MAPPING PROPERTIES OF $c_0$

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Abstract. Bessaga and Pełczyński showed that if $c_0$ embeds in the dual $X^*$ of a Banach space $X$, then $\ell^1$ embeds as a complemented subspace of $X$. Pełczyński proved that every infinite-dimensional closed linear subspace of $\ell^1$ contains a copy of $\ell^1$ that is complemented in $\ell^1$. Later, Kadec and Pełczyński proved that every non-reflexive closed linear subspace of $L^1[0,1]$ contains a copy of $\ell^1$ that is complemented in $L^1[0,1]$. In this note a traditional sliding hump argument is used to establish a simple mapping property of $c_0$ which simultaneously yields extensions of the preceding theorems as corollaries. Additional classical mapping properties of $c_0$ are briefly discussed and applications are given.

All Banach spaces in this note are defined over the real field. The canonical unit vector basis of $c_0$ will be denoted by $(e_n)$, the canonical unit vector basis of $\ell^1$ will be denoted by $(e^*_n)$, and a continuous linear transformation will be referred to as an operator. The reader is referred to Diestel [3] or Lindenstrauss and Tzafriri [8] for undefined notation and terminology.

Theorem 1. If $T: c_0 \to X$ is an operator and $(x_k^*)$ is any bounded sequence in $X^*$ so that

$$\sum_{k=1}^{\infty} |x_k^*(T(e_{n_k})) - 1| < \infty$$

for some subsequence $(T(e_{n_k}))$ of $(T(e_n))$, then there is a sequence $(w_i^*)$ in $\{x_k^* - x_j^* : k, j \in \mathbb{N}\}$ so that $(w_i^*)$ is equivalent to $(e^*_i)$ as a basic sequence and $[w_i^*]$ is complemented in $X^*$.

Proof. Let $(b_k) = T(e_{n_k})$ for $k \in \mathbb{N}$, let $C$ be a positive number so that $C > 1$ and $\|T(x)\| \leq C\|x\|$ for all $x$, and choose $B > 1$ so that

$$2\sup \|x_n^*\| < B.$$

Without loss of generality, suppose that

$$|x_n^*(b_n) - 1| < \frac{1}{BC \cdot 2^n + 4}$$

1991 Mathematics Subject Classification: Primary 46B20.
for each \( n \). Further, since \((b_n)\) is weakly null, suppose that

\[
(1) \sum_{i=1}^{n-1} |x^*_i(b_n)| < \frac{1}{BC \cdot 2^{n+5}}
\]

for each \( n \). Now let \( r_1 = 1, r_2 = 2 \), and choose \( r_3 \) and \( r_4 \) so that \( r_2 < r_3 < r_4 \) and

\[
|\langle x^*_r - x^*_s \rangle(b_{r_2})| < \frac{1}{BC \cdot 2^{1+5}}.
\]

Next choose \( r_5 \) and \( r_6 \) so that \( r_4 < r_5 < r_6 \) and

\[
|\langle x^*_r - x^*_s \rangle(b_{r_2})| < \frac{1}{BC \cdot 2^{2+5}},
\]

\[
|\langle x^*_r - x^*_s \rangle(b_{r_4})| < \frac{1}{BC \cdot 2^{4+5}}.
\]

An additional step clarifies the induction process. Choose \( r_7 \) and \( r_8 \) so that \( r_6 < r_7 < r_8 \) and

\[
|\langle x^*_r - x^*_s \rangle(b_{r_2})| < \frac{1}{BC \cdot 2^{3+5}},
\]

\[
|\langle x^*_r - x^*_s \rangle(b_{r_4})| < \frac{1}{BC \cdot 2^{5+5}},
\]

\[
|\langle x^*_r - x^*_s \rangle(b_{r_6})| < \frac{1}{BC \cdot 2^{7+5}}.
\]

Continue this construction inductively, and let \( u_n = b_{r_{2n}} \) and \( z^*_n = x^*_{r_{2n}} \) for each \( n \). Note that

\[
|z^*_n(u_n) - x^*_{r_{2i-1}}(u_n)| < \frac{1}{BC \cdot 2^{i+4}}
\]

for \( n < i \). Further,

\[
|z^*_n(u_n) - 1| < \frac{1}{BC \cdot 2^{n+4}} \quad \text{and} \quad \sum_{i=1}^{n} |z^*_i(u_{n+1})| < \frac{1}{BC \cdot 2^{(n+1)+5}}
\]

for each \( n \).

