COLLOQUIUM MATHEMATICUM

VOL. 80 1999 NO. 2

A DUALITY RESULT FOR ALMOST SPLIT SEQUENCES

BY

LIDIA ANGELERI HÜGEL AND HELMUT VALENTA (MÜNCHEN)

Abstract. Over an artinian hereditary ring R, we discuss how the existence of almost split sequences starting at the indecomposable non-injective preprojective right R-modules is related to the existence of almost split sequences ending at the indecomposable non-projective preinjective left R-modules. This answers a question raised by Simson in [27] in connection with pure semisimple rings.

1. Introduction. Simson showed in [27] that there is a close relationship between the existence of almost split sequences starting at the indecomposable non-injective preprojective modules and the validity of the pure semisimplicity conjecture (see below for the definitions). Namely, let R be an artinian hereditary left pure semisimple ring. Then there are almost split sequences starting at any indecomposable non-injective preprojective *right* R-module. Moreover, if there are also almost split sequences starting at any indecomposable non-injective preprojective *left* R-module, then R even has finite representation type, and the conjecture is verified [27, 3.1 and 1.8].

We see that in this context, tools for passing information from the left to the right side and vice versa are required. For instance, it was asked by Simson [27, 1.9] whether, over an artinian hereditary ring, the existence of almost split sequences for all preprojectives on one side can be expressed in terms of the preinjective modules on the other side. The aim of this paper is to give the following answer.

THEOREM 1.1. *The following statements are equivalent for a right artinian hereditary ring R.*

(P) *For every indecomposable preprojective non-injective right R-module A_R there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* mod-R.

(I) *R is left artinian, all indecomposable injective left R-modules are finitely generated, and for every indecomposable preinjective non-projective left R-module ${}_RC$ there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* R-mod.

1991 *Mathematics Subject Classification*: Primary 16G10, 16G70, 16E60.

Recall that a ring R is said to be *left pure semisimple* if every left R-module is a direct sum of finitely presented modules. It is well known that a ring has finite representation type if and only if it is left and right pure semisimple, and it is conjectured that one-sided pure semisimplicity is sufficient, i.e. that every left (or right) pure semisimple ring has finite representation type. This conjecture has been verified for several classes of rings (e.g. in [6], [19]), but it is still open in general (see also [25], [26], [34], [24], [16]). Recent work of Simson (e.g. in [29], [30]) shows how a potential counter-example should look like.

The role played by the preprojectives and the preinjectives in connection with the pure semisimplicity conjecture was also pointed out in [33] and [19]. In [19], Herzog showed that over a left pure semisimple ring R, the sets $\mathcal{I}_R$ of the isomorphism classes of indecomposable preinjective right R-modules and ${}_R\mathcal{P}$ of the isomorphism classes of indecomposable preprojective left R-modules are finite, and there is an injection $\mathcal{I}_R \to {}_R\mathcal{P}$ which is given by the local duality $A_R \mapsto {}_RA^+$ (see the definition in Section 3). Furthermore, if this map is even bijective and R is artinian, then R has finite representation type.

Observe that over any artinian hereditary left pure semisimple ring we always have the corresponding bijection on the other side.

Corollary 1.2. *Let R be a right artinian hereditary ring satisfying one of the equivalent conditions of Theorem* 1.1. *Then the local duality ${}_RA \mapsto A_R^+$ gives a bijection between the set ${}_R\mathcal{I}$ of the isomorphism classes of indecomposable preinjective left R-modules and the set $\mathcal{P}_R$ of the isomorphism classes of indecomposable preprojective right R-modules.*

Let us further point out that the almost split sequences considered in Theorem 1.1 are even almost split in the category of all modules. Moreover, we can interpret condition (P) in terms of the existence of a preprojective component, and condition (I) in terms of the existence of a preinjective component; see 8.3–8.5.

Finally, we remark that Theorem 1.1 extends a result of Zimmermann [32] for artinian rings R, which states that for every indecomposable projective non-injective right R-module A_R there is an almost split sequence $0 \to A \to B \to C \to 0$ in mod-R if and only if all indecomposable injective left R-modules are finitely generated. Actually, the argument used in [32] for showing this result will be one of the main tools in our proof.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we give the definition and properties of the local duality and explain its role in connection with the existence of almost split sequences. Sections 4 and 5 deal with tilting and cotilting modules over artinian rings. We need this knowledge in Sections 6 and 7, where we construct preprojective tilting and preinjective cotilting modules in order to prove Theorem 1.1 and Corollary 1.2. The proofs of the last-mentioned results and some consequences are given in Section 8.

The main results of this paper were presented at the Conference on Interaction between Ring Theory and Representations of Algebras held in Murcia, Spain, in January 1998.

2. Preliminaries. We start out with some notation. For a ring R we denote by Mod-R the category of all and by mod-R the category of all finitely presented right R-modules; the corresponding categories of left R-modules are denoted by R-Mod and R-mod. By a subcategory $\mathcal{C}$ of mod-R we always mean a *full* subcategory. We denote by $\operatorname{Add}\mathcal{C}$ (resp. $\operatorname{add}\mathcal{C}$) the subcategory of Mod-R consisting of all modules isomorphic to direct summands of (finite) direct sums of modules in $\mathcal{C}$. If $\mathcal{C}$ contains just one module M, then we write $\operatorname{Add} M$ (resp. $\operatorname{add} M$). Further, we write ind-R, resp. R-ind, for the subcategory of mod-R, resp. of R-mod, consisting of all indecomposables. Moreover, for $M \in$ mod-R we denote by $\operatorname{ind} M$ the subcategory of mod-R consisting of the modules which are isomorphic to an indecomposable direct summand of M.

Let us now recall the notions of preprojective and preinjective modules, as introduced by Auslander and Smalø in [11]. Note that in our situation they coincide with the preprojectives and preinjectives constructed by Coxeter functors, as studied in [27], and also with the preprojectives and preinjectives considered in [19] (see [3]).

Let R be a right artinian ring, and let $\mathcal{C}$ be a subcategory of mod-R which is closed under isomorphic images and direct summands. A *cover* for $\mathcal{C}$ is a subcategory $\mathcal{Y}$ of $\mathcal{C}$ consisting of indecomposable modules such that for each module C in $\mathcal{C}$ there is an epimorphism $Y \to C$ with Y in $\operatorname{add}\mathcal{Y}$. Moreover, we say that a module C in $\mathcal{C}$ is *splitting projective* in $\mathcal{C}$ if each epimorphism $X \to C$ with X in $\operatorname{add}\mathcal{C}$ is splittable, and we denote by $\mathcal{P}_0(\mathcal{C})$ the subcategory of mod-R consisting of all indecomposable splitting projectives in $\mathcal{C}$. Obviously, $\mathcal{P}_0 = \mathcal{P}_0(\text{mod-}R)$ is the category of all indecomposable projective right R-modules. We proceed by induction and set $\mathcal{P}_n = \mathcal{P}_0(\text{mod-}R_{\mathcal{P}^n})$, where $\mathcal{P}^n = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{n-1}$ and mod-$R_{\mathcal{P}^n}$ denotes the subcategory of mod-R consisting of all modules with no direct summand in $\mathcal{P}^n$. The modules in $\operatorname{add}(\bigcup_{n\in\mathbb{N}_0}\mathcal{P}_n)$ are called *preprojective.*

Furthermore, by defining $\mathcal{P}_\infty = \text{ind-}R \setminus \bigcup_{n\in\mathbb{N}_0}\mathcal{P}_n$ we obtain a partition $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\infty$ of ind-R, which is called a *preprojective partition* if $\mathcal{P}_n$ is a finite cover for mod-$R_{\mathcal{P}^n}$ for each $n \in \mathbb{N}$. It was shown in [11] that every artin algebra has a preprojective partition. For a right artinian hereditary ring, however, we know from [3] that the existence of a preprojective partition is equivalent to condition (P) in our Theorem 1.1.

Preinjective modules are defined dually. Observe that in this paper we denote by $\mathcal{I}_n$ the subcategory of preinjective *left* R-modules of level n (while the objects of $\mathcal{P}_n$ are right modules), that is, $\mathcal{I}_n$ consists of all indecomposable splitting injectives in the category $R\text{-mod}_{\mathcal{I}^n}$ of finitely generated left modules without direct summands in $\mathcal{I}^n = \mathcal{I}_0 \cup \ldots \cup \mathcal{I}_{n-1}$ over a left artinian ring R. Again, we are interested in the property that $\mathcal{I}_n$ is a finite cocover for $R\text{-mod}_{\mathcal{I}^n}$ for each $n \in \mathbb{N}$, where a *cocover* is the dual notion to a cover. In this case we say that $\mathcal{I}_0, \ldots, \mathcal{I}_\infty$ define a *preinjective partition* of R-mod.

For definitions and basic results about almost split sequences and irreducible morphisms we refer to [10]; see also [17], [28]. We adopt the notation $A = \tau C$ and $C = \tau^{-1}A$ if $0 \to A \to B \to C \to 0$ is an almost split sequence in mod-R, and define inductively τ^n and τ^{-n}. Observe that we will consider almost split sequences in the category Mod-R or in mod-R. The relationship between these two cases was explained in [32]; see 3.4 below. Details on tilting theory can be found in [22], [12].

3. The local duality. Let R be a ring, X_R a module with $S = \operatorname{End}_R X$, and ${}_SV$ a minimal injective cogenerator of S Mod. The left R-module

$$X^+ = {}_R\operatorname{Hom}_S({}_SX, {}_SV)$$

is called the *local dual of* X. If R is an artin algebra and X_R is finitely generated, then it is well known that $X^+ \cong D(X)$ for the usual duality $D : \text{mod-}R \to R\text{-mod}$.

The local duality will be one of the main tools in proving Theorem 1.1. The reason is basically given by the following results on the existence of almost split sequences. Recall that Tr denotes the Auslander–Bridger transpose [8].

THEOREM 3.1 (Auslander [7, Chapter I, 3.9]). *Let R be a semiperfect ring. For any finitely presented non-projective module C_R with local endomorphism ring there is an almost split sequence* $0 \to (\operatorname{Tr} C)^+ \to B \to C \to 0$ *in* Mod-R.

THEOREM 3.2 (Zimmermann [32, Satz 6]). *Let R be an artinian ring. For any finitely generated indecomposable non-injective right R-module A_R such that the local dual ${}_RA^+$ is a finitely generated left R-module there is an almost split sequence* $0 \to A \to B \to \operatorname{Tr} A^+ \to 0$ *in* mod-R.

Observe that this result has the following consequence.

