COLLOQUIUM MATHEMATICUM

VOL. 80 1999 NO. 2

MAPPING PROPERTIES OF c_0

BY

PAUL LEWIS (DENTON, TEX.)

Abstract. Bessaga and Pełczyński showed that if c_0 embeds in the dual X^* of a Banach space X, then ℓ^1 embeds as a complemented subspace of X. Pełczyński proved that every infinite-dimensional closed linear subspace of ℓ^1 contains a copy of ℓ^1 that is complemented in ℓ^1. Later, Kadec and Pełczyński proved that every non-reflexive closed linear subspace of $L^1[0,1]$ contains a copy of ℓ^1 that is complemented in $L^1[0,1]$. In this note a traditional sliding hump argument is used to establish a simple mapping property of c_0 which simultaneously yields extensions of the preceding theorems as corollaries. Additional classical mapping properties of c_0 are briefly discussed and applications are given.

All Banach spaces in this note are defined over the real field. The canonical unit vector basis of c_0 will be denoted by (e_n), the canonical unit vector basis of ℓ^1 will be denoted by (e_n^*), and a continuous linear transformation will be referred to as an operator. The reader is referred to Diestel [3] or Lindenstrauss and Tzafriri [8] for undefined notation and terminology.

THEOREM 1. *If $T : c_0 \to X$ is an operator and (x_k^*) is any bounded sequence in X^* so that*

$$\sum_{k=1}^{\infty} |x_k^*(T(e_{n_k})) - 1| < \infty$$

for some subsequence $(T(e_{n_k}))$ of $(T(e_n))$, then there is a sequence (w_i^*) in $\{x_k^* - x_j^* : k, j \in \mathbb{N}\}$ so that (w_i^*) is equivalent to (e_i^*) as a basic sequence and $[w_i^*]$ is complemented in X^*.

Proof. Let $(b_k) = T(e_{n_k})$ for $k \in \mathbb{N}$, let C be a positive number so that $C > 1$ and $\|T(x)\| \le C\|x\|$ for all x, and choose $B > 1$ so that

$$2 \sup \|x_n^*\| < B.$$

Without loss of generality, suppose that

$$|x_n^*(b_n) - 1| < \frac{1}{BC \cdot 2^{n+4}}$$

1991 *Mathematics Subject Classification*: Primary 46B20.

for each n. Further, since (b_n) is weakly null, suppose that

$$\sum_{i=1}^{n-1} |x_i^*(b_n)| < \frac{1}{BC \cdot 2^{n+5}} \tag{1}$$

for each n. Now let $r_1 = 1$, $r_2 = 2$, and choose r_3 and r_4 so that $r_2 < r_3 < r_4$ and

$$|(x_{r_4}^* - x_{r_3}^*)(b_{r_2})| < \frac{1}{BC \cdot 2^{1+5}}.$$

Next choose r_5 and r_6 so that $r_4 < r_5 < r_6$ and

$$\begin{aligned} |(x_{r_6}^* - x_{r_5}^*)(b_{r_2})| &< \frac{1}{BC \cdot 2^{2+5}}, \\ |(x_{r_6}^* - x_{r_5}^*)(b_{r_4})| &< \frac{1}{BC \cdot 2^{2+5}}. \end{aligned} \tag{2}$$

An additional step clarifies the induction process. Choose r_7 and r_8 so that $r_6 < r_7 < r_8$ and

$$\begin{aligned} |(x_{r_8}^* - x_{r_7}^*)(b_{r_2})| &< \frac{1}{BC \cdot 2^{3+5}}, \\ |(x_{r_8}^* - x_{r_7}^*)(b_{r_4})| &< \frac{1}{BC \cdot 2^{3+5}}, \\ |(x_{r_8}^* - x_{r_7}^*)(b_{r_6})| &< \frac{1}{BC \cdot 2^{3+5}}. \end{aligned} \tag{3}$$

