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PROJECTIVE EMBEDDINGS OF TORIC VARIETIES

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1. Introduction. The question of when a toric variety admits an equivariant embedding into projective space is well understood. A toric variety $X_{\mathbf{k}}(\Sigma)$ defined over a field $\mathbf{k}$ is determined by a complex Σ of rational cones in $\mathbb{R}^d$ and will have an algebraic embedding into projective space if there is an integral convex polytope P which is dual to Σ in an appropriate sense. Having chosen such a P, there is a natural map μ_P from $X_{\mathbf{k}}(\Sigma)$ to a certain projective space $\mathbb{P}^r_{\mathbf{k}}$, and in the event that P is large enough, this map is an algebraic embedding. In particular, if nP denotes the n-fold scaling of P, then for n sufficiently large, μ_{nP} is an algebraic embedding. In this paper, we consider the weaker question of when μ_P is injective, giving necessary and sufficient conditions on P which depend only on a certain arithmetic property of the field $\mathbf{k}$. When the field is $\mathbb{R}$ or $\mathbb{C}$, injectivity implies that the map will be a topological embedding (in the metric topology). We conclude by giving an example $\mu_P : X_{\mathbb{C}}(\Sigma) \to \mathbb{P}^r_{\mathbb{C}}$ which is a topological embedding but not an algebraic embedding and an example $\mu_P : X_{\mathbb{C}}(\Sigma) \to \mathbb{P}^r_{\mathbb{C}}$ which is not a topological embedding, but whose restriction $\mu_P : X_{\mathbb{R}}(\Sigma) \to \mathbb{P}^r_{\mathbb{R}}$ is a topological embedding.

1. Definitions

1.1. *Cones and affine toric varieties.* Let N be a free $\mathbb{Z}$-module of rank d and let M be the dual module $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Denote by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ (respectively, $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$) the vector spaces $N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M \otimes_{\mathbb{Z}} \mathbb{R}$ (resp., $N \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}$). The natural pairing $\langle\, , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ restricts to $M \times N$ and to $M_{\mathbb{Q}} \times N_{\mathbb{Q}}$.

A *cone* $\mathbf{c}$ in $N_{\mathbb{R}}$ is the convex hull of a finite set of rays passing through nonzero points of $N_{\mathbb{R}}$. All cones in this paper will be *rational*, meaning that they are the convex hulls of rays passing through points of the lattice N. A rational cone $\mathbf{c}$ can also be written dually as the intersection of a finite

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number of rational halfspaces:

$$\mathbf{c} = \bigcap_{i=1}^{r} \{q \in N_{\mathbb{R}} \mid \langle m_i, q\rangle \geq 0\}$$

where the m_i are unique, primitive lattice points of M.

A cone $\mathbf{c}$ is *strictly convex* if it contains no line in $N_{\mathbb{R}}$. For an arbitrary set $S \subset N$, we use the notation $\mathbb{R}_{\geq 0}S$ for the convex hull of the rays passing through points of S. For a strictly convex cone $\mathbf{c}$ in $N_{\mathbb{R}}$ there is a unique minimal set of primitive lattice points in N, called the *extreme set* for $\mathbf{c}$ and written $\operatorname{ext}\mathbf{c}$, such that

$$\mathbf{c} = \mathbb{R}_{\geq 0} \operatorname{ext}\mathbf{c}.$$

Any hyperplane which does not intersect the interior of $\mathbf{c}$ is called a *supporting hyperplane*, and a *face* $\mathbf{b}$ of $\mathbf{c}$, written $\mathbf{b} < \mathbf{c}$, is the intersection of $\mathbf{c}$ with a supporting hyperplane. The smallest $\mathbb{R}$-subspace of $N_{\mathbb{R}}$ containing $\mathbf{c}$ will be denoted by $\mathbb{R}\mathbf{c}$, and the *dimension* of the cone $\mathbf{c}$ is the dimension of the vector space $\mathbb{R}\mathbf{c}$. If $\mathbf{c}$ is strictly convex and the cardinality of $\operatorname{ext}\mathbf{c}$ is equal to the dimension of $\mathbf{c}$, then $\mathbf{c}$ is called *simplicial*. A cone $\mathbf{c}$ is *maximal* if $\mathbb{R}\mathbf{c} = N_{\mathbb{R}}$. A maximal, simplicial cone for which $\operatorname{ext}\mathbf{c}$ is a basis for N will be called *basic*.