PROPOSITION 3.3 (Zimmermann [32, Folgerung 9]). *Let R be an artinian ring. For every indecomposable projective non-injective right R-module A_R there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* mod-R *if and only if all indecomposable injective left R-modules are finitely generated.*

Recall that a module M is said to be *endofinite* if it has finite length as a module over its endomorphism ring. In particular, endofinite modules are pure-injective. Of course, a semiperfect ring R is left artinian if and only if every indecomposable projective right R-module is endofinite. From the next result we then conclude that the almost split sequences in the above proposition are even almost split sequences in Mod-R.

PROPOSITION 3.4 (Zimmermann [32, Proposition 3]). *Let R be a semiperfect ring. An almost split sequence $0 \to A \to B \to C \to 0$ in* mod-R *is almost split in* Mod-R *if and only if A is pure-injective.*

We now collect some basic properties of the local duality we are going to use in the sequel.

LEMMA 3.5. *Let R be a semiperfect ring with Jacobson radical J.*

(1) *Let M_R be a finitely presented module. If* $\operatorname{End}_R M$ *is local*, *then so is* $\operatorname{End}{}_R M^+$.

(2) *Let M_R, N_R be modules and assume that M_R is finitely presented. Then* $\operatorname{Hom}_R(M,N) \neq 0$ *if and only if* $\operatorname{Hom}_R(N^+, M^+) \neq 0$.

(3) *A module M_R is endofinite if and only if ${}_R M^+$ is endofinite.*

(4) *Let $e \in R$ be a local idempotent. Then $(eR/eJ)^+ \cong Re/Je$, and $(eR)^+$ is an injective envelope of Re/Je. Further*, *if R is left artinian and ${}_R I$ is an injective envelope of (Re/Je)*, *then $I^+ \cong eR$.*

(5) *Let R be left artinian. An indecomposable left R-module ${}_R A$ is injective if and only if A_R^+ is projective.*

Proof. All statements are well known (see [23, Th. 2], [32, Lemma 5]). For the reader's convenience we include the argument for (2). Let $S = \operatorname{End}_R N$, $T = \operatorname{End}_R M$, and ${}_S V, {}_T U$ be minimal injective cogenerators of S-Mod and T-Mod, respectively. By the Hom-$\otimes$-adjointness we have $\operatorname{Hom}_R(N^+, M^+) \cong \operatorname{Hom}_T({}_T M \otimes_R N^+, {}_T U)$, and since M_R is finitely presented and ${}_S V$ is injective, we have a canonical isomorphism ${}_T M \otimes_R N^+ \cong {}_T \operatorname{Hom}_S({}_S \operatorname{Hom}_R(M,N), {}_S V)$. ■

As we have seen above, the local duality extends the ordinary duality we are supplied with in the special case of artin algebras. The next result allows us to replace the Auslander–Reiten formula [9] in some arguments.

LEMMA 3.6. *Let $0 \to A \to B \to Z \to 0$ be an almost split sequence in* Mod-R, *where R is an arbitrary ring*, *and X_R a module. If* $\operatorname{Hom}_R(X,A)=0$, *then also* $\operatorname{Ext}^1_R(Z,X) = 0$. *The converse holds if Z has projective dimension at most one.*

P r o o f. A non-split exact sequence $0 \to X \to Y \to Z \to 0$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle\sigma}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & Z & \longrightarrow & 0 \end{array}$$

where $\sigma \neq 0$ because β is not a split epimorphism.

Conversely, assume that Z has projective dimension at most one and take a map $\sigma : X \to A$. If $\mathrm{Ext}^1_R(Z, X) = 0$, then the exact sequence $X \stackrel{\sigma}{\to} A \stackrel{p}{\to} \mathrm{Cok}\,\sigma \to 0$ gives rise to an isomorphism $\mathrm{Ext}^1_R(Z, p) : \mathrm{Ext}^1_R(Z, A) \cong \mathrm{Ext}^1_R(Z, \mathrm{Cok}\,\sigma)$. But this implies that p does not factor through the almost split sequence $0 \to A \to B \to Z \to 0$, hence p is an isomorphism, and $\sigma = 0$. ■

4. Some tilting theory. We have seen in the previous section (see Proposition 3.3) that the first step of our Theorem 1.1 was already proven by Zimmermann in [32, Folgerung 9]. He constructed almost split sequences starting at the projectives by using the property that their local dual is finitely generated and by applying Theorem 3.2. So, if we want to proceed similarly for the other preprojectives, we first have to show that their local duals are finitely generated. To this end, we shall need some tilting theory.

Throughout this section, let R be a ring and T_R a *tilting module* in the sense of [12], i.e. a finitely presented module of projective dimension at most one such that $\mathrm{Ext}^1_R(T, T) = 0$, and there is no non-zero module M_R satisfying $\mathrm{Hom}_R(T, M) = \mathrm{Ext}^1_R(T, M) = 0$ (see also [14, Th. 3]). We denote by $\mathrm{Gen}\, T$ the category of all T-generated modules, set $S = \mathrm{End}_R\, T$, and start out with a result connected with the previous section. It shows that the injective left modules over S can be described in terms of local duality. Observe that statement (1) is also true for $*$-modules in the sense of Colpi [13].

LEMMA 4.1. (1) ${}_R X^+ \cong \mathrm{Hom}_S({}_S T, {}_S(\mathrm{Hom}_R(T, X)_S)^+)$ *for any* $X \in \mathrm{Gen}\, T$.

(2) *If S is semiperfect and $X_R = eT$ for some local idempotent $e \in S$, then ${}_S T \otimes_R X^+$ is the injective envelope of $Se/J(S)e$.*

P r o o f. (1) Let $S' = \mathrm{End}_R\, X$ and ${}_{S'}V$ be a minimal injective cogenerator of S'-Mod. Since $\mathrm{Hom}_R(T, -) : \text{Mod-}R \to \text{Mod-}S$ is fully faithful on objects of $\mathrm{Gen}\, T$, we have $S' \cong \mathrm{End}_S \mathrm{Hom}_R(T, X) =: E$. We identify S' with E to obtain ${}_{S'}\mathrm{Hom}_R(T, X) \otimes_S T_R \cong {}_{S'}X_R$. So, we conclude that

$$\begin{aligned} {}_R X^+ &= {}_R\mathrm{Hom}_{S'}(X, {}_{S'}V) \cong {}_R\mathrm{Hom}_{S'}(\mathrm{Hom}_R(T,X)\otimes_S T, {}_{S'}V) \\ &\cong {}_R\mathrm{Hom}_S(T, {}_S\mathrm{Hom}_{S'}(\mathrm{Hom}_R(T,X), {}_{S'}V)) \\ &\cong {}_R\mathrm{Hom}_S({}_ST, {}_S(\mathrm{Hom}_R(T,X)_S)^+). \end{aligned}$$

(2) From $\mathrm{Hom}_R(T,X)\cong eS$ we deduce by 3.5 that ${}_SI := \mathrm{Hom}_R(T,X)^+$ is the injective envelope of $Se/J(S)e$. Further, we know $X^+\cong \mathrm{Hom}_S(T,I)$ by (1). Since ${}_SI$ is in the torsion class in S-Mod induced by the tilting module ${}_ST$ with $\mathrm{End}\,{}_ST\cong R$ (see [12, 1.1]), we have ${}_SI\cong {}_ST\otimes_R \mathrm{Hom}_S(T,I)\cong {}_ST\otimes_R X^+$. ∎

Next, we need some knowledge about the Ext-projectives and the Ext-injectives in torsion theories. Recall that a module X in an extension-closed subcategory $\mathcal{C}\subseteq$ Mod-R is said to be Ext-*projective in* $\mathcal{C}$ if $\mathrm{Ext}^1_R(X,C)=0$ for all $C\in\mathcal{C}$. Ext-injectives are defined dually. Moreover, for a class $\mathcal{T}\subseteq$ Mod-R we denote by $\tau_{\mathcal{T}}(X)$ the *trace* of $\mathcal{T}$ in the module X.

LEMMA 4.2 (Hoshino [20], [21]). *Let* $(\mathcal{T},\mathcal{F})$ *be a torsion theory in* Mod-R.

(1) *A module* X *is* Ext-*projective in* $\mathcal{F}$ *if and only if* $X\cong P/\tau_{\mathcal{T}}(P)$ *for some projective module* P. *Moreover*, X *is* Ext-*injective in* $\mathcal{T}$ *if and only if* $X\cong \tau_{\mathcal{T}}(I)$ *for some injective module* I.

(2) *Let* $0\to A\to B\to Z\to 0$ *be an almost split sequence in* Mod-R. *Then* Z *is* Ext-*projective in* $\mathcal{T}$ *if and only if* A *is* Ext-*injective in* $\mathcal{F}$.

Proof. The first statement in (1) is a straightforward generalization of [20, Lemma 1], the second statement is proven dually. For (2), see [21, Lemmata 2 and 3]. ∎

Let us apply this result to our situation. Consider the functors

$$F=\mathrm{Hom}_R(T,-):\text{Mod-}R\to\text{Mod-}S,\quad G=-\otimes_S T:\text{Mod-}S\to\text{Mod-}R,$$
$$F'=\mathrm{Ext}^1_R(T,-):\text{Mod-}R\to\text{Mod-}S,\quad G'=\mathrm{Tor}^S(T,-):\text{Mod-}S\to\text{Mod-}R$$

associated with our tilting module T. It is well known [12, §1] that these functors induce a pair of equivalences $(\mathcal{T}\overset{F,G}{\longleftrightarrow}\mathcal{Y},\ \mathcal{F}\overset{F',G'}{\longleftrightarrow}\mathcal{X})$ between the torsion and the torsion-free part of two torsion theories, namely $(\mathcal{T},\mathcal{F})$ in Mod-R on one side, where $\mathcal{T}=\mathrm{Ker}\,F'$ and $\mathcal{F}=\mathrm{Ker}\,F$, and $(\mathcal{X},\mathcal{Y})$ in Mod-S on the other side, where $\mathcal{X}=\mathrm{Ker}\,G$ and $\mathcal{Y}=\mathrm{Ker}\,G'$. Since the torsion class $\mathcal{T}$ contains all injectives, we immediately deduce from condition (1) in 4.2 that a module is Ext-injective in $\mathcal{T}$ if and only if it is injective. Moreover, we know by [31, 1.3] that a module is Ext-projective in $\mathcal{T}$ if and only if it lies in $\mathrm{Add}\,T$.

Next, we show that the Connecting Lemma, which is well known for artin algebras [18, 2.3], still holds in our more general context.

