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# PSEUDO-BOCHNER CURVATURE TENSOR ON HERMITIAN MANIFOLDS

BY

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**Abstract.** Our main purpose of this paper is to introduce a natural generalization $B_H$ of the Bochner curvature tensor on a Hermitian manifold $M$ provided with the Hermitian connection. We will call $B_H$ the *pseudo-Bochner curvature tensor*. Firstly, we introduce a unique tensor $P$, called the (*Hermitian*) *pseudo-curvature tensor*, which has the same symmetries as the Riemannian curvature tensor on a Kähler manifold. By using $P$, we derive a necessary and sufficient condition for a Hermitian manifold to be of pointwise constant Hermitian holomorphic sectional curvature. Our pseudo-Bochner curvature tensor $B_H$ is naturally obtained from the conformal relation for the pseudo-curvature tensor $P$ and it is conformally invariant. Moreover we show that $B_H$ is different from the Bochner conformal tensor in the sense of Tricerri and Vanhecke.

**1. Introduction.** In [2], Bochner introduced a curvature tensor $B$ on a Kähler manifold $M$ as a formal analogue of the Weyl conformal curvature tensor. Let $J$ be the complex structure of $M$, $g$ the Kähler metric and $\dim_{\mathbb{C}} M = m$. Then the Bochner curvature tensor $B$ (cf. [8]) is defined by

$$B = R - \frac{1}{2(m+2)}\, g \bigtriangleup R_1 + \frac{r}{8(m+1)(m+2)}\, g \bigtriangleup g,$$

where $R$ denotes the Riemannian curvature tensor (the curvature tensor of the Levi-Civita connection) on $M$, $R_1$ the Ricci tensor, $r$ the scalar curvature and $\cdot \bigtriangleup \cdot$ is defined as follows: For any (0,2)-tensors $a, b$ and for any vector fields $X, Y, Z, W$ on $M$, we set

$$\begin{aligned}(a \owedge b)(X,Y,Z,W) &= a(X,Z)b(Y,W) - a(X,W)b(Y,Z)\\ &\quad + b(X,Z)a(Y,W) - b(X,W)a(Y,Z)\end{aligned}$$

and

$$\overline{a}(X,Y) = a(X,JY).$$

Then we define

$$a \bigtriangleup b = a \owedge b + \overline{a} \owedge \overline{b} + 2\overline{a} \otimes \overline{b} + 2\overline{b} \otimes \overline{a}.$$

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In [9], Tricerri and Vanhecke studied the decomposition of the space $\mathcal{R}(V)$ of all curvature tensors on a Hermitian vector space $V$. They defined the *Bochner component* $\mathcal{R}_B(V)$ of $\mathcal{R}(V)$ and called the projection $B(A)$ of $A \in \mathcal{R}(V)$ on $\mathcal{R}_B(V)$ the *Bochner conformal tensor* associated with $A$. As an application, they proved that the Bochner conformal tensor $B(R)$ associated with the Riemannian curvature tensor $R$ on an almost Hermitian manifold is conformally invariant. Of course, in the Kähler case, $B(R) = B$.

In this paper, we introduce a natural generalization $B_H$ of the Bochner curvature tensor on a Hermitian manifold $M$ provided with the Hermitian connection and we will call $B_H$ the *pseudo-Bochner curvature tensor* on $M$. For this purpose, we discuss in §2 Hermitian holomorphic sectional curvature of a Hermitian manifold and derive a necessary and sufficient condition for a Hermitian manifold to be of pointwise constant Hermitian holomorphic sectional curvature. Then we introduce a unique tensor $P$ on a Hermitian manifold $M$ having the same symmetries as the Riemannian curvature tensor on a Kähler manifold. We will call this tensor $P$ the (*Hermitian*) *pseudo-curvature tensor* on $M$. In §3 our pseudo-Bochner curvature tensor $B_H$ is naturally obtained as a conformal invariant from the conformal relation for the pseudo-curvature tensor $P$.

In §4, we give some examples of Hermitian manifolds with vanishing $B_H$ and we call such manifolds *pseudo-Bochner-flat* Hermitian manifolds. In [7], we proved that the product of two Kenmotsu manifolds with constant sectional curvature $-1$ is Hermitian-flat, that is, the Hermitian curvature tensor (the curvature tensor of the Hermitian connection) vanishes. We show that this product manifold is pseudo-Bochner-flat but not Bochner-flat in the sense of Tricerri and Vanhecke.