If $\mathbf{c}$ is a cone in $N_{\mathbb{R}}$, we denote by $\mathbf{c}^{\perp}$ the vector subspace

$$\{p \in M_{\mathbb{R}} \mid \langle p, q\rangle = 0 \text{ for all } q \in \mathbf{c}\}.$$

We define the *dual cone* $\check{\mathbf{c}}$ to be the rational cone in $M_{\mathbb{R}}$ given by

$$\check{\mathbf{c}} = \{p \in M_{\mathbb{R}} \mid \langle p, q\rangle \geq 0 \text{ for all } q \in \mathbf{c}\}.$$

Notice that this duality has the following properties ([Oda]):

(i) $(\check{\mathbf{c}})\check{} = \mathbf{c}$.
(ii) If $\mathbf{b}$ is a face of $\mathbf{c}$, $\check{\mathbf{c}}$ is contained in $\check{\mathbf{b}}$.
(iii) If $\mathbf{c}$ is strictly convex, $\check{\mathbf{c}}$ is maximal.
(iv) $\mathbf{c}$ is simplicial and maximal if and only if $\check{\mathbf{c}}$ is simplicial and maximal.
(v) $\mathbf{c}$ is strictly convex and maximal if and only if $\check{\mathbf{c}}$ is strictly convex and maximal.
(vi) $\mathbf{c}$ is basic if and only if $\check{\mathbf{c}}$ is basic.

Property (ii) has a much stronger formulation, the proof of which can also be found in [Oda]. Namely, there is an inclusion reversing bijection between k-faces of $\mathbf{c}$ and codimension-k faces of $\check{\mathbf{c}}$ given by

$$\mathbf{b} \mapsto \mathbf{b}^{\perp} \cap \check{\mathbf{c}}.$$

The set-theoretical definition of a toric variety which we shall use holds over an arbitrary field $\mathbf{k}$. We refer the reader to any of a number of excellent surveys for the general (algebraic) definition ([Dan, Ful, Oda]).

If $\mathbf{c}$ is a cone in $N_{\mathbb{R}}$, then $\check{\mathbf{c}} \cap M$ has the structure of a finitely generated additive semigroup with unit 0. Likewise, any field $\mathbf{k}$ is a multiplicative semigroup with unit 1.

DEFINITION 1. The *affine toric variety* (*over* $\mathbf{k}$) *associated with* $\mathbf{c}$ is the set

$$U_{\mathbf{k}}(\mathbf{c}) = \mathrm{Hom}(\check{\mathbf{c}} \cap M, \mathbf{k})$$

where Hom denotes unitary semigroup homomorphisms.

Choosing, say n, generators for the semigroup $\check{\mathbf{c}} \cap M$ and finding all additive relations among these generators gives a presentation for the algebra $\mathbf{k}[\check{\mathbf{c}} \cap M]$ as a quotient of the polynomial ring with n indeterminates by some ideal I. We can then identify $U_{\mathbf{k}}(\mathbf{c})$ with the zero locus in $\mathbf{k}^n$ of a set of polynomials generating I. If $\mathbf{k}$ is $\mathbb{R}$ or $\mathbb{C}$, the toric variety inherits a (Hausdorff) topology from the product topology on $\mathbf{k}^n$ which is independent of the choice of generators for $\check{\mathbf{c}} \cap M$.

1.2. *Cone complexes and toric varieties.* The following fact motivates the construction of a general toric variety: if $\mathbf{b}$ is a face of $\mathbf{c}$, then the affine variety $U_{\mathbf{k}}(\mathbf{b})$ is a subset of $U_{\mathbf{k}}(\mathbf{c})$. Therefore, if two cones in $N_{\mathbb{R}}$ share a face, there is a natural way to glue the associated affine varieties together along a subset of each variety. To obtain this natural inclusion, notice that if $\mathbf{b} < \mathbf{c}$, then $\check{\mathbf{c}} \cap M$ is a subsemigroup of $\check{\mathbf{b}} \cap M$, and that this induces a map

$$U_{\mathbf{k}}(\mathbf{b}) = \mathrm{Hom}(\check{\mathbf{b}} \cap M, \mathbf{k}) \to U_{\mathbf{k}}(\mathbf{c}) = \mathrm{Hom}(\check{\mathbf{c}} \cap M, \mathbf{k}).$$

