1 would be complete if we were able to express the constants $B_1, B_2, B_3, \upsilon, \phi, \chi, \theta, \xi, r, R(0, D_0)$ in terms of a, b. Of course, there are interrelations between these constants but it seems difficult to use them to determine the constants explicitly.

We mention an interesting property of the extremal domain D_0. Suppose that Case 1 holds and let $\upsilon, \phi, \gamma_1, \gamma_2$ be as above. Each point of $\gamma_1 \setminus \{a\}$ supports two prime ends of D_0. So we can talk about the two "sides" of γ_1 and similarly for γ_2. Let f map D_0 conformally onto $\mathbb{D}$ with $f(0) = 0$. Let also E be a Borel set of prime ends of D_0. By definition, the *harmonic measure* $\omega(0, E, D_0)$ of E at 0 is the length of $f(E) \subset \partial\mathbb{D}$ divided by 2π.

PROPOSITION 2.1. *Let I be an open subarc of γ_1 (or of γ_2). Let I_+, I_- be the two sides of I. Then $\omega(0, I_+, D_0) = \omega(0, I_-, D_0)$.*

Proof. We use a technique of Lavrent'ev [8]. Jenkins [5] also used this technique.

Let $D' = D_0 \cup I$. Then D' is a doubly connected domain. Let f map D' conformally onto the annulus $A = \{z : \varrho < |z| < 1\}$ with $f(0) < 0$, where ϱ is an appropriate positive constant. Because of the conformal invariance of harmonic measure the assertion of the proposition follows at once from the following claim.

CLAIM. $f(I) = (\varrho, 1)$.

To prove the claim let $G_0 = A \setminus f(I)$ and $G_1 = A \setminus (\varrho, 1)$. By applying a circular symmetrization with respect to the negative semiaxis we obtain

$$R(f(0), G_0) \le R(f(0), G_1). \tag{2.3}$$

Let $D_1 = f^{-1}(G_1)$ and consider the Riemann mappings f_0 and f_1 that map $\mathbb{D}$ conformally onto D_0 and D_1, respectively, with $f_0(0) = 0$, $f_1(0) = 0$, $f_0'(0) > 0$, $f_1'(0) > 0$. Then

$$R(f(0), G_0) = |(f \circ f_0)'(0)| = |f'(0)| f_0'(0) = |f'(0)| R(0, D_0),$$
$$R(f(0), G_1) = |(f \circ f_1)'(0)| = |f'(0)| f_1'(0) = |f'(0)| R(0, D_1).$$

So (2.3) implies

$$R(0, D_0) \le R(0, D_1). \tag{2.4}$$

Now, since D_0 is the extremal domain for Problem 1.1, (2.4) implies $D_0 = D_1$ and the claim follows at once.

REMARK 2.2. A similar proposition holds for Cases 2 and 3.

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.2. By the method of extremal metric, there is a unique extremal domain Ω in F. But Ω must be symmetric and therefore $\Omega = D_*$.

Proof of Theorem 1.5. We use an equivalent formulation of the theorem. By applying a reflection in the unit circle we see that we have to prove the following statement:

Let $a' < -1 < 1 < b'$. Let $F_4(a', b')$ be the class of all compact sets K in $\mathbb{C}$ such that $\mathbb{D} \cup \partial\mathbb{D} \cup \{a', b'\} \subset K$ and a', b' belong to the same component *of* K. *Then*

$$\forall K \in F_4(a', b'),\ \operatorname{cap} K \geq \operatorname{cap} K^*, \tag{2.5}$$

where $K^* = \mathbb{D} \cup \partial\mathbb{D} \cup [a', -1] \cup [1, b']$. *If equality holds in* (2.5) *for some* $K \in F_4(a', b')$ *such that* $\mathbb{C}' \setminus K$ *is a regular domain then* $K = K^*$.

To prove (2.5) let $K \in F_4(a', b')$. A Steiner symmetrization with respect to the real axis shows that $\operatorname{cap} K \geq \operatorname{cap} K^*$. If $\operatorname{cap} K = \operatorname{cap} K^*$ and $\mathbb{C}' \setminus K$ is regular then by Jenkins's uniqueness result for symmetrization ([4]) we conclude that $K = K^*$ and the proof is complete.

Remark 2.3. It is interesting that the Steiner symmetrization result can only be applied to the formulation of the theorem that involves capacity and not to the original formulation that involves conformal radius.

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Department of Mathematics
University of Helsinki
FIN-00014 Helsinki, Finland
E-mail: betsakos@geom.helsinki.fi

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