LEMMA 4.3 (Connecting Lemma). *Assume that* R *is a right artinian hereditary ring. The following statements are equivalent for an almost split* *sequence* $0 \to A \stackrel{\alpha}{\to} B \to Z \to 0$ *in* Mod-S:

(1) $A \in \mathcal{Y}$, $Z \in \mathcal{X}$.

(2) $A = F(I)$ *for some indecomposable injective module* I_R.

(3) $Z = F'(P/\tau_{\mathcal{T}}(P))$ *for some indecomposable projective module* P_R. *In this case*, $P/PJ \cong \operatorname{Soc} I$.

Proof. (1)⇒(3). For all $X \in \mathcal{X}$ we have $\operatorname{Hom}_S(X, A) = 0$ and therefore by 3.6 also $\operatorname{Ext}^1_S(Z, X) = 0$. This implies $Z \cong F'(W)$ for some indecomposable Ext-projective object $W \in \mathcal{F}$. The claim then follows from 4.2.

(3)⇒(2). We deduce from 4.2 that Z is Ext-projective in $\mathcal{X}$ and therefore A is Ext-injective in $\mathcal{Y}$. But then $A = F(I)$ for some Ext-injective object $I \in \mathcal{T}$, which is even injective by our above considerations.

(2)⇒(1). Using again 4.2, we see that A is Ext-injective in $\mathcal{Y}$, and Z lies therefore in $\mathcal{X}$.

To verify the relationship between I and P, we consider the exact sequence $P/\tau_{\mathcal{T}}(P) \cong G'Z \stackrel{\delta}{\to} GA \xrightarrow{G\alpha} GB \to GZ = 0$. Since α does not split, $G\alpha$ is not an isomorphism, and $\delta \neq 0$. Moreover, we find that the canonical map $\nu : I \to I/\operatorname{Soc} I$, which lies in $\mathcal{T}$, factors through $G\alpha$, because $F\nu$ is not a split monomorphism and thus factors through α. We then obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\delta & \longrightarrow & GA & \xrightarrow{G\alpha} & GB & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Soc} I & \longrightarrow & I & \xrightarrow{\nu} & I/\operatorname{Soc} I & \longrightarrow & 0 \end{array}$$

where g is a monomorphism, hence an isomorphism. Thus we have an epimorphism $P \to \operatorname{Im}\delta \cong \operatorname{Soc} I$, and our claim is proven. ∎

We quote the following result from [31, 1.4].

LEMMA 4.4. *Let R be a right artinian ring. The functors F, G and F' carry finitely generated modules to finitely generated modules. If S is right artinian*, *then G' has this property as well.*

If R is hereditary, then we know by [12, 1.5] that the torsion theory $(\mathcal{X}, \mathcal{Y})$ is splitting. An analogous result holds if S is hereditary.

LEMMA 4.5. (1) *If S is right hereditary*, *then $(\mathcal{T}, \mathcal{F})$ is a splitting torsion theory.*

(2) *Assume that R and S are right artinian. Then the following statements are equivalent*:

(a) $(\mathcal{T} \cap \text{mod-}R, \mathcal{F} \cap \text{mod-}R)$ *is a splitting torsion theory in* mod-R.

(b) $\operatorname{Ext}^1_R(W, U) = 0$ *for all* $U \in \mathcal{T} \cap \text{mod-}R$, $W \in \mathcal{F} \cap \text{mod-}R$.

(c) *Every finitely generated module $X_S \in \mathcal{X}$ has projective dimension at most one.*

Proof. We use the same arguments as in [21].

(1) Let $U \in \mathcal{T}$ and $W \in \mathcal{F}$. By assumption, the module $X_S = F'(W)$ has a projective resolution $0 \to P_1 \to P_0 \to X_S \to 0$. Observe that $P_1, P_0 \in \mathcal{Y} = \operatorname{Ker} G'$ and $X_S \in \mathcal{X} = \operatorname{Ker} G$. So, we have an exact sequence $0 \to G'(X) \to G(P_1) \to G(P_0) \to 0$. Since $G(P_1), G(P_0) \in \operatorname{Add} T$, we know that $\operatorname{Ext}^1_R(G(P_1), U) = 0 = \operatorname{Ext}^2_R(G(P_0), U)$. Moreover, $G'(X) \cong W$, and so we obtain $\operatorname{Ext}^1_R(W, U) = 0$. This proves our claim.

(2) (a)⇔(b). Take an exact sequence $0 \to U \xrightarrow{\alpha} V \to W \to 0$ where $U \in \mathcal{T} \cap \text{mod-}R, W \in \mathcal{F} \cap \text{mod-}R$. If $(\mathcal{T} \cap \text{mod-}R, \mathcal{F} \cap \text{mod-}R)$ is a splitting torsion theory in mod-R, then $V \in \text{mod-}R$ has a decomposition $V = V_1 \oplus V_2$ with $V_1 \in \mathcal{T}$ and $V_2 \in \mathcal{F}$. So, V_2 and W belong to $\operatorname{Ker} F$, and we obtain an isomorphism $F(p_1\alpha) : F(U) \to F(V_1)$ where $p_1 : V \to V_1$ denotes the canonical projection. But since U and V_1 belong to $\mathcal{T}$, this means that $p_1\alpha$ is an isomorphism, hence α a split monomorphism, and condition (b) is verified. For the converse implication, we use the fact that R is right noetherian to see that $(\mathcal{T} \cap \text{mod-}R, \mathcal{F} \cap \text{mod-}R)$ is a torsion theory in mod-R. The splitting property then immediately follows from (b).

(b)⇒(c). Let $X_S \in \mathcal{X}$ be a finitely generated module with projective cover $0 \to K_S \to P_S \to X_S \to 0$. Observe that P, and hence also K, are in $\mathcal{Y}$. We claim that K_S is projective. First of all, since S is right noetherian, we have a presentation $S^m \to S^n \xrightarrow{f} K_S \to 0$, which yields an exact sequence $0 \to U_R \to T^n \xrightarrow{G(f)} G(K)_R \to 0$ where $U \in \mathcal{T} \cap \text{mod-}R$. Further, we have an exact sequence $0 \to G'(X) \to G(K) \to G(P) \to 0$ where $G(P) \in \operatorname{add} T$ and $G'(X) \in \mathcal{F}$ is finitely generated by Lemma 4.4. From (b) we then know that $\operatorname{Ext}^1_R(G'(X), U) = 0 = \operatorname{Ext}^1_R(G(P), U)$, hence $\operatorname{Ext}^1_R(G(K), U) = 0$. But then $G(f)$ is a splitting epimorphism, and $G(K) \in \operatorname{add} T$. This shows that K is projective.

(c)⇒(b). Let $U \in \mathcal{T} \cap \text{mod-}R$, $W \in \mathcal{F} \cap \text{mod-}R$. Then $X = F'(W) \in \mathcal{X}$ is finitely generated by Lemma 4.4 and has by assumption a projective resolution $0 \to P_1 \to P_0 \to X_S \to 0$. So, we conclude that $\operatorname{Ext}^1_R(W, U) = 0$ as in the proof of (1). ■

Finally, we will need a criterion for verifying finiteness conditions over the endomorphism ring. Let $\mathcal{C}$ be a subcategory of ind-R over an arbitrary ring R. We say that $\operatorname{add} \mathcal{C}$ *has left almost split morphisms* if for all $A \in \mathcal{C}$ there is a homomorphism $a : A \to X$ with $X \in \operatorname{add} \mathcal{C}$ such that a is not a split monomorphism, and any map $h : A \to Y$ where $Y \in \operatorname{add} \mathcal{C}$ and h is not a split monomorphism factors through a. The category $\operatorname{add} \mathcal{C}$ having right almost split morphisms is defined dually.

LEMMA 4.6. *Let R be a right artinian ring and $\mathcal{C} \subseteq$ ind-R a finite subcategory such that* $\operatorname{add} \mathcal{C}$ *has left* (*resp. right*) *almost split morphisms. Then* *for any two indecomposable modules* $X, Y \in \mathcal{C}$, $\mathrm{Hom}_R(X,Y)$ *is a finitely generated left* $\mathrm{End}_R Y$*-module* (*resp. right* $\mathrm{End}_R X$*-module*).

Proof. From the existence of left almost split morphisms in $\mathcal{C}$ we deduce inductively, for any two indecomposable modules $X, Y \in \mathcal{C}$ and any $i \in \mathbb{N}$, that the module $\mathrm{Hom}_R(X,Y)/r^i_{\mathcal{C}}(X,Y)$ (see [11, §3]) is finitely generated over $\mathrm{End}_R Y$, which gives the claim by the lemma of Harada and Sai. ∎

Let us remark that actually, under the assumptions of the above lemma, $\mathrm{Hom}_R(X,Y)$ has even finite length over $\mathrm{End}_R Y$ (resp. $\mathrm{End}_R X$). In fact, in [5] we show the following for a subcategory $\mathcal{C} \subseteq$ ind-R over an arbitrary ring R. If $\{C_i \mid i \in I\}$ is a complete irredundant set of representatives of the isomorphism classes of $\mathcal{C}$, and if we set $C = \coprod_{i\in I} C_i$ and $S = \mathrm{End}_R C$, then $\mathrm{add}\,\mathcal{C}$ has left almost split morphisms if and only if $r(C_i, C)$ is a finitely generated left S-module for all $i \in I$ (see also [16]). In the situation of Lemma 4.6, the latter means that S is left artinian, and hence so are all $\mathrm{End}_R C_i$.

We are now in a position to prove the main result of this section.

THEOREM 4.7. *Assume that* R *and* S *are right artinian hereditary rings and that* T_R *is a tilting module with* $S = \mathrm{End}_R T$. *The following statements are equivalent*:

(1) *For every indecomposable non-injective direct summand* A *of* T *there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* Mod-R *consisting of finitely generated modules.*

(2) S *is left artinian*, *and for every indecomposable direct summand* A *of* T *the local dual* ${}_RA^+$ *is finitely generated.*

Proof. (1)⇒(2). The ring S being hereditary implies by Lemma 4.5 that the torsion theory $(\mathcal{T},\mathcal{F})$ is splitting. Hence the almost split sequences considered in (1) are almost split sequences in the subcategory $\mathcal{T}$. It is then easy to see that they are mapped by F to almost split sequences in $\mathcal{Y}$, which consist of finitely generated modules by Lemma 4.4, and which are even almost split sequences in Mod-S, because R is hereditary and the torsion theory $(\mathcal{X},\mathcal{Y})$ is therefore splitting (see [12, 1.5]).