A point $x \in U_{\mathbf{k}}(\mathbf{c})$ is in the image of this map if and only if $x(p) \neq 0$ for all $p \in (\mathbf{b}^{\perp} \cap \check{\mathbf{c}}) \cap M$. Because any $p \in \check{\mathbf{b}} \cap M$ can be written as $p = p_1 - p_2$ where $p_1 \in \check{\mathbf{c}} \cap M$ and $p_2 \in (\mathbf{b}^{\perp} \cap \check{\mathbf{c}}) \cap M$, any $x \in U_{\mathbf{k}}(\mathbf{c})$ in the image of this map is the image of the unique semigroup homomorphism $x' \in U_{\mathbf{k}}(\mathbf{b})$ defined by

$$x'(p) = x(p_1)/x(p_2).$$

Notice, in particular, that the variety $U_{\mathbf{k}}(\mathbf{0})$ corresponding to the zero cone is the "algebraic torus"

$$\mathrm{Hom}(M, \mathbf{k}) = (\mathbf{k}^*)^d$$

($\mathbf{k}^*$ denotes the nonzero elements of $\mathbf{k}$). This torus sits inside every affine toric variety.

A *rational cone complex* (also called a *fan*) Σ in $N_{\mathbb{R}}$ is a collection of strictly convex rational cones satisfying the following two conditions:

(i) every face of a cone in Σ is also in Σ, and
(ii) the intersection of two cones in Σ is a face of both.

A cone complex is *complete* if the union of all of its cones is $N_{\mathbb{R}}$.

DEFINITION 2. Let Σ be a rational cone complex in $N_{\mathbb{R}}$. The *toric variety* (*over* $\mathbf{k}$) *associated with* Σ is the set $X_{\mathbf{k}}(\Sigma)$ obtained by gluing together the affine toric varieties $\{U_{\mathbf{k}}(\mathbf{c}) \mid \mathbf{c} \in \Sigma\}$ along the natural inclusions described above.

When $\mathbf{k}$ is $\mathbb{R}$ or $\mathbb{C}$, each inclusion of affine toric varieties described above is continuous and the image is dense and open. In the category of spaces, $X_{\mathbf{k}}(\Sigma)$ is the topological union or *pushout* of the directed system $\{U_{\mathbf{k}}(\mathbf{c}) \mid \mathbf{c} \in \Sigma\}$. We leave it to the reader to verify that $X_{\mathbf{k}}(\Sigma)$ is Hausdorff and compact if Σ is complete.

1.3. *Convex polytopes and maps to projective space.* Let P be an integral convex polytope in $M_{\mathbb{R}}$ (i.e., P is the convex hull of a finite subset of M). Assume that P is d-dimensional. We can always translate P so that 0 is in the interior and the vertices of P are in $M_{\mathbb{Q}}$. Hence, we can write this translate of P as the intersection of *affine* halfspaces (one for each codimension 1 face):

$$P = \bigcap_{i=1}^{k} \{p \in M_{\mathbb{R}} \mid \langle n_i, p\rangle \geq -1\}$$

where each $n_i \in N_{\mathbb{Q}}$ is uniquely determined by the chosen translation of P. We now let Σ be the collection of cones in $N_{\mathbb{R}}$ of the form

$$\mathbf{c} = \mathbb{R}_{\geq 0} S$$

where $S \subset \{n_1, \dots, n_k\}$ is such that

$$\bigcap_{n\in S} \{p \in M_{\mathbb{R}} \mid \langle n, p\rangle = -1\}$$

is an affine subspace of $M_{\mathbb{R}}$ containing a face of P.

From the above remarks it follows that Σ is a complete rational cone complex and does not depend on the choice of 0 (i.e., the translation) or on the scaling of P. The collection Σ is called the *cone complex dual to* P and the variety $X_{\mathbf{k}}(\Sigma)$ is called the *toric variety associated with* P.

In general, there are many different polytopes giving rise to the same cone complex and, hence, to the same toric variety. It turns out that different polytopes dual to the same cone complex Σ parametrize different maps from the toric variety $X_{\mathbf{k}}(\Sigma)$ to projective spaces. For a convex polytope P, let $\{m_0, m_1, \dots, m_r\}$ be the set of lattice points in P. Then there is a natural algebraic map μ_P from the toric variety associated with P to the projective space $\mathbb{P}^r_{\mathbf{k}}$ defined as follows. A maximal cone $\mathbf{c} \in \Sigma$ corresponds to a vertex, say $m_i \in P \cap M$. If $P - m_i$ is the translation of P which moves m_i to the origin, then the $\mathbb{R}_{\geq 0}$-span of $P - m_i$ is the dual cone $\check{\mathbf{c}}$. In particular, $m - m_i \in \check{\mathbf{c}} \cap M$ for all $m \in P$. The map μ_P is then defined on $U_{\mathbf{k}}(\mathbf{c})$ by

$$\mu_P(x) = [x(m_0 - m_i), x(m_1 - m_i), \dots, x(m_i - m_i), \dots, x(m_r - m_i)]$$

where $x(m_i - m_i) = x(0) = 1$. The definition of μ_P is similar for each maximal cone, and one can check that these maps agree on the overlaps.