Next let \( w^*_n = z^*_n - x^*_n = x^*_{r_{2n}} - x^*_{r_{2n-1}} \) for \( n \in \mathbb{N} \). Then

\[
|w^*_n(u_n) - 1| \leq |z^*_n(u_n) - 1| + |x^*_{r_{2n-1}}(u_n)|
\]

\[
< \frac{1}{BC \cdot 2^{n+4}} + \frac{1}{BC \cdot 2^{n+5}} = \frac{3}{BC} \cdot \frac{1}{2^{n+1}} \cdot \frac{1}{2^4}.
\]

Also, \( \|x^*_n\| \leq B \) for each \( n \).

Now suppose that \( q \in \mathbb{N} \) and \( t_i \) is a non-zero real number for \( 1 \leq i \leq q \). If \( \varepsilon_i = \text{sgn}(t_i w^*_i(u_i)) \), then
\[
\sum_{i=1}^{q} t_i w_i^* (\varepsilon_i u_1) \geq |t_1 w_1^* (u_1)| - \sum_{i=2}^{q} |w_i^* t_i (u_1)| \\
\geq |t_1| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^2} \cdot \frac{1}{2^4} \right) \\
- \left( \frac{|t_2|}{BC \cdot 2^{1+5}} + \ldots + \frac{|t_q|}{BC \cdot 2^{(q-1)+5}} \right).
\]

Further,
\[
\sum_{i=1}^{q} t_i w_i^* (\varepsilon_2 u_2) \\
= |t_2 w_2^* (u_2)| - \sum_{i=1, i \neq 2}^{q} t_i w_i^* (\varepsilon_2 u_2) \\
\geq |t_2| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^2} \cdot \frac{1}{2^4} \right) \\
- \left( \frac{|t_1|}{BC \cdot 2^{2+5}} + \frac{|t_3|}{BC \cdot 2^{2+5}} + \ldots + \frac{|t_q|}{BC \cdot 2^{(q-1)+5}} \right).
\]

(Observe that
\[
|w_1^* (u_2)| = |(x_2^* - x_1^*)(b_{r_4})| \leq |x_2^*(b_{r_4})| + |x_1^*(b_{r_4})| \\
< \frac{1}{BC \cdot 2^{r_4+5}} + \frac{1}{BC \cdot 2^{r_4+5}} < \frac{1}{BC \cdot 2^{2+5}}
\]
from (1). Also,
\[
|w_2^* (u_2)| = |(x_{r_6}^* - x_{r_5}^*)(b_{r_4})| \leq \frac{1}{BC \cdot 2^{2+5}}
\]
from (2), and
\[
|(x_{r_8}^* - x_{r_7}^*)(b_{r_4})| < \frac{1}{BC \cdot 2^{4+5}}
\]
from (3).)

In general,
\[
\left\langle \sum_{i=1}^{q} \varepsilon_i u_i, \sum_{n=1}^{q} t_n w_n^* \right\rangle \\
\geq |t_1| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^2} \cdot \frac{1}{2^4} \right) \\
- |t_1| \left( \frac{1}{BC \cdot 2^{r_4+5}} + \frac{1}{BC \cdot 2^{r_6+5}} + \ldots + \frac{1}{BC \cdot 2^{r_2q+5}} \right) \\
+ |t_2| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^3} \cdot \frac{1}{2^4} \right)
\]
\[-|t_2| \left( \frac{1}{BC \cdot 2^{2}+4} + \frac{1}{BC \cdot 2^{r_6+5}} + \ldots + \frac{1}{BC \cdot 2^{r_q+5}} \right) \]
\[+ |t_3| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^4} \cdot \frac{1}{2^7} \right) \]
\[-|t_3| \left( \frac{2}{BC \cdot 2^{4}+4} + \frac{1}{BC \cdot 2^{r_8+5}} + \frac{1}{BC \cdot 2^{r_{10}+5}} + \ldots + \frac{1}{BC \cdot 2^{r_q+5}} \right) \]
\[+ |t_4| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^5} \cdot \frac{1}{2^7} \right) \]
\[-|t_4| \left( \frac{3}{BC \cdot 2^{4}+4} + \frac{1}{BC \cdot 2^{r_{10}+5}} + \ldots + \frac{1}{BC \cdot 2^{r_q+5}} \right) + \ldots \]
\[+ |t_q| \left( 1 - \frac{3}{BC} \cdot \frac{1}{2^{q+1}} \cdot \frac{1}{2^4} \right) - |t_q| \left( \frac{q-1}{BC \cdot 2^{q+4}} \right). \]