We claim that for every indecomposable non-injective projective right S-module P_S there is an almost split sequence $0 \to P \overset{\alpha}{\to} B \to Z \to 0$ in mod-S. In fact, $P = F(A)$ for some indecomposable direct summand of T, and the claim has just been proven above in case A is not injective. So, let us consider the case that A is injective and denote by Q the projective cover of $\mathrm{Soc}\,A$. Assume for a moment that Q, and therefore also $\mathrm{Soc}\,A$, are in $\mathcal{T}$. For the embedding $i : \mathrm{Soc}\,A \to A$ in $\mathcal{T}$ we then have $0 \neq F(i) : F(\mathrm{Soc}\,A)_S \to F(A) = P_S$ in $\mathcal{Y}$, and the composition f of $F(i)$ with an injective envelope $P_S \overset{a}{\to} I_S$ is a monomorphism. Moreover, we know by the choice of P that a is not a split monomorphism, hence $G(a) : GF(A) \to G(I)$ cannot be injective and $G(f)$ must be zero. For the canonical epimorphism $\nu : I \to I/\tau_{\mathcal{X}}(I)$ we then obtain $G(\nu f) = 0$, and since $I/\tau_{\mathcal{X}}(I) \in \mathcal{Y}$, we deduce $\nu f = 0$ and $\operatorname{Im} f \subseteq \tau_{\mathcal{X}}(I)$. Thus we have an exact sequence $0 \to F(\operatorname{Soc} A) \xrightarrow{f} \tau_{\mathcal{X}}(I) \to K \to 0$ where $\tau_{\mathcal{X}}(I)$ and K are in $\mathcal{X}$, which yields an exact sequence $0 \to G'(\tau_{\mathcal{X}}(I)) \to G'(K) \to GF(\operatorname{Soc} A) \to 0$. But this implies the existence of a non-zero map $Q \to G'(K)$ where $Q \in \mathcal{T}$ and $G'(K) \in \mathcal{F}$, a contradiction. So, we conclude that Q does not lie in $\mathcal{T}$. Since $(\mathcal{T}, \mathcal{F})$ is splitting, we then have $Q \in \mathcal{F}$, and $F'(Q) = F'(Q/\tau_{\mathcal{T}}(Q))$ is an indecomposable non-projective module from $\mathcal{X}$, which is finitely generated by 4.4, and has an almost split sequence $0 \to A' \to B \to F'(Q) \to 0$ in Mod-S by 3.1. From the Connecting Lemma 4.3 it now follows that $A' \cong F(A) = P$, and the claim is proven.

As a consequence, we deduce that S is two-sided artinian. Indeed, if $B_S = B' \oplus Y$ where B'_S is projective and Y_S has no non-zero projective direct summands, then S being hereditary implies that any map $h : P \to P'$ with P'_S projective which factors through $\alpha : P \to B$ actually factors through $\alpha' : P \xrightarrow{\alpha} B \xrightarrow{\mathrm{pr}} B'$. So, $\operatorname{add} \mathcal{P}_0(S)$ is a category with left almost split morphisms, and we deduce from Lemma 4.6 that all indecomposable projective right S-modules are endofinite. In particular, S is then also left artinian.

In other words, recalling Proposition 3.3, we have shown that all indecomposable injective left S-modules are finitely generated. But by Lemma 4.1 this means that all modules of the form ${}_S T \otimes_R A^+$ for some $A \in \operatorname{ind} T$ are finitely generated. Now, these are modules in the torsion class induced by the tilting module ${}_S T$ over the left artinian ring S with $\operatorname{End}{}_S T \cong R$ (see [12, 1.1]), and so we know by Lemma 4.4 that the last statement is equivalent to the second condition in (2).

(2)⇒(1). As we have just observed, condition (2) means that for every indecomposable non-injective projective right S-module P_S there is an almost split sequence $0 \to P \to Y \to Z \to 0$ in Mod-S consisting of finitely generated modules (see 3.3 and the subsequent remark). Now, if $A \in \operatorname{ind} T$ is not injective, then neither is $F(A)$, because otherwise A would be Ext-injective in $\mathcal{T}$, hence injective by our observation after Lemma 4.2. So, there is an almost split sequence starting at $F(A)$, which lies in $\mathcal{Y}$ by the Connecting Lemma 4.3, and using similar arguments as above, we see that it is mapped by G to the required almost split sequence. ■

5. Some cotilting theory. We call a left module ${}_R Q$ over an arbitrary ring R a *cotilting module* if it is a module of injective dimension at most one such that $\operatorname{Ext}^1_R(Q, Q) = 0$, and there is no non-zero module ${}_R M$ satisfying $\operatorname{Hom}_R(M, Q) = \operatorname{Ext}^1_R(M, Q) = 0$.

Observe that our definition is weaker than the one introduced by Colpi, D'Este and Tonolo in [15]. In particular, we do not know whether every module cogenerated by Q is also copresented by Q. We are going to see, however, that under some additional assumptions, every finitely generated module X which is cogenerated by Q is *finitely copresented* by Q, i.e. there is an exact sequence $0 \to X \to Q^n \to Q^m$ for some $n, m \in \mathbb{N}$. In this respect, we will therefore still have a dual behaviour to the tilting case.

Before starting, we fix some notation. For a left module ${}_RQ$, we denote by ${}^\perp Q$ the kernel of the functor $\mathrm{Ext}^1_R(-, Q)$ and by $\mathrm{Cogen}\, Q$ the category of the modules cogenerated by Q. Observe that for a cotilting module Q, it follows from our definition that ${}^\perp Q$ contains every module finitely cogenerated by Q. Finally, let $\mathrm{Rej}_Q(X) = \bigcap\{\mathrm{Ker}\, f \mid f \in \mathrm{Hom}_R(X, Q)\}$ denote the *reject* of Q in a module X.

PROPOSITION 5.1. *Let ${}_RQ$ be a module and $S = (\mathrm{End}\, {}_RQ)^{\mathrm{op}}$.*

(1) *Assume that $\mathrm{Ext}^1_R(Q, Q) = 0$, and let ${}_RX \in \mathrm{Cogen}\, Q$. The right S-module $\mathrm{Hom}_R(X, Q)_S$ is finitely generated if and only if there is an exact sequence $0 \to X \to Q^n \to L \to 0$ for some $n \in \mathbb{N}$ and some $L \in {}^\perp Q$.*

(2) *Assume that Q is a module of injective dimension at most one such that there is no non-zero module ${}_RM$ satisfying $\mathrm{Hom}_R(M, Q) = \mathrm{Ext}^1_R(M, Q) = 0$. If ${}_RX$ and $X/\mathrm{Rej}_Q(X)$ are contained in ${}^\perp Q$, then $X \in \mathrm{Cogen}\, Q$.*

(3) *Assume that Q is cotilting. If R is left artinian, or if $\mathrm{Hom}_R(X, Q)_S$ is finitely generated for any finitely generated module X, then $\mathrm{Cogen}\, Q \cap R\text{-mod} = {}^\perp Q \cap R\text{-mod}$.*

Proof. (1) The "if" part follows by applying the functor $\mathrm{Hom}_R(-, Q)$ on the given sequence. The "only if" part is shown using arguments from [15, 1.8]. We take maps $f_1, \ldots, f_n$ generating $\mathrm{Hom}_R(X, Q)_S$ and consider the exact sequence $0 \to X \xrightarrow{f} Q^n \to L \to 0$ induced by them. Observe that f is a monomorphism since ${}_RX \in \mathrm{Cogen}\, Q$. Applying the functor $\mathrm{Hom}_R(-, Q)$, we get an exact sequence $0 \to \mathrm{Hom}_R(L, Q) \to \mathrm{Hom}_R(Q^n, Q) \xrightarrow{f^*} \mathrm{Hom}_R(X, Q) \xrightarrow{\delta} \mathrm{Ext}^1_R(L, Q) \to \mathrm{Ext}^1_R(Q^n, Q) = 0$, where f^* is an epimorphism by construction, hence $\mathrm{Ext}^1_R(L, Q) = 0$.

(2) We proceed as in [15, 1.7]. We have to show that $X' = \mathrm{Rej}_Q(X) = 0$. To this end, we apply the functor $\mathrm{Hom}_R(-, Q)$ on the exact sequence $0 \to X' \to X \xrightarrow{b} X/X' \to 0$ to get an exact sequence $0 \to \mathrm{Hom}_R(X/X', Q) \xrightarrow{b^*} \mathrm{Hom}_R(X, Q) \to \mathrm{Hom}_R(X', Q) \xrightarrow{\delta} \mathrm{Ext}^1_R(X/X', Q) = 0$. Now, b^* is an isomorphism by construction, hence $\mathrm{Hom}_R(X', Q) = 0$. Since moreover ${}^\perp Q$ is closed under submodules, also $\mathrm{Ext}^1_R(X', Q) = 0$, thus $X' = 0$ by assumption.

(3) In the first case we will use the fact that every finitely generated module ${}_RX$ is artinian, hence every factor module of X is finitely cogenerated,

and in particular finitely cogenerated by Q whenever it lies in $\mathrm{Cogen}\, Q$ (see [1, 10.10, 10.2]). In the second case we will use (1).

Take now a finitely generated module X. We have seen that in both cases X being in $\mathrm{Cogen}\, Q$ implies that it is finitely cogenerated by Q and therefore is in ${}^{\perp}Q$. Conversely, if $X \in {}^{\perp}Q$, then again by the above considerations, the factor module $X/\mathrm{Rej}_Q(X) \in \mathrm{Cogen}\, Q$ is contained in ${}^{\perp}Q$, thus $X \in \mathrm{Cogen}\, Q$ by (2). ∎

In [4], we call a module ${}_RQ$ *finitely cotilting* if Q is a finitely generated cotilting module such that $\mathrm{Hom}_R(X,Q)_{(\mathrm{End}\, {}_RQ)^{\mathrm{op}}}$ is finitely generated for any finitely generated module X. The following observation is an immediate consequence of Proposition 5.1.

COROLLARY 5.2. *Assume that ${}_RQ$ is finitely cotilting. Then every finitely generated module X which is cogenerated by Q is finitely copresented by Q.*

In Sections 7 and 8, we are going to consider modules which are finitely cotilting as well as tilting. In fact, we have the following relationship.