2. Projective embeddings

2.1. *Algebraic embeddings.* Our main result concerns the question of when the map $\mu_P : X_{\mathbf{k}}(\Sigma) \to \mathbb{P}^r_{\mathbf{k}}$, defined in the previous section, is injective. We first state the known theorem on algebraic projective embeddings. For the proof, the reader is referred to [Oda]. Recall that a polynomial map $f : X \to \mathbb{P}^r_{\mathbf{k}}$ is an *algebraic embedding* if there is a projective variety $Y \subset \mathbb{P}^r_{\mathbf{k}}$ such that $f(X) = Y$ and $f : X \to Y$ is an isomorphism.

THEOREM 1. *The map $\mu_P : X_{\mathbf{k}}(\Sigma) \to \mathbb{P}^r_{\mathbf{k}}$ is an algebraic embedding if and only if for each vertex $m_0 \in P$ the set $\{m \in M \mid m + m_0 \in P \cap M\}$ generates the semigroup $\check{\mathbf{c}} \cap M$ where $\check{\mathbf{c}}$ is the cone dual to m_0.*

REMARK 1. For nonsingular toric varieties, this theorem implies that any μ_P will be an algebraic embedding. This follows from the fact that $X_{\mathbf{k}}(\Sigma)$ is nonsingular if and only if every maximal cone of Σ (and its dual cone) is basic (see [Oda]). This means that the extreme set of the dual of every maximal cone $\mathbf{c}$ is a lattice basis for M and, hence, generates the semigroup $\check{\mathbf{c}} \cap M$. If P is a lattice polytope P dual to such a Σ, then $\{m \in M \mid m + m_0 \in P \cap M\}$ contains the extreme set for the cone $\check{\mathbf{c}}$ dual to m_0 for all vertices $m_0 \in P$ and, therefore, satisfies the hypotheses of Theorem 1.

2.2. *Injectivity of μ_P.* Since an algebraic embedding is injective, the conditions on P in Theorem 1 are sufficient for μ_P to be injective; however, one can give weaker conditions which depend on a certain arithmetic property of the field. Let $\mathcal{R}(\mathbf{k})$ denote the set of primes p in $\mathbb{Z}$ such that the power map $x \mapsto x^p$ does not define an automorphism of the group $\mathbf{k}^*$. For example, $\mathcal{R}(\mathbb{R})$ is $\{2\}$, $\mathcal{R}(\mathbb{C})$ is the set of all primes, and $\mathcal{R}(\mathbb{F}_q)$ is the set of prime divisors of $q - 1$.

For each l-face σ of a convex integral polytope P, we denote by $\mathbb{R}\sigma$ the unique l-dimensional subspace of $M_{\mathbb{R}}$ which is parallel to σ (i.e., the $\mathbb{R}$-linear extension of $\sigma - p$ for some $p \in \sigma$). We let $\mathbb{Z}\sigma$, then, be the rank-l unimodular sublattice $\mathbb{R}\sigma \cap M$ of M. We now state our main result.

THEOREM 2. *Let P be an integral convex polytope, and let $\mu_P : X_{\mathbf{k}}(\Sigma) \to \mathbb{P}^r_{\mathbf{k}}$ be the corresponding map to projective space. If for every pair (v, σ), where σ is a face of P and v is a vertex of σ, the image of $\{m \in M \mid m + v \in \sigma \cap M\}$ generates $\mathbb{Z}\sigma \otimes \mathbb{Z}_p$ as a $\mathbb{Z}_p$-vector space for all $p \in \mathcal{R}(\mathbf{k})$, then μ_P is injective. The converse holds if $\mathbf{k}$ contains a nontrivial pth root of 1 for every $p \in \mathcal{R}(\mathbf{k})$.*

Just as we defined the sublattice $\mathbb{Z}\sigma$ for a face σ of P, for any convex l-dimensional cone $\mathbf{c} \in M$, we let $\mathbb{Z}\mathbf{c}$ denote the rank-l unimodular sublattice $\mathbb{R}\mathbf{c} \cap M$. In particular, $\mathbb{Z}\mathbf{c} = M$ for every maximal cone $\mathbf{c}$. For any finite subset S of M, we denote by $\mathbb{Z}_{\geq 0}S$ the subsemigroup of M generated by S (including 0), and for $x \in \mathrm{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k})$ we define the *support of x in S*, written $\operatorname{supp} x$, to be the set $\{m \in S \mid x(m) \neq 0\}$. Theorem 2 is a straightforward consequence of the following affine version.