Continue this construction inductively, and let $u_n = b_{r_{2n}}$ and $z_n^* = x_{r_{2n}}^*$ for each n. Note that

$$|z_i^*(u_n) - x_{r_{2i-1}}^*(u_n)| < \frac{1}{BC \cdot 2^{i+4}}$$

for $n < i$. Further,

$$|z_n^*(u_n) - 1| < \frac{1}{BC \cdot 2^{n+4}} \quad \text{and} \quad \sum_{i=1}^{n} |z_i^*(u_{n+1})| < \frac{1}{BC \cdot 2^{(n+1)+5}}$$

for each n.

Next let $w_n^* = z_n^* - x_{r_{2n-1}}^* = x_{r_{2n}}^* - x_{r_{2n-1}}^*$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} |w_n^*(u_n) - 1| &\le |z_n^*(u_n) - 1| + |x_{r_{2n-1}}^*(u_n)| \\ &< \frac{1}{BC \cdot 2^{n+4}} + \frac{1}{BC \cdot 2^{n+5}} = \frac{3}{BC} \cdot \frac{1}{2^{n+1}} \cdot \frac{1}{2^4}. \end{aligned}$$

Also, $\|x_n^*\| \le B$ for each n.

Now suppose that $q \in \mathbb{N}$ and t_i is a non-zero real number for $1 \le i \le q$. If $\varepsilon_i = \operatorname{sgn}(t_i w_i^*(u_i))$, then

$$\begin{aligned}\sum_{i=1}^{q} t_i w_i^*(\varepsilon_i u_1) &\geq |t_1 w_1^*(u_1)| - \sum_{i=2}^{q} |w_i^* t_i(u_1)| \\ &\geq |t_1|\Big(1 - \frac{3}{BC}\cdot\frac{1}{2^2}\cdot\frac{1}{2^4}\Big) \\ &\quad - \Big(\frac{|t_2|}{BC\cdot 2^{1+5}} + \ldots + \frac{|t_q|}{BC\cdot 2^{(q-1)+5}}\Big).\end{aligned}$$

Further,

$$\begin{aligned}&\sum_{i=1}^{q} t_i w_i^*(\varepsilon_2 u_2) \\ &\quad = |t_2 w_2^*(u_2)| - \sum_{i=1, i\neq 2}^{q} t_i w_i^*(\varepsilon_2 u_2) \\ &\quad \geq |t_2|\Big(1 - \frac{3}{BC}\cdot\frac{1}{2^3}\cdot\frac{1}{2^4}\Big) \\ &\qquad - \Big(|t_1|\frac{1}{BC}\cdot\frac{1}{2^{2+5}} + |t_3|\frac{1}{BC}\cdot\frac{1}{2^{2+5}} + \ldots + |t_q|\frac{1}{BC}\cdot\frac{1}{2^{(q-1)+5}}\Big).\end{aligned}$$

(Observe that

$$\begin{aligned}|w_1^*(u_2)| &= |(x_2^* - x_1^*)(b_{r_4})| \leq |x_2^*(b_{r_4})| + |x_1^*(b_{r_4})| \\ &< \frac{1}{BC\cdot 2^{r_4+5}} + \frac{1}{BC\cdot 2^{r_4+5}} < \frac{1}{BC\cdot 2^{2+5}}\end{aligned}$$

from (1). Also,

$$|w_3^*(u_2)| = |(x_{r_6}^* - x_{r_5}^*)(b_{r_4})| \leq \frac{1}{BC\cdot 2^{2+5}}$$

from (2), and

$$|(x_{r_8}^* - x_{r_7}^*)(b_{r_4})| < \frac{1}{BC\cdot 2^{3+5}}$$

from (3).)