PROPOSITION 5.3. *Let R be a left artinian hereditary ring. Then every cotilting module ${}_RQ$ such that ${}_RQ_{(\mathrm{End}\, {}_RQ)^{\mathrm{op}}}$ is finitely generated on both sides is a tilting module.*

Proof. We only have to take a left R-module ${}_RM$ satisfying $\mathrm{Hom}_R(Q,M) = \mathrm{Ext}^1_R(Q,M) = 0$ and to verify that $M = 0$. Observe that $R \in {}^{\perp}Q \cap R\text{-mod} = \mathrm{Cogen}\, Q \cap R\text{-mod}$ by 5.1(3), and that $\mathrm{Hom}_R(R,Q)_{(\mathrm{End}\, {}_RQ)^{\mathrm{op}}}$ is finitely generated. So, we obtain as above an exact sequence $0 \to R \to Q^n \to L \to 0$ where $L \in {}^{\perp}Q$ is finitely generated artinian, hence even contained in Q^m, for some $n, m \in \mathbb{N}$. Applying the functor $\mathrm{Hom}_R(-,M)$, we see that $\mathrm{Hom}_R(R,M) \xrightarrow{\delta} \mathrm{Ext}^1_R(L,M)$ is an isomorphism. Now, since R is hereditary, ${}^{\perp}M$ is closed under submodules. Thus $\mathrm{Ext}^1_R(Q^m,M) = 0$ implies $\mathrm{Ext}^1_R(L,M) = 0$, and $M \cong \mathrm{Hom}_R(R,M) = 0$. ∎

6. Preprojective tilting modules. Throughout this section let R be a right artinian hereditary ring. It is well known that the isomorphism classes of the indecomposable projective modules can then be partially ordered by setting $P \le Q$ if $\mathrm{Hom}_R(P,Q) \ne 0$. The following condition allows us to order the modules in $\mathcal{P}_n$ (see [2]). For $n \in \mathbb{N}_0$ we say that R is (right) *$\mathcal{P}_n$-hereditary* if it has the following property: If C is a module in $\mathcal{P}_n$, then every finitely generated indecomposable module X with a non-zero morphism $X \to C$ is in $\mathcal{P}^{n+1}$. In particular, every non-zero morphism $P \to Q$ where P and Q are in $\mathcal{P}_n$ must then be injective. So, the isomorphism classes of $\mathcal{P}_n$ can be partially ordered by setting $P \le Q$ if $\mathrm{Hom}_R(P,Q) \ne 0$. We will denote by $\mathcal{P}_n \max$ (resp. $\mathcal{P}_n \setminus \max$) the subcategory of $\mathcal{P}_n$ consisting of those modules which are maximal (resp. non-maximal) with respect to this order. Further, let us write $\mathcal{P}^n\mathrm{inj}$ for the subcategory of $\mathcal{P}^n$ consisting of the injective objects. Take now a complete irredundant set $P_1, \dots, P_r$ of modules in $\mathcal{P}_{n+1} \cup \mathcal{P}_n\mathrm{max} \cup \mathcal{P}^n\mathrm{inj}$ and put $T_{n+1} = \bigoplus_{i=1}^r P_i$. Finally, let T_0 be a minimal progenerator of Mod-R, and define $\mathcal{P}_{-1} = \emptyset$.

The modules T_n were studied in [3]. More precisely, it was established that R satisfies condition (P) in Theorem 1.1 if and only if for all $n \in \mathbb{N}_0$ the module T_n is tilting and R is $\mathcal{P}_n$-hereditary, and for every indecomposable preprojective non-projective module C there is an almost split sequence $0 \to A \to B \to C \to 0$ in mod-R. With the arguments used there we can now prove the following.

PROPOSITION 6.1. *Let R be a right artinian hereditary ring. Assume that $n \geq 0$ and that for all non-injective modules $A \in \mathcal{P}^n$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* mod-R. *Then*:

(1) *For all $X, Y \in \mathcal{P}^n$ the bimodule* $\mathrm{Hom}_R(X,Y)$ *is finitely generated over the skew-fields* $\mathrm{End}_R X$ *and* $\mathrm{End}_R Y$.

(2) $\mathrm{End}_R T_i$ *is right artinian and hereditary for all $0 \leq i < n$.*

(3) *$\mathcal{P}_i$ is a finite cover for* mod-$R_{\mathcal{P}^i}$, *R is $\mathcal{P}_i$-hereditary, and T_i is a tilting module for all $0 \leq i \leq n$.*

(4) *For all $C \in \bigcup_{i=1}^n \mathcal{P}_i$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* mod-R.

(5) $\mathcal{P}_{i+1} = \{\tau^{-1}X \mid X \in \mathcal{P}_{i-1}\mathrm{max}$ *non-injective, or* $X \in \mathcal{P}_i \setminus \mathrm{max}\}$ *for all $0 \leq i < n$.*

Proof. We only give the inductive step for (1) and (2), the other statements are proven as in [3, 2.1 and 2.9]. Let $n \geq 0$, and assume that the statements (1)–(5) are proven for n. Observe that $\mathrm{add}\,\mathcal{P}^{n+1}$ is then a finite category with left almost split morphisms and right almost split morphisms. Indeed, this follows from the existence of the almost split sequences and the fact that R is $\mathcal{P}_i$-hereditary for all $0 \leq i \leq n$. By Lemma 4.6 the bimodule $\mathrm{Hom}_R(X,Y)$ is then finitely generated over $\mathrm{End}_R X$ and $\mathrm{End}_R Y$ for all $X, Y \in \mathcal{P}_n$. That the endomorphism rings of modules in $\mathcal{P}_n$ are skew-fields follows from the fact that R is $\mathcal{P}_n$-hereditary. So, we have proven (1).

Now we verify (2). Let $S = \mathrm{End}_R T_n$, and let $(\mathcal{T}, \mathcal{F})$ in Mod-R and $(\mathcal{X}, \mathcal{Y})$ in Mod-S be the torsion theories induced by the tilting module T_n. It follows from (1) that $r(X,Y)_{\mathrm{End}_R X}$ is finitely generated for all indecomposable direct summands X and Y of T_n, hence the Jacobson radical $J(S) = r(T_n, T_n)$ is a finitely generated right S-module. Then the semiprimary ring S is even right artinian. In order to prove that S is hereditary, it is enough to show that every indecomposable finitely generated right S-module X_S has projective dimension at most one. Since R is hereditary, we know by [12, 1.5] that the torsion theory $(\mathcal{X}, \mathcal{Y})$ is splitting. If $X \in \mathcal{Y}$, then the claim is shown as in [22, 4.1(6)]. Moreover, we know from Lemma 4.5 that the claim for $X \in \mathcal{X}$ holds true if and only if $\mathrm{Ext}^1_R(W,U) = 0$ for all $U \in \mathcal{T} \cap \text{mod-}R$ and $W \in \mathcal{F} \cap \text{mod-}R$. Now, if W is not projective, then we know by 3.1 and 3.6 that this condition is satisfied whenever $\mathrm{Hom}_R(U, (\mathrm{Tr}\, W)^+) = 0$. So, it remains to prove that the latter follows from our induction assumption. Indeed, since $\mathcal{P}_n$ is a cover for mod-$R_{\mathcal{P}^n}$, we have mod-$R_{\mathcal{P}^n} \cup \mathcal{P}_{n-1}\mathrm{max} \subseteq \mathcal{T}$, and every finitely generated module $W \in \mathcal{F}$ has to lie in $\mathcal{P}^{n-1} \cup \mathcal{P}_{n-1} \setminus \mathrm{max}$. From condition (5) we then deduce $(\mathrm{Tr}\, W)^+ \in \mathcal{P}^{n-1}$, and using the fact that $U \in \mathcal{T}$ is generated by T_n and that R is $\mathcal{P}_i$-hereditary for all $i < n-1$, we conclude $\mathrm{Hom}_R(U, (\mathrm{Tr}\, W)^+) = 0$. ∎

COROLLARY 6.2. *Let the assumptions be as in Proposition* 6.1 *with* $n \geq 1$. *Then R is left artinian, all $X \in \mathcal{P}^n$ are endofinite with ${}_RX^+$ finitely generated and $X^{++} \cong X$, and the almost split sequences considered in* 6.1 *are even almost split in* Mod-R.

P r o o f. It follows from (1) in 6.1 that all modules in $\mathcal{P}^n$ are endofinite. In particular, this holds true for all indecomposable projectives, and we deduce that R is left artinian. Also, since all modules in $\mathcal{P}^n$ are pure-injective, the almost split sequences considered are even almost split in Mod-R by Proposition 3.4. Moreover, every module $X \in \mathcal{P}^n$ is a direct summand of some T_i, $0 \leq i < n$, and we know by (2) and (3) in 6.1 that Theorem 4.7 applies to the T_i. So, ${}_RX^+$ is finitely generated. Finally, X being pure-injective with ${}_RX^+$ finitely presented implies $X^{++} \cong X$ by [32, Lemma 5]. ∎

COROLLARY 6.3. *Let the assumptions be as in Proposition* 6.1. *Then for all $X \in \mathcal{P}^n$ and all $Y \in$* mod-R, $\mathrm{Hom}_R(X,Y)$ *is a right* $\mathrm{End}_R\, X$*-module of finite length.*

P r o o f. The module X is an indecomposable direct summand of one of the tilting modules T_i, $0 \leq i < n$. Now, $S = \mathrm{End}_R\, T_i$ is right artinian by 6.1, and we know from 4.4 that the functor $F = \mathrm{Hom}_R(T_i, -) : \text{Mod-}R \to \text{Mod-}S$ carries finitely generated modules to finitely generated modules. Hence $F(Y)_S$ is a module of finite length. Let $e \in S$ be the idempotent associated with the direct summand X of T_i. Then $F(Y)e\,{}_{eSe}$, and thus also $\mathrm{Hom}_R(X,Y)_{\mathrm{End}_R X}$, has finite length as well. ∎

7. Preinjective cotilting modules. We now deal with the dual situation of preinjective left modules as considered in condition (I) of Theorem 1.1. Throughout this section, we assume R to be a left artinian ring such that all indecomposable injective left R-modules are finitely generated. We are going to collect some results which we will need in the sequel. Most of them are dual versions of results from [3], and therefore we will often omit the proofs.

LEMMA 7.1 (cp. [3, 1.2]). *Let* $N \in R$-mod *and* $X \in R$-Mod. *Then* $X \in \operatorname{Cogen} N$ *provided there is* $n \in \mathbb{N}$ *such that* $\mathcal{I}^n$ *is finite and every morphism* $h : X \to I$ *with* $I \in \mathcal{I}^n \setminus \operatorname{ind} N$ *factors through a morphism* $\beta : B \to I$ *which is not a split epimorphism and where* B *is a finite direct sum of modules in* $\mathcal{I}^n \cup \operatorname{ind} N$.