THEOREM 3. *Let $\mathbf{c}$ be a maximal, strictly convex cone in $N_{\mathbb{R}}$ with dual cone $\check{\mathbf{c}}$. Let S be a subset of $\check{\mathbf{c}} \cap M$ which contains $\operatorname{ext} \check{\mathbf{c}}$. If for every face $\mathbf{b} < \check{\mathbf{c}}$, the image of $S \cap \mathbf{b}$ generates $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$ for all $p \in \mathcal{R}(\mathbf{k})$, then the natural map*

$$\phi : U_{\mathbf{k}}(\mathbf{c}) = \mathrm{Hom}(\check{\mathbf{c}} \cap M, \mathbf{k}) \to \mathrm{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k})$$

is injective. The converse holds if $\mathbf{k}$ contains a nontrivial pth root of 1 for every $p \in \mathcal{R}(\mathbf{k})$.

LEMMA 1. *Let $\mathbf{c}$ be a strictly convex, l-dimensional cone in $M_{\mathbb{R}}$, let p be a point of M in the relative interior (denoted by int) of $\mathbf{c}$, and let m_0 be an element of $\operatorname{ext} \mathbf{c}$. Then there is a map $a : \operatorname{ext} \mathbf{c} \to \mathbb{Q}_{\geq 0}$ such that $a(m_0) > 0$ and*

$$p = \sum_{m \in \operatorname{ext} \mathbf{c}} a(m) \cdot m.$$

Proof. The proof is by induction on $l = \dim \mathbf{c}$. If $l = 0$, then p is a positive integral multiple of m_0. Hence, $\operatorname{ext} \mathbf{c} = \{m_0\}$ and we can take $a(m_0)$ to be this integral multiple. More generally, the ray starting at m_0 and passing through p will intersect $\operatorname{int} \mathbf{b}$ for a unique proper face $\mathbf{b}$ of $\mathbf{c}$. Let p' be this intersection point and notice that there are positive rational constants α and β such that

$$p = \alpha m_0 + \beta p'.$$

Next choose $m_0' \in \operatorname{ext} \mathbf{b}$. By induction, there is a map $a' : \operatorname{ext} \mathbf{b} \to \mathbb{Q}_{\geq 0}$ such that $a'(m_0') > 0$ and

$$p' = \sum_{m \in \operatorname{ext} \mathbf{b}} a'(m) \cdot m.$$

Finally, define $a(m_0) = \alpha$, $a(m) = \beta \cdot a'(m)$ for $m \in \operatorname{ext} \mathbf{b}$, and $a(m) = 0$ otherwise. ∎

LEMMA 2. *Let $\check{\mathbf{c}}$ be a maximal, strictly convex cone in $M_{\mathbb{R}}$. If S is any finite subset of $\check{\mathbf{c}} \cap M$ containing $\operatorname{ext} \check{\mathbf{c}}$ and $x \in \mathrm{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k})$, then $\operatorname{supp} x = S \cap \mathbf{b}$ for some face $\mathbf{b}$ of $\check{\mathbf{c}}$.*

Proof. Each $p \in S$ is in the relative interior of exactly one face of $\check{\mathbf{c}}$. If $p \in \operatorname{supp} x$ and $p \in \operatorname{int} \mathbf{b}$ then $\mathbf{b} \cap S$ is contained in $\operatorname{supp} x$. Indeed, suppose that $p \in \operatorname{int} \mathbf{b}$ for some l-face $\mathbf{b}$ and $p \in \operatorname{supp} x$. For each $m_0 \in \operatorname{ext} \mathbf{b}$, we can write

$$p = \sum_{m \in \operatorname{ext} \mathbf{b}} a(m) \cdot m$$

with a as in Lemma 1. But $x(p) \neq 0$ implies, then, that $x(m_0) \neq 0$ since all coefficients $a(m)$ are nonnegative and $a(m_0)$ is strictly positive. It follows that $\operatorname{ext} \mathbf{b}$ is contained in the support of x, and since any other point in $\mathbf{b} \cap S$ is a nonnegative rational combination of these extreme points, it must also be an element of $\operatorname{supp} x$.