Note that
\[\frac{3}{BC \cdot 2^{2}+4} + \frac{1}{BC \cdot 2^{r_4+5}} + \ldots + \frac{1}{BC \cdot 2^{r_q+5}} \leq \frac{2}{BC \cdot 2^4}, \]
\[\frac{1}{BC \cdot 2^{4}+4} + \frac{1}{BC \cdot 2^{r_{6}+5}} + \ldots + \frac{1}{BC \cdot 2^{r_{2q}+5}} \leq \frac{2}{BC \cdot 2^4}, \]
\[\frac{1}{BC \cdot 2^{q+1}+2^4} + \frac{q-1}{BC \cdot 2^{q+4}} \leq \frac{2}{BC \cdot 2^4}. \]

Consequently,
\[\langle \sum_{i=1}^{q} \varepsilon_i u_i, \sum_{n=1}^{q} t_n w_n^* \rangle \geq \left( \sum_{i=1}^{q} |t_i| \right) \left( 1 - \frac{2}{BC \cdot 2^4} \right) > 0. \]

Thus \(\sum_{i=1}^{q} \varepsilon_i u_i \neq 0\), and
\[\left\| \sum_{i=1}^{q} t_i w_i^* \right\| \geq \left( \frac{1}{\left\| \sum_{i=1}^{q} \varepsilon_i u_i \right\|} \right) \langle \sum_{i=1}^{q} \varepsilon_i u_i, \sum_{n=1}^{q} t_n w_n^* \rangle \]
\[\geq \left( \sum_{i=1}^{q} |t_i| \right) \left( \left( 1 - \frac{1}{BC \cdot 2^3} \right) c^{-1} \right). \]

Hence
\[\left( \left( 1 - \frac{1}{BC \cdot 2^3} \right) c^{-1} \right) \left( \sum_{i=1}^{q} |t_i| \right) \leq \left\| \sum_{i=1}^{q} t_i w_i^* \right\| \leq B \sum_{i=1}^{q} |t_i|, \]
and \((w_n^*) \sim (e_i^*)\).

Next we show that \([w_n^*]\) is complemented in \(X^*\). Suppose that \(v^* = \sum_{n=1}^{\infty} t_n w_n^*\), and let \(U : X^* \to [w_n^*]\) be defined by
\[ U(x^*) = \sum_n x^*(u_n)w^*_n. \]

Since \((u_n)\) is a subsequence of \((b_k)\) and \(\sum b_k\) is weakly unconditionally convergent, it is clear that \(U\) is well defined, continuous, and linear. Now observe that

\[
\|v^* - U(v^*)\| = \left\| \sum t_n w^*_n - \sum t_n U(w^*_n) \right\|
= \left\| \sum_{n=1}^{\infty} t_n w^*_n - \sum_{n=1}^{\infty} t_n \left( \sum_{k=1}^{\infty} w^*_n(u_k)w^*_k \right) \right\|
= \left\| \sum_{n=1}^{\infty} t_n w^*_n - \sum_{n=1}^{\infty} t_n w^*_n(u_n)w^*_n - \sum_{n=1}^{\infty} t_n \left( \sum_{k=1, k \neq n}^{\infty} w^*_n(u_k)w^*_k \right) \right\|
\leq \sum_{n=1}^{\infty} |t_n| |1 - w^*_n(u_n)| \cdot \|w^*_n\| + \sum_{n=1}^{\infty} |t_n| \left( \sum_{k=1, k \neq n}^{\infty} |w^*_n(u_k)|B \right)
\leq \sum_{n=1}^{\infty} |t_n| \left( \sup_k \left\{ |1 - w^*_k(u_k)| + \sum_{i=1, i \neq k}^{\infty} |w^*_k(u_i)| \right\} \right) B.
\]

Also,

\[
\sum_{n=1}^{\infty} |t_n| \leq \frac{c}{\|v^*\|}.n
\]

Further,

\[
\sup_k |1 - w^*_k(u_k)| \leq \frac{3}{BC \cdot 2^{24}},
\]

and \(\|w^*_n\| \leq B\) for each \(k\).