Proof. With arguments dual to those used in [3, 1.2] we show that for every morphism $h : X \to I$ where $I \in \mathcal{I}^n \setminus \operatorname{ind} N$ there is a morphism $g : Y \to I$ such that $Y \in \operatorname{add} N$ and h factors through f. In particular, any embedding $h : X \to \prod_{k\in K} I_k$ with injective modules I_k, $k \in K$, can be factored through a map $\prod_{k\in K} g_k : \prod_{k\in K} Y_k \to \prod_{k\in K} I_k$ where $Y_k \in \operatorname{add} N$. Since there are $m_k \in \mathbb{N}$ with a split monomorphism $\prod_{k\in K} Y_k \to \prod_{k\in K} N^{m_k}$, we obtain a monomorphism $X \to \prod_{k\in K} N^{m_k}$, which proves $X \in \operatorname{Cogen} N$. ■

We immediately see that (I) is a sufficient condition for R-mod having a preinjective partition.

THEOREM 7.2 (cp. [3, 1.3]). *Let* R *be a left artinian ring such that all indecomposable injective left* R*-modules are finitely generated. Further*, *let* $n \in \mathbb{N}$, *and assume that for each non-projective module* ${}_RC \in \mathcal{I}^n$ *there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* R-mod. *Then* $\mathcal{I}_i$ *is a finite cocover for* $R\text{-mod}_{\mathcal{I}^i}$ *for all* $0 \le i \le n$.

Moreover, we obtain a criterion for endofiniteness of the preinjective left R-modules.

PROPOSITION 7.3. *Let the assumptions be as in Theorem* 7.2, *and assume further that* R *is hereditary. Then all* $C \in \mathcal{I}^n$ *are endofinite.*

Proof. Let ${}_RQ$ be a minimal injective cogenerator of R-Mod and let $R' = (\operatorname{End}{}_RQ)^{\mathrm{op}}$. By assumption, Q induces a Morita duality $D := \operatorname{Hom}_R(-,Q) : R\text{-mod} \to \text{mod-}R'$, and R' is a right artinian hereditary ring. Observe that for all $i \in \mathbb{N}_0$, an indecomposable left R-module ${}_RC$ lies in $\mathcal{I}_i(R)$ if and only if $D(C)_{R'}$ lies in $\mathcal{P}_i(R')$. So, R' satisfies the assumptions of Proposition 6.1, and we know by Corollary 6.3 that $\operatorname{Hom}_{R'}(X,Y)$ is a right $\operatorname{End}_{R'} X$-module of finite length for all $X \in \mathcal{P}^n(R')$ and for all $Y \in \text{mod-}R'$. In particular, $\operatorname{Hom}_{R'}(D(C),Q)_{\operatorname{End}_{R'} D(C)}$ has finite length for all ${}_RC \in \mathcal{I}^n(R)$, and since D induces isomorphisms $\operatorname{Hom}_R(R,C) \cong \operatorname{Hom}_{R'}(D(C),Q)$ and $(\operatorname{End}{}_RC)^{\mathrm{op}} \cong \operatorname{End}_{R'} D(C)$, we conclude that also $C_{(\operatorname{End}{}_RC)^{\mathrm{op}}} \cong \operatorname{Hom}_R(R,C)_{(\operatorname{End}{}_RC)^{\mathrm{op}}}$ has finite length. ■

For $n \in \mathbb{N}_0$ we will say that R is (left) $\mathcal{I}_n$*-hereditary* if it has the following property: If ${}_RA$ is a module in $\mathcal{I}_n$, then every finitely generated indecomposable module ${}_RX$ with a non-zero morphism $A \to X$ is in $\mathcal{I}^{n+1}$. Again, this implies that every non-zero morphism $P \to Q$ where P and Q are in $\mathcal{I}_n$ must be surjective, and so the isomorphism classes of $\mathcal{I}_n$ can be partially ordered by setting $P \geq Q$ if $\mathrm{Hom}_R(P,Q) \neq 0$. We will denote by $\mathcal{I}_n\mathrm{max}$, resp. $\mathcal{I}_n \setminus \mathrm{max}$, the subcategory of $\mathcal{I}_n$ consisting of those modules which are maximal, resp. non-maximal, with respect to this order. Observe that using the same arguments as in [2, 4.3] we obtain the following criterion for maximality.

LEMMA 7.4. *Let R be an artinian ring and $n \in \mathbb{N}_0$.*

(1) *Assume that R is (right) $\mathcal{P}_i$-hereditary for all $0 \leq i \leq n$. Then a module C_R in $\mathcal{P}_n$ is maximal in $\mathcal{P}_n$ with respect to "$\leq$" if and only if there is no irreducible morphism $C \to D$ in* mod-R *where $D \in \mathcal{P}_n$.*

(2) *Assume that R is (left) $\mathcal{I}_i$-hereditary for all $0 \leq i \leq n$. Then a module ${}_RA$ in $\mathcal{I}_n$ is maximal in $\mathcal{I}_n$ with respect to "$\leq$" if and only if there is no irreducible morphism $B \to A$ in R-*mod *where $B \in \mathcal{I}_n$.*

We include a well known description of the maximal injectives and the maximal projectives.

LEMMA 7.5. *Let R be an artinian hereditary ring.*

(1) *An injective module ${}_RA$ is maximal injective if and only if its socle is projective.*

(2) *A projective module C_R is maximal projective if and only if its top is injective.*

(3) *A module ${}_RA$ is maximal injective if and only if A^+_R is maximal projective.*

Proof. (1) Consider an injective module ${}_RA$ and a projective cover P of $\mathrm{Soc}\,A$. If P is not simple, then we can find a simple module S contained in $\mathrm{Rad}\,P$, and thus a non-zero map $f : P \to I$ where I is an injective envelope of S. Since $S \not\subseteq \mathrm{Ker}\,f$, we see that $\mathrm{Ker}\,f$ is properly contained in $\mathrm{Rad}\,P$, and so we have a proper epimorphism $P/\mathrm{Ker}\,f \to \mathrm{Soc}\,A$ which can be extended to a non-zero non-isomorphism $I \to A$, showing that A is not maximal. The converse implication is left to the reader.

(2) is proven by dual arguments.

(3) We know by Lemma 3.5 that ${}_RA$ is injective if and only if A^+_R is projective, and that the socle of A is projective if and only if the top of A^+_R is injective. ■

A construction dual to the one considered in Section 6 will now provide us with cotilting preinjective left modules. We denote the subcategory of $\mathcal{I}^n$ consisting of the projective objects by $\mathcal{I}^n\mathrm{proj}$, take a complete irredundant set $I_1, \ldots, I_r$ of modules in $\mathcal{I}_{n+1} \cup \mathcal{I}_n\mathrm{max} \cup \mathcal{I}^n\mathrm{proj}$ and put $Q_{n+1} = \bigoplus_{i=1}^r I_i$. Let us apply the results of Section 5.

PROPOSITION 7.6 (cp. [3, 2.2]). *Let $n \in \mathbb{N}_0$ such that $\mathcal{I}_i$ is a finite cocover for R-*$\mathrm{mod}_{\mathcal{I}^i}$*, R is $\mathcal{I}_i$-hereditary and Q_i is a finitely cotilting mod-*

ule for all $0 \le i \le n$. *Then for each* $A \in R\text{-ind} \setminus \mathcal{I}^{n+1}$ *there is a non-split monomorphism* $f : A \to I$ *such that* $I \in \operatorname{add} \mathcal{I}_n$ *and all morphisms* $h : A \to X$ *where* $X \in \mathcal{I}^{n+1}$ *factor through* f.

Proof. We know by 5.2 that all finitely generated modules which are cogenerated by some Q_i are even finitely copresented by Q_i. In particular, since $\mathcal{I}_n$ is a cocover for $R\text{-mod}_{\mathcal{I}^n}$, the module A is finitely copresented by Q_n, and we have a non-split exact sequence $0 \to A \xrightarrow{f} I \xrightarrow{p} L \to 0$ where $I \in \operatorname{add} Q_n$ and $L \in \operatorname{Cogen} Q_n \cap R\text{-mod}$. Further, $\mathcal{I}_i$ being a cocover for $R\text{-mod}_{\mathcal{I}^i}$ and Q_i being finitely cotilting imply by 5.1(3) that $\operatorname{Cogen} Q_n \cap R\text{-mod} \subseteq \operatorname{Cogen} Q_i \cap R\text{-mod} = {}^{\perp}Q_i \cap R\text{-mod}$ for all $0 \le i \le n$. So, the functor $\operatorname{Hom}_R(-, X)$ is exact on our sequence whenever $X \in \mathcal{I}_i \subseteq \operatorname{ind} Q_i$ for some $0 \le i \le n$, which means that f has the stated factorization property. As in [3, 2.2], we can now show that if we choose the above sequence with I of minimal length, then $I \in \operatorname{add} \mathcal{I}_n$. ■

PROPOSITION 7.7. *Let the assumptions be as in Proposition* 7.6. *Then*:

(1) (cp. [3, 2.3]) *For each* $A \in \mathcal{I}_{n+1}$ *there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* R-mod *where* $C \in \mathcal{I}^n \cup \mathcal{I}_n \setminus \max$.

(2) (cp. [3, 2.4]) $\operatorname{Ext}^1_R(Q_{n+1}, Q_{n+1}) = 0$.

(3) (cp. [3, 2.5]) *Suppose that* $\mathcal{I}_{n+1}$ *is a finite cocover for* $R\text{-mod}_{\mathcal{I}^{n+1}}$ *and* Q_{n+1} *is a finitely cotilting module. Then* R *is* $\mathcal{I}_{n+1}$*-hereditary.*

(4) (cp. [3, 2.7]) *Suppose that* $\mathcal{I}_{n+1}$ *is a finite cocover for* $R\text{-mod}_{\mathcal{I}^{n+1}}$. *Then for each* $C \in \mathcal{I}^{n+2} \setminus \operatorname{ind} Q_{n+1}$ *there is a non-split exact sequence* $0 \to A \to B \to C \to 0$ *in* R-mod *such that* $\operatorname{Ext}^1_R(X, A) = 0$ *for all* $X \in {}^{\perp}Q_{n+1}$.