Let $\mathcal{B}$ be the collection of all faces $\mathbf{b}$ such that $\mathbf{b} \cap S \subset \operatorname{supp} x$. It will suffice, then, to show that $\mathcal{B}$ has a unique maximal element. We will show that if $\mathbf{b}_1 \cap S$ and $\mathbf{b}_2 \cap S$ are both in $\operatorname{supp} x$, then so is $\mathbf{b}_3 \cap S$ where $\mathbf{b}_3$ is the smallest face of $\check{\mathbf{c}}$ containing $\mathbf{b}_1$ and $\mathbf{b}_2$. If

$$p_1 = \sum_{m \in \operatorname{ext} \mathbf{b}_1} m \quad \text{and} \quad p_2 = \sum_{m \in \operatorname{ext} \mathbf{b}_2} m \tag{1}$$

then $p_1 \in \operatorname{int} \mathbf{b}_1$, $p_2 \in \operatorname{int} \mathbf{b}_2$, and $p_1 + p_2 \in \operatorname{int} \mathbf{b}_3$. For any $m_0 \in \operatorname{ext} \mathbf{b}_3$, we can find (by Lemma 1) $a : \operatorname{ext} \mathbf{b}_3 \to \mathbb{Q}_{\geq 0}$ with $a(m_0) > 0$ such that

$$p_1 + p_2 = \sum_{m \in \operatorname{ext} \mathbf{b}_3} a(m) \cdot m.$$

Substituting the expressions (1) for p_1 and p_2, we have

$$\sum_{m \in \operatorname{ext} \mathbf{b}_1} m + \sum_{m \in \operatorname{ext} \mathbf{b}_2} m = \sum_{m \in \operatorname{ext} \mathbf{b}_3} a(m) \cdot m. \tag{2}$$

Multiplying both sides by a suitable positive integer D to clear denominators (so that all terms of equation (2) are in the semigroup $\mathbb{Z}_{\geq 0} S$) and applying x to the result gives the equation

$$\prod_{m \in \operatorname{ext} \mathbf{b}_1} x(m)^D \prod_{m \in \operatorname{ext} \mathbf{b}_2} x(m)^D = x(m_0)^{a(m_0) D} \prod_{m \in \operatorname{ext} \mathbf{b}_3 \setminus \{m_0\}} x(m)^{a(m) D}.$$

Since the left hand side of this equation is nonzero, $x(m_0)$ is nonzero. Repeating the argument for all extreme points of $\mathbf{b}_3$ gives $\operatorname{ext} \mathbf{b}_3 \subset \operatorname{supp} x$, and using the argument of the previous paragraph, we have $\mathbf{b}_3 \cap S \subset \operatorname{supp} x$. It follows that there is a unique maximal element of $\mathcal{B}$. ■

Lemma 3. *Let* $\mathbf{b}$ *be a convex cone in* $M_{\mathbb{R}}$, *and let* $\mathcal{R}$ *be any collection of prime numbers. If* S *is any finite subset of* $\mathbf{b} \cap M$ *whose image generates* $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$ *for all* $p \in \mathcal{R}$, *then for any* $m \in \mathbf{b} \cap M$ *there exist integers* b *and* a_s *(for all* $s \in S$*) such that*

$$bm = \sum_{s \in S} a_s s,$$

and $b \not\equiv 0 \bmod p$ *for all* $p \in \mathcal{R}$.

P r o o f. Let $p \in \mathcal{R}$. Since S generates $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$, S contains a rational basis for $\mathbb{Z}\mathbf{b} \otimes \mathbb{Q}$ whose image is a $\mathbb{Z}_p$-basis for $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$. This means we can find relatively prime integers $b(p)$ and $a_s(p)$ (with all but $\dim \mathbf{b}$ of the $a_s(p)$'s equal to zero) such that

$$b(p)m = \sum a_s(p)s. \tag{3}$$

Moreover, we know $b(p) \not\equiv 0 \bmod p$ since reducing (3) mod p would give a nontrivial linear dependence for our $\mathbb{Z}_p$-basis. Do this for every $p \in \mathcal{R}$, and let b be the greatest common divisor of $\{b(p) \mid p \in \mathcal{R}\}$. Then b can be written as a $\mathbb{Z}$-linear combination of the $b(p)$'s (finitely many since the gcd of any set of integers is the gcd of a finite subset). Taking the same $\mathbb{Z}$-linear combination of the equations (3) gives the desired integers b and a_s, for $s \in S$. ∎