In general,

$$\begin{aligned}&\Big\langle \sum_{i=1}^{q} \varepsilon_i u_i, \sum_{n=1}^{q} t_n w_n^* \Big\rangle \\ &\qquad \geq |t_1|\Big(1 - \frac{3}{BC}\cdot\frac{1}{2^2}\cdot\frac{1}{2^4}\Big) \\ &\qquad\quad - |t_1|\Big(\frac{1}{BC\cdot 2^{r_4+5}} + \frac{1}{BC\cdot 2^{r_6+5}} + \ldots + \frac{1}{BC\cdot 2^{r_{2q}+5}}\Big) \\ &\qquad\quad + |t_2|\Big(1 - \frac{3}{BC}\cdot\frac{1}{2^3}\cdot\frac{1}{2^4}\Big)\end{aligned}$$

$$
\begin{aligned}
&-|t_2|\Big(\frac{1}{BC\cdot 2^{2+4}}+\frac{1}{BC\cdot 2^{r_6+5}}+\ldots+\frac{1}{BC\cdot 2^{r_{2q}+5}}\Big)\\
&+|t_3|\Big(1-\frac{3}{BC}\cdot\frac{1}{2^4}\cdot\frac{1}{2^4}\Big)\\
&-|t_3|\Big(2\frac{1}{BC\cdot 2^{3+4}}+\frac{1}{BC\cdot 2^{r_8+5}}+\frac{1}{BC\cdot 2^{r_{10}+5}}+\ldots+\frac{1}{BC\cdot 2^{r_{2q}+5}}\Big)\\
&+|t_4|\Big(1-\frac{3}{BC}\cdot\frac{1}{2^5}\cdot\frac{1}{2^4}\Big)\\
&-|t_4|\Big(3\frac{1}{BC\cdot 2^{4+4}}+\frac{1}{BC\cdot 2^{r_{10}+5}}+\ldots+\frac{1}{BC\cdot 2^{r_{2q}+5}}\Big)+\ldots+\\
&+|t_q|\Big(1-\frac{3}{BC}\cdot\frac{1}{2^{q+1}}\cdot\frac{1}{2^4}\Big)-|t_q|\Big(\frac{q-1}{BC\cdot 2^{q+4}}\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{3}{BC\cdot 2^2 2^4}+\frac{1}{BC\cdot 2^{r_4+5}}+\ldots+\frac{1}{BC\cdot 2^{r_{2q}}}&\le\frac{2}{BC\cdot 2^4},\\
\frac{3}{BC\cdot 2^3 2^4}+\frac{1}{BC\cdot 2^{2+4}}+\frac{1}{BC\cdot 2^{r_6+5}}+\ldots+\frac{1}{BC\cdot 2^{r_{2q}+5}}&\le\frac{2}{BC\cdot 2^4},\\
&\ldots\\
\frac{3}{BC\cdot 2^{q+1}2^4}+\frac{q-1}{BC\cdot 2^{q+4}}&\le\frac{2}{BC\cdot 2^4}.
\end{aligned}
$$

Consequently,

$$
\Big\langle\sum_{i=1}^q\varepsilon_i u_i,\sum_{n=1}^q t_n w_n^*\Big\rangle\ge\Big(\sum_{i=1}^q|t_i|\Big)\Big(1-\frac{2}{BC\cdot 2^4}\Big)>0.
$$

Thus $\sum_{i=1}^q\varepsilon_i u_i\ne 0$, and

$$
\begin{aligned}
\Big\|\sum_{i=1}^q t_i w_i^*\Big\|&\ge\Big(1\Big/\Big\|\sum_{i=1}^q\varepsilon_i u_i\Big\|\Big)\Big\langle\sum_{i=1}^q\varepsilon_i u_i,\sum_{n=1}^q t_n w_n^*\Big\rangle\\
&\ge\Big(\sum_{i=1}^q|t_i|\Big)\Big(\Big(1-\frac{1}{BC\cdot 2^3}\Big)c^{-1}\Big).
\end{aligned}
$$

Hence

$$
\Big(\Big(1-\frac{1}{BC\cdot 2^3}\Big)c^{-1}\Big)\Big(\sum_{i=1}^q|t_i|\Big)\le\Big\|\sum_{i=1}^q t_i w_i^*\Big\|\le B\sum_{i=1}^q|t_i|,
$$

and $(w_i^*)\sim(e_i^*)$.