Next note that

\[
\sum_{k=2}^{\infty} |w^*_k(u_k)| = \sum_{k=2}^{\infty} |(x^*_2 - x^*_1)(u_k)|
= |(x^*_2 - x^*_1)(T(e_{r_4}))| + |(x^*_2 - x^*_1)(T(e_{r_5}))| + \ldots
\leq |x^*_2 T(e_{r_4})| + |x^*_1 T(e_{r_5})| + (|x^*_2 T(e_{r_4})| + |x^*_1 T(e_{r_5})|) + \ldots
< \frac{1}{BC \cdot 2^{r_4 + 5}} + \frac{1}{BC \cdot 2^{r_5 + 5}} + \ldots < \frac{1}{BC \cdot 2^{r_4 + 4}} < \frac{1}{BC \cdot 2^4}.
\]

A similar argument shows that

\[
\sum_{i=1, i \neq k}^{\infty} |w^*_k(u_i)| < \frac{1}{BC \cdot 2^4}
\]
for each $k$. Thus

$$
\| v^* - U(v^*) \| \leq \frac{c}{1 - \frac{BC}{2^2}} \| v^* \| \left( \frac{3}{BC \cdot 2^2} + \frac{1}{BC \cdot 2^4} \right) B < \frac{1}{7} \| v^* \|.
$$

If $U_1 = U|_{[w^*_1]}$, then $\| \text{Identity} - U_1 \| |w^*_1| < 1$, and $U_1$ is invertible on $[w^*_1]$. It is easy to see that $U_1^{-1} U$ is a projection from $X^*$ onto $[w^*_1]$. □

REM ARK. (a) The operator $T : c_0 \to X$ satisfies the hypotheses of Theorem 1 if and only if $\lim \inf \| T(e_n) \| > 0$. H. Rosenthal [11] has given a penetrating study of the situation in which $T : \ell^\infty(\Gamma) \to X$ is an operator so that $\inf_{e \in \Gamma} \| T(e) \| > 0$.

(b) If $(x^*_k)$ is $w^*$-null, the proof of Theorem 1 makes it clear that we may choose the sequence $(w^*_k)$ in the conclusion of the theorem to be $w^*$-null.

As the following corollaries indicate, Theorem 1 unifies and extends several classical results.

**Corollary 2** ([1, Thm. 4], [3, p. 48]). If $c_0$ embeds isomorphically in the dual $X^*$ of the Banach space $X$, then $X$ contains a copy of $\ell^1$ which is complemented (in $X^{**}$ and thus) in $X$.

**Proof.** If $T : c_0 \to X^*$ is an isomorphism, then let $(x_n)$ be a bounded sequence in $X$ ($\subseteq X^{**}$) so that $\sum_{n=1}^\infty |x_n(T(e_n)) - 1| = 0$. Apply Theorem 1 to the sequence $(x_n)$. □

**Corollary 3** ([10], [3, p. 72]). If $\ell^1$ is a quotient of $X$, then $X$ contains a copy of $\ell^1$ which is complemented in $X^{**}$.

**Proof.** If $T : X \to \ell^1$ is a surjective operator, then $T^* : \ell^\infty \to X^*$ is an isomorphism. Hence $T^*_c$ is an isomorphism. □

If $\Sigma$ is a $\sigma$-algebra, $(\mu_n)$ is a bounded sequence in $\text{cabv}(\Sigma, X)$, and $0 < \varepsilon < \delta$, then $(\mu_n)$ is said to be $(\delta, \varepsilon)$-relatively disjoint [11] if there is a pairwise disjoint sequence $(A_n)$ in $\Sigma$ so that

$$
|\mu_n|(A_n) > \delta \quad \text{and} \quad \sum_{m=1, m \neq n}^\infty |\mu_n|(A_m) < \varepsilon
$$

for each $n$. Further, $(\mu_n)$ is said to be relatively disjoint if it is $(\delta, \varepsilon)$-relatively disjoint for some pair $(\delta, \varepsilon)$. Rosenthal [11] and Kadec and Pelczyński [7] showed that if $(\mu_n)$ is a relatively disjoint sequence in $\text{cabv}(\Sigma, X)$, then $(\mu_n) \sim (e^*_n)$ and $[\mu_n]$ is complemented in $\text{cabv}(\Sigma, X)$.