Proof of Theorem 3. Let $x \in \mathrm{Hom}(\mathbb{Z}_{\geq 0}S, \mathbb{R})$ and find (by Lemma 2) an l-face $\mathbf{b}$ of $\check{\mathbf{c}}$ such that $\operatorname{supp} x = S \cap \mathbf{b}$. We will define a unique element $y \in U_{\mathbf{k}}(\mathbf{c})$ such that $\phi(y) = x$. For $m \notin \mathbf{b} \cap M$, we let $y(m) = 0$. If $m \in \mathbf{b} \cap M$, then by Lemma 3, there exist integers b and a_s $(s \in S \cap \mathbf{b})$ such that

$$bm = \sum_{s \in S \cap \mathbf{b}} a_s s$$

and $b \not\equiv 0 \bmod p$ for all $p \in \mathcal{R}(\mathbf{k})$. The fact that $b \not\equiv 0 \bmod p$ for all $p \in \mathcal{R}(\mathbf{k})$ implies that every nonzero element of $\mathbf{k}$ has a unique bth root. Since $x(s) \neq 0$ for all $s \in S \cap \mathbf{b}$, there will be a unique nonzero element $y(m) \in \mathbf{k}$ satisfying the equation

$$y(m)^b = \prod_{s \in S \cap \mathbf{b}} x(s)^{a_s}.$$

The function $y : \check{\mathbf{c}} \cap M \to \mathbf{k}$ thus defined is a semigroup homomorphism satisfying $\phi(y) = x$, and is the unique such semigroup homomorphism since any y in the preimage of x must satisfy the above equation for $y(m)$. Hence, ϕ is injective.

For the converse, let $p \in \mathcal{R}(\mathbf{k})$ be such that $S \cap \mathbf{b}$ does not generate $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$, and let ξ be a nontrivial pth root of 1. Let $x \in \mathrm{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k})$ be the semigroup homomorphism given by $x(m) = 1$ for $m \in \mathbb{Z}_{\geq 0}S \cap \mathbf{b}$ and $x(m) = 0$ otherwise. The point $y \in U_{\mathbf{k}}(\mathbf{c})$, given by $y(m) = 1$ for all $m \in \mathbf{b} \cap M$ and $y(m) = 0$ otherwise, is clearly a preimage of x. We will show that x has more than one preimage. Since the map $\mathbf{b} \cap M \to \mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$ is surjective while the map $\mathbb{Z}_{\geq 0}S \to \mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$ is not, we can find a primitive m_0 in $\mathbf{b} \cap M$ such that $m_0 \notin \mathbb{Z}_{\geq 0}S$ and m_0 is nonzero mod p. Define $y' \in U_{\mathbf{k}}(\mathbf{c})$ by letting $y'(m) = y(m)$ for all $m \in \mathbb{Z}_{\geq 0}S$, $y'(m_0) = \xi$, and choosing any consistent extension to $\check{\mathbf{c}} \cap M$. Then y and y' are both preimages of x but $y(m_0) \neq y'(m_0)$. ∎

Proof of Theorem 2. Let $\mathbf{c}$ be a maximal cone of Σ and let $\mu_{\mathbf{c}}$ be the restriction of μ_P to $U_{\mathbf{k}}(\mathbf{c})$. As we said above, $\check{\mathbf{c}}$ is the $\mathbb{R}_{\geq 0}$-span of the translated polytope $P - v$ where v is the vertex of P dual to the cone $\mathbf{c}$. Let S be the set $(P - v) \cap M$. For each face $\mathbf{b}$ of $\check{\mathbf{c}}$, the set $\mathbf{b} \cap S$ is precisely $\{m \in M \mid m + v \in \sigma \cap M\}$ where σ is the face of P corresponding to $\mathbf{b}$. Moreover, the map $\mu_{\mathbf{c}}$ factors through the natural map ϕ of Theorem 3, giving the following commutative diagram:

$$\begin{array}{ccc} U_{\mathbf{k}}(\mathbf{c}) & \xrightarrow{\mu_{\mathbf{c}}} & \mathbb{P}^r_{\mathbf{k}} \\ & \searrow^{\phi} \qquad \nearrow & \\ & \operatorname{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k}) & \end{array}$$

Since $\mathbb{Z}\sigma = \mathbb{Z}\mathbf{b}$, the hypotheses of our theorem guarantee that $\mathbf{b} \cap S$ generates $\mathbb{Z}\mathbf{b} \otimes \mathbb{Z}_p$ for all $p \in \mathcal{R}(\mathbf{k})$; hence, by Theorem 3, the map ϕ is injective. But it is clear that the natural map $\operatorname{Hom}(\mathbb{Z}_{\geq 0}S, \mathbf{k}) \to \mathbb{P}^r_{\mathbf{k}}$ is injective, because each element of S corresponds to a distinct projective coordinate. It follows that $\mu_{\mathbf{c}}$ is injective.