Next we show that $[w_n^*]$ is complemented in X^*. Suppose that $v^*=\sum_{n=1}^\infty t_n w_n^*$, and let $U:X^*\to[w_n^*]$ be defined by

$$U(x^*) = \sum_n x^*(u_n) w_n^*.$$

Since (u_n) is a subsequence of (b_k) and $\sum b_k$ is weakly unconditionally convergent, it is clear that U is well defined, continuous, and linear. Now observe that

$$\begin{aligned}
&\|v^* - U(v^*)\| \\
&\quad = \Big\| \sum t_n w_n^* - \sum t_n U(w_n^*) \Big\| \\
&\quad = \Big\| \sum_{n=1}^{\infty} t_n w_n^* - \sum_{n=1}^{\infty} t_n \Big(\sum_{k=1}^{\infty} w_n^*(u_k) w_k^* \Big) \Big\| \\
&\quad = \Big\| \sum_{n=1}^{\infty} t_n w_n^* - \sum_{n=1}^{\infty} t_n w_n^*(u_n) w_n^* - \sum_{n=1}^{\infty} t_n \Big(\sum_{k=1,\, k\neq n}^{\infty} w_n^*(u_k) w_k^* \Big) \Big\| \\
&\quad \le \sum_{n=1}^{\infty} |t_n| \cdot |1 - w_n^*(u_n)| \cdot \|w_n^*\| + \sum_{n=1}^{\infty} |t_n| \Big(\sum_{k=1,\, k\neq n}^{\infty} |w_n^*(u_k)| B \Big) \\
&\quad \le \sum_{n=1}^{\infty} |t_n| \Big(\sup_k \Big\{ |1 - w_k^*(u_k)| + \sum_{i=1,\, i\neq k}^{\infty} |w_k^*(u_i)| \Big\} \Big) B.
\end{aligned}$$

Also,

$$\sum_{n=1}^{\infty} |t_n| \le \frac{c}{1 - \frac{1}{BC\cdot 2^3}} \|v^*\|.$$

Further,

$$\sup_k |1 - w_k^*(u_k)| \le \frac{3}{BC \cdot 2^2 2^4},$$

and $\|w_k^*\| \le B$ for each k.

Next note that

$$\begin{aligned}
\sum_{k=2}^{\infty} |w_1^*(u_k)| &= \sum_{k=2}^{\infty} |(x_2^* - x_1^*)(u_k)| \\
&= |(x_2^* - x_1^*)(T(e_{r_4}))| + |(x_2^* - x_1^*)(T(e_{r_6}))| + \dots \\
&\le (|x_2^* T(e_{r_4})| + |x_1^* T(e_{r_4})|) + (|x_2^* T(e_{r_6})| + |x_1^* T(e_{r_6})|) + \dots \\
&< \frac{1}{BC \cdot 2^{r_4+5}} + \frac{1}{BC \cdot 2^{r_6+5}} + \dots < \frac{1}{BC \cdot 2^{r_4+4}} < \frac{1}{BC \cdot 2^4}.
\end{aligned}$$

A similar argument shows that

$$\sum_{i=1,\, i\neq k}^{\infty} |w_k^*(u_i)| < \frac{1}{BC \cdot 2^4}$$

for each k. Thus

$$\|v^* - U(v^*)\| \le \frac{c}{1 - \frac{1}{BC\cdot 2^3}} \|v^*\| \left(\frac{3}{BC \cdot 2^2 2^4} + \frac{1}{BC \cdot 2^4} \right) B < \frac{1}{7} \|v^*\|.$$