Throughout this paper, we work in $C^\infty$-category and deal with connected complex manifolds of complex dimension $\geq 2$ without boundary only.

**2. Hermitian holomorphic sectional curvature.** Let $M$ be a complex $m$-dimensional Hermitian manifold with the complex structure $J$ and the Hermitian metric $g$, that is, $g$ is a Riemannian metric on $M$ such that $g(JX, JY) = g(X, Y)$ for all vector fields $X, Y$ on $M$. The *Hermitian connection* $D$ of $M$ is defined by the following equation (see [7]):

$$4g(D_X Y, Z) = 2Xg(Y, Z) - 2JXg(JY, Z) + g(\mathcal{V}(X, Y), Z) - g(\mathcal{V}(X, Z), Y) \tag{2.1}$$

for all vector fields $X, Y, Z$ on $M$, where $\mathcal{V}(X, Y) = [JX, JY] + [X, Y] - J[X, JY] + J[JX, Y]$. The Hermitian connection $D$ is a unique affine connection such that both the metric tensor $g$ and the complex structure $J$ are parallel and the torsion tensor $T$ satisfies $T(JX, Y) = JT(X, Y)$ for all vector fields $X, Y$ on $M$. As is well known, a Hermitian manifold is Kähler if and only if the Hermitian connection is torsion-free, that is, the Hermitian connection coincides with the Levi-Civita connection.

Let $H$ be the Hermitian curvature tensor (the curvature tensor of the Hermitian connection $D$) on $M$, i.e.,

$$H(X,Y) = [D_X, D_Y] - D_{[X,Y]}$$

for all vector fields $X, Y$ on $M$. Then we have

Proposition 2.1 (cf. [7]). *The Hermitian curvature tensor $H$ has the following properties: For all vector fields $X, Y, Z, W$ on $M$,*

$$H(X,Y,Z,W) = -H(Y,X,Z,W) = -H(X,Y,W,Z),$$
$$H(JX,JY,Z,W) = H(X,Y,JZ,JW) = H(X,Y,Z,W),$$
$$\mathfrak{S}_{X,Y,Z}\{H(X,Y)Z - T(T(X,Y),Z) - (D_X T)(Y,Z)\} = 0 \quad (\textit{First Bianchi identity}),$$
$$\mathfrak{S}_{X,Y,Z}\{(D_X H)(Y,Z) + H(T(X,Y),Z)\} = 0 \quad (\textit{Second Bianchi identity}),$$

*where $H(X,Y,Z,W) = g(H(Z,W)Y,X)$ and $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to $X, Y, Z$.*

Now, let us consider the *Hermitian holomorphic sectional curvature* of $M$. For each unit vector $X$ in the tangent space $T_xM$, the Hermitian holomorphic sectional curvature $\mathcal{H}(X)$ for the holomorphic plane spanned by $X$ and $JX$ is given by

$$\mathcal{H}(X) = H(X,JX,X,JX).$$

If $\mathcal{H}(X)$ is constant for all unit vectors $X$ in $T_xM$ at each point $x \in M$, then $M$ is said to be *of pointwise constant Hermitian holomorphic sectional curvature*. Moreover, if $\mathcal{H}(X)$ is constant for all $x \in M$, then $M$ is said to be *of constant Hermitian holomorphic sectional curvature.*

In [1], Balas studied Hermitian manifolds $M$ of constant Hermitian holomorphic sectional curvature. Then he introduced a tensor $K$ of type $(0,4)$ on $M$, called the *Kähler-symmetric part* of the Hermitian curvature tensor $H$. The tensor $K$ is given by

$$K(X,Y,Z,W) = \tfrac{1}{4}\{H(X,Y,Z,W) + H(Z,W,X,Y) + H(X,W,Z,Y) + H(Z,Y,X,W)\}$$

for all vector fields $X, Y, Z, W$ on $M$. It has the following properties:

$$K(X,Y,Z,W) = K(Y,X,W,Z), \qquad K(X,Y,Z,W) = K(Z,W,X,Y),$$
$$\mathfrak{S}_{Y,Z,W}\, K(X,Y,Z,W) = 0, \qquad K(JX,JY,Z,W) = K(X,Y,JZ,JW),$$
$$K(X,Y,X,Y) = H(X,Y,X,Y).$$

From the algebraic discussion of local components of this tensor in complex local coordinates, Balas derived a necessary and sufficient condition for $M$

to be of constant Hermitian holomorphic sectional curvature. But the tensor $K$ does not satisfy the identities $K(X,Y,Z,W) = -K(Y,X,Z,W)$ and $K(JX,JY,Z,W) = K(X,Y,Z,W)$.