It remains to show that if x_1 and x_2 are two points of $X_{\mathbf{k}}(\Sigma)$ with $x_i \in U_{\mathbf{k}}(\mathbf{c}_i)$ (for maximal cones $\mathbf{c}_1$ and $\mathbf{c}_2$ of Σ) such that $\mu_P(x_1) = \mu_P(x_2)$, then $x_1 = x_2$. We will show that if x_2 has the same image as x_1, then $x_1 \in \operatorname{Im}\{U_{\mathbf{k}}(\mathbf{c}_1 \cap \mathbf{c}_2) \to U_{\mathbf{k}}(\mathbf{c}_1)\}$; in other words, $x_1 \in U_{\mathbf{k}}(\mathbf{c}_1) \cap U_{\mathbf{k}}(\mathbf{c}_2)$. But then x_1 and x_2 are both in $U_{\mathbf{k}}(\mathbf{c}_2)$ and by the previous paragraph they must coincide.

Let $\mathbf{b}$ be the common face $\mathbf{c}_1 \cap \mathbf{c}_2$ and recall from the beginning of Section 1.2 that x_1 is in the image of the inclusion $U_{\mathbf{k}}(\mathbf{b}) \to U_{\mathbf{k}}(\mathbf{c}_1)$ if and only if the support of x_1 (in $\check{\mathbf{c}}_1 \cap M$) is precisely the set $(\mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1) \cap M$. Let v_1 and v_2 be the vertices of P dual to the cones $\mathbf{c}_1$ and $\mathbf{c}_2$. Then $x_1(v_2 - v_1) \neq 0$, since $x_1(v_2 - v_1)$ and $x_2(v_2 - v_2) = 1$ are the same projective coordinate for the point $\mu_P(x_1) = \mu_P(x_2) \in \mathbb{P}^r_{\mathbf{k}}$. Since $v_2 - v_1 \in \operatorname{int} \mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1$, it follows from Lemma 2 (or its proof, rather) that $\operatorname{supp} x_1 = (\mathbf{b}^{\perp} \cap \check{\mathbf{c}}_1) \cap M$ as desired.

The proof of the converse is similar to the proof of the converse in Theorem 3. ■

2.3. *Topological embeddings and examples.* For applications of Theorem 2, we single out the cases $\mathbf{k} = \mathbb{R}$ and $\mathbf{k} = \mathbb{C}$. For these fields, every $p \in \mathcal{R}(\mathbf{k})$ has a nontrivial pth root of 1. Moreover, since $X_{\mathbf{k}}(\Sigma)$ is compact, and $\mathbb{P}^r_{\mathbf{k}}$ is Hausdorff, μ_P is a topological embedding if it is injective.

Corollary 1. *The map $\mu_P : X_{\mathbb{C}}(\Sigma) \to \mathbb{P}^r_{\mathbb{C}}$ is a topological embedding if and only if for every pair (v, σ), where σ is a face of P and v is a vertex of σ, the image of $\{m \in M \mid m + v \in \sigma \cap M\}$ generates the lattice $\mathbb{Z}\sigma$.*

Example. Let $M = \mathbb{Z}^3$, and let P be the convex hull of the set

$$S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,2), (1,1,3)\}.$$

Then $\mu_P : X_{\mathbb{C}}(\Sigma) \to \mathbb{P}^4_{\mathbb{C}}$ is a topological embedding, but not an algebraic embedding (the semigroup for the maximal cone dual to $(0,0,0)$ is not generated as a semigroup by S).

COROLLARY 2. *The map* $\mu_P : X_{\mathbb{R}}(\Sigma) \to \mathbb{P}^r_{\mathbb{R}}$ *is a topological embedding if and only if for every pair* (v, σ), *where* σ *is a face of* P *and* v *is a vertex of* σ, *the image of* $\{m \in M \mid m + v \in \sigma \cap M\}$ *generates* $\mathbb{Z}\sigma \otimes \mathbb{Z}_2$ *as a* $\mathbb{Z}_2$*-vector space.*

EXAMPLE 2. Let $M = \mathbb{Z}^3$, and let P be the convex hull of the set

$$S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,3)\}.$$

Then $\mu_P : X_{\mathbb{R}}(\Sigma) \to \mathbb{P}^3_{\mathbb{R}}$ is a topological embedding while $\mu_P : X_{\mathbb{C}}(\Sigma) \to \mathbb{P}^3_{\mathbb{C}}$ is not.

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