If $U_1 = U_{|[w_i^*]}$, then $\|\text{Identity} - U_1\|_{|[w_i^*]} < 1$, and U_1 is invertible on $[w_i^*]$. It is easy to see that $U_1^{-1}U$ is a projection from X^* onto $[w_i^*]$. ■

Remark. (a) The operator $T : c_0 \to X$ satisfies the hypotheses of Theorem 1 if and only if $\liminf \|T(e_n)\| > 0$. H. Rosenthal [11] has given a penetrating study of the situation in which $T : \ell^\infty(\Gamma) \to X$ is an operator so that $\inf_{\gamma\in\Gamma} \|T(e_\gamma)\| > 0$.

(b) If (x_k^*) is w^*-null, the proof of Theorem 1 makes it clear that we may choose the sequence (w_i^*) in the conclusion of the theorem to be w^*-null.

As the following corollaries indicate, Theorem 1 unifies and extends several classical results.

Corollary 2 ([1, Thm. 4], [3, p. 48]). *If c_0 embeds isomorphically in the dual X^* of the Banach space X, then X contains a copy of ℓ^1 which is complemented* (*in X^{**} and thus*) *in X.*

Proof. If $T : c_0 \to X^*$ is an isomorphism, then let (x_n) be a bounded sequence in X ($\subseteq X^{**}$) so that $\sum_{n=1}^\infty |x_n(T(e_n))-1| = 0$. Apply Theorem 1 to the sequence (x_n). ■

Corollary 3 ([10], [3, p. 72]). *If ℓ^1 is a quotient of X, then X contains a copy of ℓ^1 which is complemented in X^{**}.*

Proof. If $T : X \to \ell^1$ is a surjective operator, then $T^* : \ell^\infty \to X^*$ is an isomorphism. Hence $T^*_{|c_0}$ is an isomorphism. ■

If Σ is a σ-algebra, (μ_n) is a bounded sequence in $\text{cabv}(\Sigma, X)$, and $0 < \varepsilon < \delta$, then (μ_n) is said to be (δ,ε)-*relatively disjoint* [11] if there is a pairwise disjoint sequence (A_n) in Σ so that

$$|\mu_n|(A_n) > \delta \quad \text{and} \quad \sum_{m=1,\, m\ne n}^\infty |\mu_n|(A_m) < \varepsilon$$

for each n. Further, (μ_n) is said to be *relatively disjoint* if it is (δ,ε)-relatively disjoint for some pair (δ,ε). Rosenthal [11] and Kadec and Pełczyński [7] showed that if (μ_n) is a relatively disjoint sequence in $\text{cabv}(\Sigma, X)$, then $(\mu_n) \sim (e_n^*)$ and $[\mu_n]$ is complemented in $\text{cabv}(\Sigma, X)$.

If $\mathcal{A}$ is an algebra of subsets of Ω, then $\text{fabv}(\mathcal{A}, X)$ denotes the Banach space (total variation norm) of all finitely additive set functions $m : \mathcal{A} \to X$ which have finite variation. Both [4] and [6] contain an extensive discussion of spaces of measures. In addition, we note that [4] includes a detailed presentation of results related to the Radon–Nikodym property. Note that part (i) of Corollary 4 below contains an extension of Proposition 3.1 of [11] to the setting of finitely additive set functions defined on an algebra of sets. Further, we remark that in a classic paper Kadec and Pełczyński [7, Theorem 6] showed that if Y is any non-reflexive closed linear subspace of $L^1[0,1]$, then Y contains a copy of ℓ^1 which is complemented in $L^1[0,1]$. Part (v) of the next corollary shows that if X and X^* have the Radon–Nikodym property, then any non-reflexive closed linear subspace of $L^1(\mu, X)$ contains a copy of ℓ^1 which is complemented in $L^1(\mu, X)$.