We now introduce a tensor $P$ of type $(0,4)$ on $M$ defined by

$$P(X,Y,Z,W) = \tfrac{1}{4}[3\{L(X,Y,Z,W) + L(X,Y,JZ,JW)\} - H(X,Y,Z,W) - H(Z,W,X,Y)], \tag{2.2}$$

for all vector fields $X,Y,Z,W$ on $M$, where $L$ is the tensor introduced in Appendix of [7] as follows:

$$L(X,Y,Z,W) = \tfrac{2}{3}\{K(X,Y,Z,W) - K(Y,X,Z,W)\}.$$

By (A.3)–(A.6) of [7] and Proposition 2.1, we can easily see that $P$ has the following properties:

$$P(X,Y,Z,W) = -P(Y,X,Z,W) = -P(X,Y,W,Z), \tag{2.3}$$

$$P(X,Y,Z,W) = P(Z,W,X,Y), \tag{2.4}$$

$$\mathfrak{S}_{Y,Z,W}\, P(X,Y,Z,W) = 0, \tag{2.5}$$

$$P(JX,JY,Z,W) = P(X,Y,JZ,JW) = P(X,Y,Z,W) \tag{2.6}$$

for all vector fields $X,Y,Z,W$ on $M$, and in particular we have

$$P(X,JX,X,JX) = H(X,JX,X,JX) \tag{2.7}$$

for all vector fields $X$ on $M$. On the other hand, the tensor $g \triangle g$ satisfies all the identities (2.3)–(2.6) and

$$(g \triangle g)(X,JX,X,JX) = 8g(X,X)^2. \tag{2.8}$$

Therefore, from (2.7), (2.8) and Proposition 7.1 of Chapter IX in [5], we conclude

THEOREM 2.1. *A Hermitian manifold $M$ is of pointwise constant Hermitian holomorphic sectional curvature $c$ if and only if $P = \frac{1}{8}cg \triangle g$.*

We call the tensor $P$ defined by (2.2) the (*Hermitian*) *pseudo-curvature tensor* of the Hermitian connection $D$. We define the (*Hermitian*) *pseudo-Ricci tensor* $P_1$ and the (*Hermitian*) *pseudo-scalar curvature* $p$ as follows:

$$P_1(X,Y) = \tfrac{1}{2}\operatorname{tr}[Z \to P(X,JY)JZ], \qquad p = \operatorname{tr} P_1,$$

where the tensor $P$ of type $(1,3)$ is defined by $g(P(X,Y)Z,W) = P(W,Z,X,Y)$. $P_1$ is symmetric and compatible with $J$, and so we can associate with $P_1$ a 2-form $\varrho$ in the usual manner: $\varrho = \overline{P}_1$. We call $\varrho$ the (*Hermitian*) *pseudo-Ricci form* which is not closed in general.

If $M$ has pointwise constant Hermitian holomorphic sectional curvature, then we have

THEOREM 2.2. *Let $M$ be a Hermitian manifold of pointwise constant Hermitian holomorphic sectional curvature $c$. Then*

$$P_1 = \frac{(m+1)c}{2}\,g, \qquad p = m(m+1)c.$$

REMARK 2.1. We note that, on a Kähler manifold, the pseudo-quantities $P$, $P_1$, $\varrho$ and $p$ defined above coincide with the curvature tensor $R$, the Ricci tensor $R_1$, the Ricci form $\gamma$ and the scalar curvature $r$ of the Levi-Civita connection respectively.