If $\mathcal{A}$ is an algebra of subsets of $\Omega$, then $\text{fabv}(\mathcal{A}, X)$ denotes the Banach space (total variation norm) of all finitely additive set functions $m : \mathcal{A} \to X$ which have finite variation. Both [4] and [6] contain an extensive discussion of spaces of measures. In addition, we note that [4] includes a detailed presentation of results related to the Radon–Nikodym property. Note that
part (i) of Corollary 4 below contains an extension of Proposition 3.1 of [11] to the setting of finitely additive set functions defined on an algebra of sets. Further, we remark that in a classic paper Kadec and Pełczyński [7, Theorem 6] showed that if \( Y \) is any non-reflexive closed linear subspace of \( L^1[0,1] \), then \( Y \) contains a copy of \( \ell^1 \) which is complemented in \( L^1[0,1] \). Part (v) of the next corollary shows that if \( X \) and \( X^* \) have the Radon–Nikodym property, then any non-reflexive closed linear subspace of \( L^1(\mu, X) \) contains a copy of \( \ell^1 \) which is complemented in \( L^1(\mu, X) \).

**Corollary 4.** (i) If \( (\mu_n) \) is any bounded sequence in \( \text{fabv}(A, X) \) for which there is a pairwise disjoint sequence \( (A_n) \) in \( A \) and an \( \varepsilon > 0 \) so that

\[
|\mu_n|(A_n) > \varepsilon
\]

for each \( n \), then there is a sequence \( (\nu_i) \) in \( \{\mu_n - \mu_k : k, n \in \mathbb{N}\} \) so that \( (\nu_i) \sim (e^*_i) \) and \( [\nu_i] \) is complemented in \( \text{fabv}(A, X) \).

(ii) If \( K \) is a relatively weakly compact subset of \( \text{fabv}(A, X) \) and \( (A_i) \) is a pairwise disjoint sequence of members of \( A \), then \( \lim_{i} |\mu|(A_i) = 0 \) uniformly for \( \mu \in K \).

(iii) If \( K \) is a relatively weakly compact subset of \( \text{cabv}(\Sigma, X) \), then \( \{||\mu|| : \mu \in K\} \) is uniformly countably additive.

(iv) If \( \mu \) is a finite positive measure on \( \Sigma \) and \( K \) is a relatively weakly compact subset of the space \( L^1(\mu, X) \) of Bochner integrable functions, then \( K \) is uniformly integrable.

(v) If \( Y \) is a closed linear subspace of \( \text{fabv}(A, X) \), \( Y \) is not reflexive, and \( X \) and \( X^* \) have the Radon–Nikodym property, then \( Y \) contains a copy of \( \ell^1 \) which is complemented in \( \text{fabv}(A, X) \).

**Proof.** (i) For each \( n \) let \( (A_{n_i})_{i=1}^{k_n} \) be a partition of \( A_n \) and \( (x^*_{n_i})_{i=1}^{k_n} \) be points in the unit ball of \( X^* \) so that

\[
\sum_{i=1}^{k_n} x^*_{n_i} \mu_n(A_{n_i}) > \varepsilon.
\]

Now define the \( X^* \)-valued simple function \( s_n \) by

\[
s_n = \sum_{i=1}^{k_n} \chi_{A_{n_i}} x^*_{n_i},
\]

and observe that \( \int s_n \, d\mu_n > \varepsilon \). Define \( T : c_0 \to \text{fabv}(A, X)^* \) by

\[
T((\gamma_n)) = \sum_{n} \gamma_n s_n.
\]

Then \( T \) is an operator. Normalize and use Theorem 1 to conclude that some sequence \( (\nu_i) \) in \( \{\mu_n - \mu_k : n, k \in \mathbb{N}\} \) is equivalent to \( (e^*_n) \) and that \( [\nu_n] \) is complemented in \( \text{fabv}(A, X) \).
(ii) Suppose that $\varepsilon > 0$ and $(\mu_i)$ is a sequence in $K$ so that $|\mu_i|(A_i) > \varepsilon$ for each $i$. Part (i) ensures that $(\varepsilon^*_n)$ is equivalent to some sequence in $K - K$. However, this is impossible since $K - K$ is relatively weakly compact.