COROLLARY 4. (i) *If* (μ_n) *is any bounded sequence in* $\mathrm{fabv}(\mathcal{A}, X)$ *for which there is a pairwise disjoint sequence* (A_n) *in* $\mathcal{A}$ *and an* $\varepsilon > 0$ *so that*

$$|\mu_n|(A_n) > \varepsilon$$

for each n, *then there is a sequence* (ν_i) *in* $\{\mu_n - \mu_k : k, n \in \mathbb{N}\}$ *so that* $(\nu_i) \sim (e_i^*)$ *and* $[\nu_i]$ *is complemented in* $\mathrm{fabv}(\mathcal{A}, X)$.

(ii) *If* K *is a relatively weakly compact subset of* $\mathrm{fabv}(\mathcal{A}, X)$ *and* (A_i) *is a pairwise disjoint sequence of members of* $\mathcal{A}$, *then* $\lim_i |\mu|(A_i) = 0$ *uniformly for* $\mu \in K$.

(iii) *If* K *is a relatively weakly compact subset of* $\mathrm{cabv}(\Sigma, X)$, *then* $\{|\mu| : \mu \in K\}$ *is uniformly countably additive.*

(iv) *If* μ *is a finite positive measure on* Σ *and* K *is a relatively weakly compact subset of the space* $L^1(\mu, X)$ *of Bochner integrable functions*, *then* K *is uniformly integrable.*

(v) *If* Y *is a closed linear subspace of* $\mathrm{fabv}(\mathcal{A}, X)$, Y *is not reflexive*, *and* X *and* X^* *have the Radon–Nikodym property*, *then* Y *contains a copy of* ℓ^1 *which is complemented in* $\mathrm{fabv}(\mathcal{A}, X)$.

Proof. (i) For each n let $(A_{n_i})_{i=1}^{k_n}$ be a partition of A_n and $(x_{n_i}^*)_{i=1}^{k_n}$ be points in the unit ball of X^* so that

$$\sum_{i=1}^{k_n} x_{n_i}^* \mu_n(A_{n_i}) > \varepsilon.$$

Now define the X^*-valued simple function s_n by

$$s_n = \sum_{i=1}^{k_n} \chi_{A_{n_i}} x_{n_i}^*,$$

and observe that $\int s_n \, d\mu_n > \varepsilon$. Define $T : c_0 \to \mathrm{fabv}(\mathcal{A}, X)^*$ by

$$T((\gamma_n)) = \sum_n \gamma_n s_n.$$

Then T is an operator. Normalize and use Theorem 1 to conclude that some sequence (ν_i) in $\{\mu_n - \mu_k : n, k \in \mathbb{N}\}$ is equivalent to (e_n^*) and that $[\nu_n]$ is complemented in $\mathrm{fabv}(\mathcal{A}, X)$.

(ii) Suppose that $\varepsilon > 0$ and (μ_i) is a sequence in K so that $|\mu_i|(A_i) > \varepsilon$ for each i. Part (i) ensures that (e_n^*) is equivalent to some sequence in $K-K$. However, this is impossible since $K - K$ is relatively weakly compact.

(iii) Since each member of K is a countably additive measure on a σ-algebra, $|K| = \{|\mu| : \mu \in K\}$ is uniformly countably additive if and only if $\lim_i |\mu|(A_i) = 0$ uniformly for $\mu \in K$ whenever (A_i) is a pairwise disjoint sequence from Σ. Deny the uniform countable additivity of $|K|$, repeat the same construction as in (i), and obtain the same contradiction as in (ii).

(iv) If $f \in L^1(\mu, X)$ and $A \in \Sigma$, put

$$\nu_f(A) = \int_A f \, d\mu.$$

It is well known that $\lim_{\mu(A)\to 0} |\nu_f|(A) = 0$ uniformly for $f \in K$ (i.e., K is uniformly integrable) if and only if $\{|\nu_f| : f \in K\}$ is uniformly countably additive. Appeal to (iii).