**3. Pseudo-Bochner curvature tensor.** Consider a conformal change $g' = e^{-\sigma} g$ of the Hermitian metric $g$ on $M$, where $\sigma$ is a function on $M$. For every object related to $g'$ we shall add the symbol $'$. Then the Hermitian connections $D'$, $D$ are connected by the following equation:

$$D'_X Y = D_X Y - \tfrac{1}{2}\, d\sigma(X)Y - \tfrac{1}{2}\, d^{\mathbf{c}}\sigma(X)JY, \tag{3.1}$$

where $d^{\mathbf{c}}\sigma(X) = -d\sigma(JX)$. From this equation, we obtain the relation between their Hermitian curvature tensors $H$ and $H'$:

$$e^{\sigma} H' = H - \Omega \otimes dd^{\mathbf{c}}\sigma, \tag{3.2}$$

where $\Omega$ denotes the Kähler form, i.e., $\Omega = \overline{g}$. From (3.2), for the pseudo-curvature tensors, we obtain

$$e^{\sigma} P' = P + \tfrac{1}{8}\, g \triangle \overline{dd^{\mathbf{c}}\sigma}. \tag{3.3}$$

From (3.3), for the pseudo-Ricci tensors we obtain

$$P'_1 = P_1 + \frac{m+2}{4}\,\overline{dd^{\mathbf{c}}\sigma} + \frac{1}{8}\,(\mathrm{tr}\,\overline{dd^{\mathbf{c}}\sigma})g. \tag{3.4}$$

Here we used the equality $dd^{\mathbf{c}}\sigma(JX, JY) = dd^{\mathbf{c}}\sigma(X, Y)$. Moreover, from (3.4), for the pseudo-scalar curvatures we obtain

$$e^{-\sigma} p' - p = \frac{m+1}{2}(\mathrm{tr}\,\overline{dd^{\mathbf{c}}\sigma}). \tag{3.5}$$

Substitution of (3.5) into (3.4) gives

$$\overline{dd^{\mathbf{c}}\sigma} = \frac{4}{m+2}\left\{\left(P'_1 - \frac{p'}{4(m+1)}\,g'\right) - \left(P_1 - \frac{p}{4(m+1)}\,g\right)\right\}. \tag{3.6}$$

Moreover, substitution of (3.6) into (3.3) yields the conformal invariance of the tensor $B_H$ defined by

$$B_H = P - \frac{1}{2(m+2)}\, g \triangle P_1 + \frac{p}{8(m+1)(m+2)}\, g \triangle g. \tag{3.7}$$

We can easily check that $B_H$ satisfies (2.3)–(2.6), i.e.,

$$B_H(X, Y, Z, W) = -B_H(Y, X, Z, W) = -B_H(X, Y, W, Z), \tag{3.8}$$

$$B_H(X, Y, Z, W) = B_H(Z, W, X, Y), \tag{3.9}$$

$$(3.10)\qquad \mathfrak{S}_{Y,Z,W}\, B_H(X,Y,Z,W)=0,$$

$$(3.11)\qquad B_H(JX,JY,Z,W)=B_H(X,Y,JZ,JW)=B_H(X,Y,Z,W).$$

Moreover $B_H$ satisfies

$$(3.12)\qquad \mathrm{tr}[Z\to B_H(Z,X)Y]=0,$$

where the tensor $B_H$ of type $(1,3)$ is defined by $g(B_H(X,Y)Z,W) = B_H(W,Z,X,Y)$. If the metric $g$ is Kähler, the tensor $B_H$ coincides with the original Bochner curvature tensor $B$ mentioned in the introduction. We call $B_H$ defined by (3.7) the *pseudo-Bochner curvature tensor* on $M$. Summing up, we conclude

THEOREM 3.1. *The pseudo-Bochner curvature tensor $B_H$ on a Hermitian manifold is conformally invariant.*

**4. Examples of pseudo-Bochner-flat Hermitian manifolds.** We call Hermitian manifolds with vanishing $B_H$ *pseudo-Bochner-flat*. In the same way as in the Kähler case (cf. [8]), we can prove the following theorems.

THEOREM 4.1. *Every Hermitian manifold of pointwise constant Hermitian holomorphic sectional curvature is pseudo-Bochner-flat.*

THEOREM 4.2. *A pseudo-Bochner-flat Hermitian manifold has pointwise constant Hermitian holomorphic sectional curvature if and only if the pseudo-Ricci tensor satisfies the Einstein condition*, *i.e.*, $P_1=(p/(2m))g$.