Proof. All proofs are straightforward dualizations of the corresponding proofs in [3]. The only point we have to take care of is in the proof of (4). As in [3, 2.7], we proceed by induction on n, and in the induction step we distinguish two cases. Let us look at the first case, namely when C lies in $\mathcal{I}^{n-1} \cup \mathcal{I}_{n-1} \setminus \max$. Since R is assumed to be $\mathcal{I}_i$-hereditary for all $0 \le i < n$, we have $\operatorname{Hom}_R(C, Q_n) = 0$. By the definition of a (finitely) cotilting module, there is no non-zero module ${}_RM$ satisfying $\operatorname{Hom}_R(M, Q_n) = \operatorname{Ext}^1_R(M, Q_n) = 0$. Hence we have $\operatorname{Ext}^1_R(C, Q_n) \ne 0$, and there is a non-split exact sequence $0 \to A \to B \to C \to 0$ where $A \in \operatorname{ind} Q_n$. Now we can conclude the proof as in [3, 2.7]. ■

COROLLARY 7.8. *Let* $n \in \mathbb{N}_0$ *such that* $\mathcal{I}_i$ *is a finite cocover for* $R\text{-mod}_{\mathcal{I}^i}$, R *is* $\mathcal{I}_i$*-hereditary and* Q_i *is a finitely cotilting module for all* $0 \le i \le n$. *Suppose further that* $\mathcal{I}_{n+1}$ *is a finite cocover for* $R\text{-mod}_{\mathcal{I}^{n+1}}$. *Then* Q_{n+1} *is a cotilting module.*

Proof. Since R is hereditary and $\operatorname{Ext}^1_R(Q_{n+1}, Q_{n+1}) = 0$ by 7.7(2), it only remains to verify that there is no non-zero module ${}_RM$ satisfying

$\mathrm{Hom}_R(M, Q_{n+1}) = \mathrm{Ext}^1_R(M, Q_{n+1}) = 0$. To this end, it suffices to show that ${}^{\perp}Q_{n+1} \subseteq \mathrm{Cogen}\, Q_{n+1}$. This is deduced from 7.7(4) and Lemma 7.1 as in [3, p. 10]. ■

8. The main results. We now apply the results of the previous sections in order to pass information from the preprojective right modules to the preinjective left modules. Let Q_n, $n \in \mathbb{N}$, be as in Section 7, and put $\mathcal{I}_{-1} = \emptyset$.

PROPOSITION 8.1. *Let R be a right artinian hereditary ring. Assume that $n \geq 0$ and that for all non-injective modules $A \in \mathcal{P}^{n+2}$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* mod-R. *Then*:

(1) *R is left artinian, and any minimal injective cogenerator ${}_RQ_0$ of* R-Mod *is finitely generated.*

(2) *For all non-projective modules ${}_RC \in \mathcal{I}^n$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* R-Mod *consisting of finitely generated modules.*

(3) *$\mathcal{I}_i$ is a finite cocover for* R-$\mathrm{mod}_{\mathcal{I}^i}$, *$R$ is $\mathcal{I}_i$-hereditary, and Q_i is an endofinite cotilting module for all $0 \leq i \leq n$.*

(4) *For all $A \in \bigcup_{i=1}^n \mathcal{I}_i$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* R-mod.

(5) *$\mathcal{I}_{i+1} = \{\tau X \mid X \in \mathcal{I}_{i-1}\max$ non-projective, or $X \in \mathcal{I}_i \setminus \max\}$ for all $0 \leq i < n$.*

(6) *For all finitely generated indecomposable left R-modules ${}_RA$ and for all $0 \leq i \leq n$, the module A lies in $\mathcal{I}_i$ (and is maximal) if and only if A^+_R lies in $\mathcal{P}_i$ (and is maximal). Moreover, A is in $\mathcal{I}_{n+1}$ if A^+_R is in $\mathcal{P}_{n+1}$.*

Proof. For (1), note that R is left artinian by Corollary 6.2. Moreover, all indecomposable injective left R-modules are finitely generated by Proposition 3.3.

We will now prove (2)–(6) by induction on n. To this end, we will employ Theorems 3.1 and 3.2. In particular, we will repeatedly use 3.1 to obtain almost split sequences $0 \to C^+ \to E \to \mathrm{Tr}\, C \to 0$ in Mod-R for some indecomposable non-projective modules ${}_RC$.

Let us start with $n = 0$. Then $\mathcal{I}^n = \emptyset$, and the statements in (3) are true by assumption on R. It remains to prove (6). The first statement is shown in Lemmata 3.5 and 7.5. Assume that A^+_R is in $\mathcal{P}_1$ and set $C = \mathrm{Tr}\, A^+$. We know by Proposition 6.1 and Corollary 6.2 that there is an almost split sequence $0 \to C^+ \to E \to A^+ \to 0$ in Mod-R where $C^+ \in \mathcal{P}_0 \setminus \max$, hence $C \in \mathcal{I}_0 \setminus \max$. Further, we have an almost split sequence $0 \to A \to B \to C \to 0$ in R-mod by Theorem 3.2. Since B must have an injective direct summand by 7.4, we conclude that $A \in \mathcal{I}_1$.

Let now $n \geq 0$ such that for all non-injective modules $A \in \mathcal{P}^{n+3}$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in mod-R. Then statements

(2)–(6) hold for n by the induction assumption and are to be verified for $n+1$. We need the following information from 6.1 and 6.2.

(i) For all $0 \le i \le n+3$, the ring R is $\mathcal{P}_i$-hereditary, and even more, if $C \in \mathcal{P}_i$, then every indecomposable module $X \in \text{Mod-}R$ with a non-zero morphism $X \to C$ lies in $\mathcal{P}^{i+1}$. Indeed, the latter observation is proved by the same arguments as in [2, 4.1] using the fact that the almost split sequences ending at the modules from $\mathcal{P}_i$ are even almost split in Mod-R.

(ii) If $0 \to A \to B \to C \to 0$ is an almost split sequence in mod-R, and $0 \le i \le n+2$, then $A \in \mathcal{P}_{i-1}\max \cup \mathcal{P}_i \setminus \max$ iff $C \in \mathcal{P}_{i+1}$. In particular, $C \in \mathcal{P}^{n+3}$ if $A \in \mathcal{P}^{n+1}$.

(iii) All $X \in \mathcal{P}^{n+3}$ are endofinite with ${}_RX^+$ finitely generated and $X^{++} \cong X$.

For statement (2), we only have to consider the case $C \in \mathcal{I}_n$, which means $C^+ \in \mathcal{P}_n$ with an almost split sequence $0 \to C^+ \to E \to \operatorname{Tr} C \to 0$ in Mod-R. But then we know by (ii) that $\operatorname{Tr} C \in \mathcal{P}^{n+3}$, hence $(\operatorname{Tr} C)^+$ is finitely generated by (iii), and our claim follows from Auslander's Theorem 3.1.

Next, we know from 7.2 that $\mathcal{I}_{n+1}$ is a finite cocover for $R\text{-mod}_{\mathcal{I}^{n+1}}$. Since the Q_i, $0 \le i \le n$, are all endofinite, hence finitely cotilting, we deduce by 7.8 that Q_{n+1} is a cotilting module. Moreover, we infer from 7.7(1) that for every $A \in \mathcal{I}_{n+1}$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in R-mod where $C \in \mathcal{I}^n \cup \mathcal{I}_n \setminus \max$, which yields statement (4) and the inclusion "$\subseteq$" in (5). We can also show that $A^+ \in \mathcal{P}_{n+1}$. In fact, we deduce from (6) that $C^+ \in \mathcal{P}_{n-1}\max \cup \mathcal{P}_n \setminus \max$. Considering the almost split sequence $0 \to C^+ \to E \to \operatorname{Tr} C \to 0$ in Mod-R, and applying (ii), we then obtain $\operatorname{Tr} C \in \mathcal{P}_{n+1}$. So, we know by (iii) that $(\operatorname{Tr} C)^+$ is finitely generated, hence isomorphic to A, and $\operatorname{Tr} C \cong (\operatorname{Tr} C)^{++} \cong A^+$, which proves the claim.

In particular, it follows from (iii) that the local dual of any module $A \in \mathcal{I}_{n+1}$ is endofinite, hence A itself is also endofinite by Lemma 3.5. So, we conclude that Q_{n+1} is endofinite since each of its indecomposable direct summands is. Then R is $\mathcal{I}_{n+1}$-hereditary by 7.7(3), and so all statements in (3) are proven.

Further, since all Q_i are finitely generated over R and over their endomorphism rings, we know by 5.3 that they are also tilting modules. Hence the numbers of isomorphism classes in $\operatorname{ind} Q_{n+1}$ and in $\operatorname{ind} Q_n$ coincide, and a counting argument as in the proof of [3, 2.1] shows that the number of isomorphism classes in $\mathcal{I}_{n+1}$ equals the number of isomorphism classes of non-projective modules in $\mathcal{I}_{n-1}\max \cup \mathcal{I}_n \setminus \max$. This completes the proof of (5).

Finally, we prove (6). By the induction assumption and our above considerations we know that ${}_RA$ lies in $\mathcal{I}_{n+1}$ if and only if A^+_R lies in $\mathcal{P}_{n+1}$. Assume now that $A \in \mathcal{I}_{n+1} \setminus \max$. Then there is an irreducible morphism $Y \to A$

in R-mod for some $Y \in \mathcal{I}_{n+1}$ by 7.4, and A is not projective. Consider the almost split sequence $0 \to A^+ \to E \to \operatorname{Tr} A \to 0$ in Mod-R consisting of finitely generated modules. We claim that $Y^+ \in \operatorname{ind} E$. Indeed, we have an almost split sequence $0 \to Y \to B \to C \to 0$ in R-mod where $C \cong \operatorname{Tr} Y^+$ by 3.2. Hence there is an irreducible morphism $A \to \operatorname{Tr} Y^+$ in R-mod, and by properties of Tr, also an irreducible morphism $Y^+ \to \operatorname{Tr} A$ in mod-R, which proves the claim. This yields an irreducible morphism $A^+ \to Y^+$ where $A^+, Y^+ \in \mathcal{P}_{n+1}$, and we conclude that $A^+ \in \mathcal{P}_{n+1} \setminus \max$. The other implication is shown similarly using the fact that preprojectives in $\mathcal{P}^{n+3}$ are reflexive with respect to local duality by (iii).