(iii) Since each member of $K$ is a countably additive measure on a $\sigma$-algebra, $|K| = \{|\mu| : \mu \in K\}$ is uniformly countably additive if and only if $\lim_{\mu_i(A_i)} = 0$ uniformly for $\mu \in K$ whenever $(A_i)$ is a pairwise disjoint sequence from $\Sigma$. Deny the uniform countable additivity of $|K|$, repeat the same construction as in (i), and obtain the same contradiction as in (ii).

(iv) If $f \in L^1(\mu, X)$ and $A \in \Sigma$, put $\nu_f(A) = \frac{1}{\text{C}(K)} \int_A f \, d\mu$.

It is well known that $\lim_{\mu(A) \to 0} |\nu_f|(A) = 0$ uniformly for $f \in K$ (i.e., $K$ is uniformly integrable) if and only if $\{|\nu_f| : f \in K\}$ is uniformly countably additive. Appeal to (iii).

(v) If $Y$ is not reflexive, then $B_Y$ is not relatively weakly compact in $\text{fabv}(A, X)$. By Theorem 4.1 of Brooks and Dinculeanu [2], there is a pairwise disjoint sequence $(A_i)$ in $A$, an $\varepsilon > 0$, and a sequence $(\mu_i)$ in $B_Y$ so that $|\mu_i|(A_i) > \varepsilon$ for each $i$. The construction in (i) above shows that $Y$ contains a copy of $\ell^1$ which is complemented in $\text{fabv}(A, X)$.

In the following corollary, $\mathcal{P}$ denotes the $\sigma$-algebra of all subsets of $\mathbb{N}$.

**Corollary 5** ([9, Lemma 2], [3, p. 74]). Every infinite-dimensional closed linear subspace of $\ell^1$ contains a copy of $\ell^1$ which is complemented in $\text{fabv}(\mathcal{P})$ and thus in $\ell^1$.

**Proof.** Every infinite-dimensional subspace of $\ell^1$ is non-reflexive. □

**Corollary 6** ([4, p. 149]). If $(\Omega, \Sigma, \mu)$ is a finite measure space and $X^*$ is a quotient of $L^\infty(\mu)$, then either $X$ is reflexive or $X$ contains a copy of $\ell^1$ which is complemented in $X^{**}$. Consequently, if $X^{**}$ is contained in $L^1(\mu)$, then $X$ is reflexive or $\ell^1$ is a complemented subspace of $X$.

**Proof.** If $T : L^\infty(\mu) \to X^*$ is a surjection and $X$ is not reflexive, then $T$ is not weakly compact. Hence $T$ is not unconditionally converging and is an isomorphism on a copy of $c_0$. Thus $X$ contains a copy of $\ell^1$ which is complemented in $X^{**}$.

If $L : X^{**} \to L^1(\mu)$ is an isomorphism, then $L^* : L^\infty(\mu) \to X^{***}$ is a surjection, $X^*$ is a quotient of $L^\infty(\mu)$, and $X$ is reflexive or $X$ contains a complemented copy of $\ell^1$. □

If $T : c_0 \to X$ is an isomorphism, classical techniques of Singer [13] can be used to easily produce complemented copies of both $c_0$ and $\ell^1$.

**Theorem 7.** If $T : c_0 \to X$ is an isomorphism, $(f^*_n)$ is any bounded sequence in $X^*$ so that
\[ f_n^*(T(e_m)) = \delta_{nm}, \]
and \((h_k^*)\) is any subsequence of \((f_n^*)\), then \([h_k^*]\) is complemented in \(X^*\). Further, if \((h_k^*)\) is \(w^*\)-null in \(X^*\) and \((y_k)\) is the corresponding subsequence of \((T(e_n))\), then \([y_k]\) is complemented in \(X\).