EXAMPLE 4.1. We call Hermitian manifolds with $H=0$ *Hermitian-flat*. The Iwasawa manifold $M$ is a compact complex manifold defined by $M=G/\Gamma$, where

$$G=\left\{\begin{pmatrix}1 & z^1 & z^2\\ 0 & 1 & z^3\\ 0 & 0 & 1\end{pmatrix} : z^i\in\mathbb{C}\right\},$$

$$\Gamma=\left\{\begin{pmatrix}1 & \alpha^1 & \alpha^2\\ 0 & 1 & \alpha^3\\ 0 & 0 & 1\end{pmatrix} : \alpha^i\in\mathbb{Z}+\sqrt{-1}\,\mathbb{Z}\right\}.$$

In [1], Balas showed that the Iwasawa manifold $M$ is Hermitian-flat. (See also [7].) Obviously, such manifolds are pseudo-Bochner-flat. In [3], Cordero, Fernández and Gray introduced the generalized Iwasawa manifold as a generalization of the Iwasawa manifold above. They proved that the generalized Iwasawa manifold has no Kähler structure, even though it has a symplectic structure and a complex structure. We can introduce a Hermitian-flat metric on the generalized Iwasawa manifold in the same way as [7].

EXAMPLE 4.2. In [4], Ganchev, Ivanov and Mihova introduced a special class of Hermitian manifolds and called them *anti-Kähler* manifolds. A Hermitian manifold $M$ is anti-Kähler if and only if the connection $\widetilde{D}$ defined by $\widetilde{D} = D - \frac{1}{2}T$, where $T$ is the torsion tensor of the Hermitian connection $D$, is flat. They showed that an anti-Kähler metric of pointwise nonzero constant Hermitian holomorphic sectional curvature is a certain conformal change of a Kähler metric of nonzero constant holomorphic sectional curvature. Moreover, using this fact, they constructed an anti-Kähler metric of pointwise positive (resp. negative) constant Hermitian holomorphic sectional curvature on the open unit ball $\mathbb{D}^m$ in $\mathbb{C}^m$ (resp. on $\mathbb{C}^m$). By Theorem 4.1, such manifolds are pseudo-Bochner-flat.

EXAMPLE 4.3. In [7], we studied Hermitian manifolds which are locally conformal to Hermitian-flat manifolds. Such manifolds are called *locally conformal Hermitian-flat* ones. Of course, locally conformal Kähler-flat manifolds (see [10]) are contained in a class of such manifolds. The Hopf manifolds $S^1 \times S^{2m-1}$ are of this type, where $S^k$ denotes the standard $k$-dimensional sphere. In [6], we constructed a locally conformal Hermitian-flat metric on a noncompact complex manifold $\mathbb{R}^{m-1} \times T^{m+1}$, where $T^{m+1}$ denotes the $(m+1)$-dimensional torus. From the conformal invariance of $B_H$, locally conformal Hermitian-flat manifolds are pseudo-Bochner-flat.

On a Kähler manifold, both our pseudo-Bochner curvature tensor $B_H$ and the Bochner conformal tensor $B(R)$ of Tricerri and Vanhecke [9] coincide with the original Bochner curvature tensor $B$. Moreover, since they are both conformally invariant, it is also trivial that $B_H$ coincides with $B(R)$ on a locally conformal Kähler manifold. But, on a general Hermitian manifold, $B_H$ is not equal to $B(R)$. We shall show this fact by giving an example of Hermitian-flat manifolds with $B(R) \neq 0$.

Now we recall the definition of the Bochner conformal tensor of Tricerri and Vanhecke [9]. Let $M$ be a complex $m$-dimensional Hermitian manifold provided with the complex structure $J$ and the metric $g$. We denote by $R$ the Riemannian curvature tensor on $M$, that is,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

where $\nabla$ is the Levi-Civita connection of $g$. The Riemannian curvature tensor $R$ of type $(0,4)$ is given by $R(X,Y,Z,W) = g(R(Z,W)Y,X)$. If $m > 3$, then the Bochner conformal tensor $B(R)$ associated with $R$ is given by

$$\begin{aligned} B(R) = {} & R + \frac{1}{4(m+2)(m-2)}\, g \triangle R_1 - \frac{2m-3}{4(m-1)(m-2)}\, g \owedge R_1 \\ & + \frac{1}{4(m+1)(m-2)}\, g \triangle (R_1 J) - \frac{1}{4(m-1)(m-2)}\, g \owedge (R_1 J) \\ & - \frac{2m^2-5}{4(m+1)(m+2)(m-2)}\, g \triangle R_1^* + \frac{2m-1}{4(m+1)(m-2)}\, g \owedge R_1^* \end{aligned} \tag{4.1}$$