(v) If Y is not reflexive, then B_Y is not relatively weakly compact in $\mathrm{fabv}(\mathcal{A},X)$. By Theorem 4.1 of Brooks and Dinculeanu [2], there is a pairwise disjoint sequence (A_i) in $\mathcal{A}$, an $\varepsilon > 0$, and a sequence (μ_i) in B_Y so that $|\mu_i|(A_i) > \varepsilon$ for each i. The construction in (i) above shows that Y contains a copy of ℓ^1 which is complemented in $\mathrm{fabv}(\mathcal{A}, X)$. ■

In the following corollary, $\mathcal{P}$ denotes the σ-algebra of all subsets of $\mathbb{N}$.

COROLLARY 5 ([9, Lemma 2], [3, p. 74]). *Every infinite-dimensional closed linear subspace of ℓ^1 contains a copy of ℓ^1 which is complemented in* $\mathrm{fabv}(\mathcal{P})$ *and thus in ℓ^1.*

Proof. Every infinite-dimensional subspace of ℓ^1 is non-reflexive. ■

COROLLARY 6 ([4, p. 149]). *If (Ω, Σ, μ) is a finite measure space and X^* is a quotient of $L^\infty(\mu)$, then either X is reflexive or X contains a copy of ℓ^1 which is complemented in X^{**}. Consequently, if X^{**} is contained in $L^1(\mu)$, then X is reflexive or ℓ^1 is a complemented subspace of X.*

Proof. If $T : L^\infty(\mu) \to X^*$ is a surjection and X is not reflexive, then T is not weakly compact. Hence T is not unconditionally converging and is an isomorphism on a copy of c_0. Thus X contains a copy of ℓ^1 which is complemented in X^{**}.

If $L : X^{**} \to L^1(\mu)$ is an isomorphism, then $L^* : L^\infty(\mu) \to X^{***}$ is a surjection, X^* is a quotient of $L^\infty(\mu)$, and X is reflexive or X contains a complemented copy of ℓ^1. ■

If $T : c_0 \to X$ is an isomorphism, classical techniques of Singer [13] can be used to easily produce complemented copies of both c_0 and ℓ^1.

THEOREM 7. *If $T : c_0 \to X$ is an isomorphism, (f_n^*) is any bounded sequence in X^* so that*

$$f_n^*(T(e_m)) = \delta_{nm},$$

and (h_k^*) is any subsequence of (f_n^*), then $[h_k^*]$ is complemented in X^*. Further, if (h_k^*) is w^*-null in X^* and (y_k) is the corresponding subsequence of $(T(e_n))$, then $[y_k]$ is complemented in X.

Proof. Suppose that $T, (f_n^*)$, and (h_k^*) are as in the first statement in the theorem. Let C be a bound for $(\|f_n^*\|)$, let (y_k^*) be the sequence of coefficient functionals for the basic sequence (y_k) (which is equivalent to (e_k)), and choose positive numbers A and B so that

$$A\sum|\alpha_i| \le \Big\|\sum \alpha_i y_i^*\Big\| \le B\sum|\alpha_i|$$

for each finite sequence $(\alpha_1, \ldots, \alpha_m)$ of real numbers. Therefore

$$A\sum|\alpha_i| \le \Big\|\sum \alpha_i h^*_{i|[y_n]}\Big\| \le \Big\|\sum \alpha_i h_i^*\Big\| \le C\sum|\alpha_i|.$$

As noted on p. 91 of Singer [13],

$$\Big\{f^* \in X^* : \sum_{k=1}^{\infty} f^*(y_k)h_k^* \text{ converges}\Big\} = [y_k]^{\perp} + [h_k^*].$$

Since $(y_k) \sim (e_k)$ and $(h_k^*) \sim (e_k^*)$, we have $[y_k]^{\perp} + [h_k^*] = X^*$. Further, if (h_k^*) is w^*-null, then

$$\Big\{x \in X : \sum_{k=1}^{\infty} h_k^*(x)y_k \text{ converges}\Big\} = [y_k] + [h_k^*]_{\perp} = X.$$