**Proof.** Suppose that \(T, (f_n^*), \) and \((h_k^*)\) are as in the first statement in the theorem. Let \(C\) be a bound for \(||f_n^*||\), let \((y_k^*)\) be the sequence of coefficient functionals for the basic sequence \((y_k)\) (which is equivalent to \((e_k)\)), and choose positive numbers \(A\) and \(B\) so that

\[
A \sum |\alpha_i| \leq \left\| \sum \alpha_i y_i^* \right\| \leq B \sum |\alpha_i|
\]

for each finite sequence \((\alpha_1, \ldots, \alpha_m)\) of real numbers. Therefore

\[
A \sum |\alpha_i| \leq \left\| \sum \alpha_i h_i^*[y_n] \right\| \leq \left\| \sum \alpha_i h_i^* \right\| \leq C \sum |\alpha_i|.
\]

As noted on p. 91 of Singer [13],

\[
\left\{ f^* \in X^* : \sum_{k=1}^{\infty} f^*(y_k)h_k^* \text{ converges} \right\} = [y_k]^\perp + [h_k^*].
\]

Since \((y_k) \sim (e_k)\) and \((h_k^*) \sim (e_k^*)\), we have \([y_k]^\perp + [h_k^*] = X^*\). Further, if \((h_k^*)\) is \(w^*\)-null, then

\[
\left\{ x \in X : \sum_{k=1}^{\infty} h_k^*(x) y_k \text{ converges} \right\} = [y_k] + [h_k^*] = X.
\]

Consequently, each of these direct sums is closed. Straightforward closed graph arguments show that these direct sums are also topological. \(\blacksquare\)

We remark that if \(X\) is separable (and \(T\) and \((f_n^*)\) have the same meaning as in the statement of Theorem 7), then Veech’s proof [15] of Sobczyk’s theorem [14], [3, p. 71] simply shows that there is a bounded sequence \((g_n^*)\) in \([T(e_n)])^\perp \) so that \((f_n^* - g_n^*)\) is \(w^*\)-null. Certainly \((T(e_n), f_n^* - g_n^*)\) is biorthogonal in this case.

The next corollary shows that a result of Saab and Saab [12] dealing with complemented copies of \(c_0\) in injective tensor products is an immediate consequence of Theorem 7. Chapter 8 of [4] contains an excellent discussion of the least crossnorm tensor product completion of Banach spaces.

**Corollary 8 ([12]).** If \(X\) contains a copy of \(c_0\), \(Y\) is an infinite-dimensional Banach space and \(Z = X \otimes_\lambda Y\) is the least crossnorm tensor product completion of \(X\) and \(Y\), then \(Z\) contains a complemented copy of \(c_0\).

**Proof.** Let \((x_n)\) be a sequence in \(X\) so that \((x_n) \sim (e_n)\), let \((x_n^*)\) be a bounded sequence in \(X^*\) so that \(x^*(x_m) = \delta_{nm}\), and let \((y_n^*)\) be a \(w^*\)-null sequence in \(Y^*\) so that \(||y_n^*|| = 1\) for each \(n\). (The Josefson–Nissenzweig Theorem [3] guarantees the existence of \((y_n^*)\).) Choose a sequence \((y_n)\) in \(Y\)
so that \( \|y_n\| \leq 3/2 \) and \( y_n(y_n^*) = 1 \) for each \( n \). Then \( (x_n^* \otimes y_n^*) \) is a \( w^* \)-null sequence in \( Z^* \); \( (x_n \otimes y_n) \sim (e_n) \), and \( x_n^* \otimes y_n^* (x_m \otimes y_m) = x_n^* (x_n) y_n^* (y_m) = \delta_{nm} \). Now appeal to Theorem 7. ■

We note that precisely the same argument yields the next result.

**Corollary 9.** If the Banach space \( X \) contains a copy of \( c_0 \) and \( Y \) is an infinite-dimensional space, then the Banach space \( K(X^*, Y) \) of compact operators from \( X^* \) to \( Y \) contains a complemented copy of \( c_0 \).

REFERENCES


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Received 10 November 1998