$$- \frac{3}{4(m+1)(m+2)(m-2)} g \triangle (R_1^* J) + \frac{3}{4(m+1)(m-2)} g \triangle (R_1^* J)$$
$$- \frac{3mr - (2m^2 - 3m + 4)r^*}{16(m+1)(m+2)(m-1)(m-2)} g \triangle g + \frac{r - r^*}{8(m-1)(m-2)} g \owedge g,$$

where

$$R_1(X,Y) = \mathrm{tr}[Z \to R(Z,X)Y], \qquad R_1^*(X,Y) = \mathrm{tr}[Z \to R(X,JZ)JY],$$
$$(R_1 J)(X,Y) = R_1(JX,JY), \qquad (R_1^* J)(X,Y) = R_1^*(JX,JY),$$
$$r = \mathrm{tr}\, R_1, \qquad r^* = \mathrm{tr}\, R_1^*.$$

The symbols $R_1$, $R_1^*$, $R_1 J$, $R_1^* J$, $r$, $r^*$ correspond to the symbols $\varrho(R)$, $\varrho^*(R)$, $\varrho(L_3 R)$, $\varrho^*(L_3 R)$, $\tau$, $\tau^*$ respectively in [9]. And for any $(0,2)$-tensor $S$, $g \owedge S$ (resp. $g \triangle S$) corresponds to $\varphi(S)$ (resp. $\varphi(S) + \psi(S)$). Thus $g \owedge g$ (resp. $g \triangle g$) corresponds to $2\pi_1$ (resp. $2(\pi_1 + \pi_2)$).

In [7], we showed that the product of two Kenmotsu manifolds with constant sectional curvature $-1$ is Hermitian-flat, and hence it is pseudo-Bochner-flat. We now show that this product manifold is not Bochner-flat in the sense of Tricerri and Vanhecke. Let $(M', \phi', \xi', \eta', g')$ (resp. $(M'', \phi'', \xi'', \eta'', g'')$) be a Kenmotsu manifold with constant sectional curvature $-1$, that is, $R' = -\frac{1}{2} g' \owedge g'$ (resp. $R'' = -\frac{1}{2} g'' \owedge g''$), where $R'$ (resp. $R''$) denotes the Riemannian curvature tensor on $M'$ (resp. $M''$). Then the product $M = M' \times M''$ provided with the metric $g = g' + g''$ and the complex structure $J = \phi' - \eta'' \otimes \xi' + \phi'' + \eta' \otimes \xi''$ is a Hermitian manifold, and the Riemannian curvature tensor $R$ on $M$ is given by

$$R = R' + R'' = -\tfrac{1}{2}(g' \owedge g' + g'' \owedge g''). \tag{4.2}$$

For simplicity, we assume that $\dim M' = \dim M'' = 2k+1$, $k > 1$. Thus we have $\dim_{\mathbb{C}} M = m = 2k+1$. From (4.2), we then obtain

$$R_1 = R_1 J = -(m-1)g, \qquad R_1^* = R_1^* J = -g + \eta' \otimes \eta' + \eta'' \otimes \eta'', \tag{4.3}$$
$$r = -2m(m-1), \qquad r^* = -2(m-1). \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.1), we obtain

$$\begin{aligned} B(R) = & -\frac{1}{2}(g' \owedge g' + g'' \owedge g'') \\ & + \frac{(m+4)(m-1)}{8(m+1)(m+2)(m-2)} g \triangle g + \frac{m-3}{4(m-2)} g \owedge g \\ & - \frac{m-1}{2(m+2)(m-2)} g \triangle (\eta' \otimes \eta' + \eta'' \otimes \eta'') \\ & + \frac{1}{2(m-2)} g \owedge (\eta' \otimes \eta' + \eta'' \otimes \eta'') \\ \neq & \ 0. \end{aligned}$$

REMARK 4.1. By a direct computation, we can also check that a Hermitian-flat metric on the Iwasawa manifold or the generalized Iwasawa manifold (Example 4.1) is not Bochner-flat in the sense of Tricerri and Vanhecke.

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