Consequently, each of these direct sums is closed. Straightforward closed graph arguments show that these direct sums are also topological. ■

We remark that if X is separable (and T and (f_n^*) have the same meaning as in the statement of Theorem 7), then Veech's proof [15] of Sobczyk's theorem [14], [3, p. 71] simply shows that there is a bounded sequence (g_n^*) in $[T(e_n))]^{\perp}$ so that $(f_n^* - g_n^*)$ is w^*-null. Certainly $(T(e_n), f_n^* - g_n^*)$ is biorthogonal in this case.

The next corollary shows that a result of Saab and Saab [12] dealing with complemented copies of c_0 in injective tensor products is an immediate consequence of Theorem 7. Chapter 8 of [4] contains an excellent discussion of the least crossnorm tensor product completion of Banach spaces.

COROLLARY 8 ([12]). *If X contains a copy of c_0, Y is an infinite-dimensional Banach space and $Z = X \otimes_\lambda Y$ is the least crossnorm tensor product completion of X and Y, then Z contains a complemented copy of c_0.*

Proof. Let (x_n) be a sequence in X so that $(x_n) \sim (e_n)$, let (x_n^*) be a bounded sequence in X^* so that $x^*(x_m) = \delta_{nm}$, and let (y_n^*) be a w^*-null sequence in Y^* so that $\|y_n^*\| = 1$ for each n. (The Josefson–Nissenzweig Theorem [3] guarantees the existence of (y_n^*).) Choose a sequence (y_n) in Y

so that $\|y_n\| \le 3/2$ and $y_n(y_n^*) = 1$ for each n. Then $(x_n^* \otimes y_n^*)$ is a w^*-null sequence in Z^*, $(x_n \otimes y_n) \sim (e_n)$, and $x_n^* \otimes y_n^*(x_m \otimes y_m) = x_n^*(x_n)y_n^*(y_m) = \delta_{nm}$. Now appeal to Theorem 7. ∎

We note that precisely the same argument yields the next result.

COROLLARY 9. *If the Banach space X contains a copy of c_0 and Y is an infinite-dimensional space*, *then the Banach space $K(X^*, Y)$ of compact operators from X^* to Y contains a complemented copy of c_0.*

REFERENCES

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.

[2] J. Brooks and N. Dinculeanu, *Strong additivity*, *absolute continuity*, *and compactness in spaces of measures*, J. Math. Anal. Appl. 45 (1974), 156–175.

[3] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, New York, 1984.

[4] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.

[5] N. Dunford, *A mean ergodic theorem*, Duke Math. J. 5 (1939), 635–646.

[6] N. Dunford and J. Schwartz, *Linear Operators. Part I*, Interscience, New York, 1958.

[7] M. I. Kadec and A. Pełczyński, *Bases*, *lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), 161–176.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.

[9] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209–228.

[10] —, *On strictly singular and strictly cosingular operators. II*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 37–41.

[11] H. Rosenthal, *On relatively disjoint families of measures*, *with some applications to Banach space theory*, Studia Math. 37 (1970), 13–36.

[12] E. Saab and P. Saab, *On complemented copies of c_0 in injective tensor products*, in: Contemp. Math. 52, Amer. Math. Soc., 1986, 131–135.

[13] I. Singer, *Bases in Banach Spaces II*, Springer, Berlin, 1981.

[14] A. Sobczyk, *Projections of the space m on its subspace c*, Bull. Amer. Math. Soc. 47 (1941), 78–106.

[15] W. Veech, *Short proof of Sobczyk's theorem*, Proc. Amer. Math. Soc. 28 (1971), 627–628.

University of North Texas
Denton, Texas
E-mail: lewis@unt.edu

Received 10 November 1998