It remains to prove that $A \in \mathcal{I}_{n+2}$ whenever $A^+ \in \mathcal{P}_{n+2}$. Consider $C = \operatorname{Tr} A^+$ and the almost split sequence $0 \to C^+ \to B \to A^+ \to 0$ in Mod-R. Then C^+ is finitely generated by 6.1 and 6.2, and by (ii) we have $C^+ \in \mathcal{P}_n\max \cup \mathcal{P}_{n+1} \setminus \max$, which implies $C \in \mathcal{I}_n\max \cup \mathcal{I}_{n+1} \setminus \max$. Moreover, by 3.2 we have an almost split sequence $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ in R-mod. We then know by (5) that $A \notin \mathcal{I}^{n+2}$, and so we only have to show that all monomorphisms $f : A \to Y \in R\text{-mod}_{\mathcal{I}^{n+2}}$ split. Suppose that such an f does not split. Put $B = B' \oplus I$ where $B' \in R\text{-mod}_{\mathcal{I}^{n+2}}$ and all indecomposable direct summands of I are in $\mathcal{I}^{n+2}$. Then f factors through a and even through $\operatorname{pr}_{B'} a$, since R is $\mathcal{I}_i$-hereditary for all $i < n+2$ and therefore $\operatorname{Hom}_R(I, Y) = 0$. In particular, $B' \neq 0$ and $\operatorname{pr}_{B'} a$ is a monomorphism.

Consider first the case $C \in \mathcal{I}_{n+1} \setminus \max$. By Lemma 7.4 the module I then has a direct summand $X \in \mathcal{I}_{n+1}$. But we have seen above that $\operatorname{pr}_{B'} a$, and therefore also $b|_I$, are monomorphisms. Hence $b|_X : X \to C$ is a non-split monomorphism with $X, C \in \mathcal{I}_{n+1}$, a contradiction.

Now we pass to the case $C \in \mathcal{I}_n\max$. Take a module $X \in \operatorname{ind} B$. We have $\operatorname{Hom}_R(A, X) \neq 0$, hence $\operatorname{Hom}_R(X^+, A^+) \neq 0$ by 3.5, and from (i) we conclude that $X^+ \in \mathcal{P}^{n+3}$. Observe further that X cannot be injective, because even in case $n = 0$ we have $C \in \mathcal{I}_0\max$. So, there is an almost split sequence $0 \to X \to E \to \operatorname{Tr} X^+ \to 0$ in R-mod by Theorem 3.2, and we have $C \in \operatorname{ind} E$. Since R is $\mathcal{I}_n$-hereditary, we obtain $\operatorname{Tr} X^+ \in \mathcal{I}^n \cup \mathcal{I}_n \setminus \max$, thus $X \in \mathcal{I}^{n+2}$ by (5). But this contradicts $B' \neq 0$.

So, we conclude that all monomorphisms $f : A \to Y \in R\text{-mod}_{\mathcal{I}^{n+2}}$ split, and $A \in \mathcal{I}_{n+2}$. ■

Proposition 8.2. *Let R be a ring satisfying condition* (I). *Then*:

(i) *For all $A \in \mathcal{I}_n$, $n \geq 1$, there is an almost split sequence $0 \to A \to B \to C \to 0$ in R-mod.*

(ii) *$\mathcal{I}_n$ is a finite cocover for $R\text{-mod}_{\mathcal{I}^n}$, R is $\mathcal{I}_n$-hereditary, and Q_n is an endofinite cotilting module for all $n \in \mathbb{N}_0$.*

(iii) $\mathcal{I}_{n+1} = \{\tau X \mid X \in \mathcal{I}_{n-1}\max$ *non-projective*, *or* $X \in \mathcal{I}_n \setminus \max\}$ *for all $n \in \mathbb{N}_0$.*

In particular, the almost split sequences considered are even almost split in R-Mod, *and* ${}_R(\operatorname{Tr} C)^+$ *is finitely generated for every indecomposable preinjective non-projective left* R*-module* ${}_RC$.

Proof. The proof of (i)–(iii) is by induction on n. We deduce from 7.3 that the Q_i are endofinite, and show the other statements as in 8.1.

Now it is easy to check that the almost split sequences considered consist of preinjective modules, which are endofinite by 7.3, hence pure-injective. We then conclude from 3.4 that the almost split sequences are even almost split in R-Mod. The last statement then follows from 3.1. ∎

We can finally complete the proof of our theorem.

Proof of Theorem 1.1. That (P) implies (I) follows immediately from Proposition 8.1. We now assume (I) and show that (P) is satisfied by induction on n. For $n = 0$ the claim follows from 3.3. Let $n = 1$, and consider a non-injective module $A \in \mathcal{P}_1$. By 3.1 there is an almost split sequence $0 \to (\operatorname{Tr} A)^+ \to E \to A \to 0$ in Mod-R, and by 6.1 and 6.2 we have $(\operatorname{Tr} A)^+ \in \mathcal{P}_0 \setminus \max$, hence $(\operatorname{Tr} A) \in \mathcal{I}_0$ by 3.5. Thus $A^+ \cong (\operatorname{Tr}(\operatorname{Tr} A))^+$ is finitely generated by 8.2, and we obtain the desired almost split sequence from 3.2. Let now $n \geq 0$, and assume that for all non-injective modules $A \in \mathcal{P}^{n+2}$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in mod-R. We consider a non-injective module $A \in \mathcal{P}_{n+2}$. By 3.1 there is an almost split sequence $0 \to (\operatorname{Tr} A)^+ \to E \to A \to 0$ in Mod-R, and we know from 6.1 and 6.2 that $(\operatorname{Tr} A)^+ \in \mathcal{P}_n\max \cup \mathcal{P}_{n+1} \setminus \max$. Then Proposition 8.1(6) yields $\operatorname{Tr} A \in \mathcal{I}_n\max \cup \mathcal{I}_{n+1}$, and we conclude the proof by the same arguments as above. ∎

Proof of Corollary 1.2. We immediately infer from 8.1 and 6.2 that the local duality ${}_RA \mapsto A_R^+$ gives a bijection between the sets of isomorphism classes in $\mathcal{I}_n$ and in $\mathcal{P}_n$ for all n, thus also between the sets ${}_R\mathcal{I}$ of the isomorphism classes of all indecomposable preinjective left R-modules and $\mathcal{P}_R$ of the isomorphism classes of all indecomposable preprojective right R-modules. ∎

It is well known that the preprojective modules over a hereditary artin algebra Λ form a preprojective component of the Auslander–Reiten quiver of Λ [22]. Indeed, we can interpret condition (P) of Theorem 1.1 in terms of the existence of such a component. A subcategory $\mathcal{C}$ of ind-R over a right artinian ring R is called a *preprojective component in* mod-R if it satisfies the following conditions:

(a) For any $X \in \mathcal{C}$ there are a left almost split morphism $X \to Z$ and a right almost split morphism $Y \to X$ in mod-R.

(b) If $X \to Y$ is an irreducible map in mod-R with one of the modules lying in $\mathcal{C}$, then both modules are in $\mathcal{C}$.

(c) The Auslander–Reiten quiver of $\mathcal{C}$ is connected and has no oriented cycles.

(d) For every $Z \in \mathcal{C}$ there is $m \geq 0$ such that $\tau^m Z \in \mathcal{P}_0$.

If in addition the almost split sequences arising from condition (a) are even almost split in Mod-R, then $\mathcal{C}$ is called a *preprojective component in* Mod-R. *Preinjective components in* R-mod and R-Mod are defined dually.

PROPOSITION 8.3. *Let R be a right artinian hereditary indecomposable ring. Then the following statements are equivalent*:

(P) *For every indecomposable preprojective non-injective right R-module A_R there is an almost split sequence $0 \to A \to B \to C \to 0$ in* mod-R.

(PC) *There is a preprojective component in* mod-R *containing all indecomposable projective modules.*

Proof. (P)⇒(PC). Choose the category of all indecomposable preprojective modules and use [3, 2.1 and 2.9].

(PC)⇒(P). Condition (P) follows from (a), because all indecomposable preprojective modules are contained in $\mathcal{C}$ by induction. In fact, if $\mathcal{P}^n \subseteq \mathcal{C}$, then we know from (a) and the proof of [3, 1.3] that all modules in $\mathcal{P}_n$ are irreducible successors of some module in $\mathcal{P}^n$, hence lie in $\mathcal{C}$ by (b). Note that by [3, 2.1 and 2.9] and (c) we even know that $\mathcal{C}$ consists of all indecomposable preprojective modules. ∎

Applying Proposition 8.2 and using dual arguments we can prove the following.

PROPOSITION 8.4. *Let R be a left artinian hereditary indecomposable ring*, *and assume that all indecomposable injective left R-modules are finitely generated. Then the following statements are equivalent*:

(I′) *For every indecomposable preinjective non-projective left R-module ${}_RC$ there is an almost split sequence $0 \to A \to B \to C \to 0$ in* R-mod.

(IC) *There is a preinjective component in* R-mod *containing all indecomposable injective modules.*

We then obtain the following consequence of Theorem 1.1.

COROLLARY 8.5. *Let R be an artinian hereditary indecomposable ring*, *and assume that all indecomposable injective left R-modules are finitely generated. Then the following statements are equivalent*:

(PC) *There is a preprojective component in* mod-R *containing all indecomposable projective modules.*

(PC′) *There is a preprojective component in* Mod-R *containing all indecomposable projective modules.*

(IC) *There is a preinjective component in* R-mod *containing all indecomposable injective modules.*

(IC′) *There is a preinjective component in* R-Mod *containing all indecomposable injective modules.*

Proof. The equivalence of (PC) and (PC′) follows from the fact that the almost split sequences in mod-R for the preprojective right modules are even almost split in Mod-R; see [3, 2.8] or Corollary 6.2. Similarly we obtain the equivalence of (IC) and (IC′); see Proposition 8.2. Moreover, we know from Theorem 1.1, 8.3 and 8.4 that (PC) is equivalent to (IC). ■

In the preprojective and in the preinjective component, the local duality and the Auslander–Bridger transpose behave as in the artin algebra case.

Corollary 8.6. *Let* R *be an artinian hereditary indecomposable ring satisfying one of the equivalent conditions in Theorem* 1.1, *and let* Δ *be the Gabriel quiver of* R. *We denote by* $\mathbf{P}$ *the preprojective component of* mod-R *and by* $\mathbf{I}$ *the preinjective component of* R-mod, *and identify them with their Auslander–Reiten quivers. Then there exist monomorphisms of translation quivers without valuation* $\phi : \mathbf{P} \to \mathbb{N}_0\Delta$ *and* $\psi : \mathbf{I} \to (-\mathbb{N}_0)\Delta^{\mathrm{op}}$. *Moreover*, *if we identify along* ϕ *and* ψ, *then the local duality gives rise to a map* $\mathbb{N}_0\Delta \to (-\mathbb{N}_0)\Delta^{\mathrm{op}}, (n, x) \mapsto (-n, x)$, *and the Auslander–Bridger transpose gives rise to a map* $\mathbb{N}\Delta \to (-\mathbb{N}_0)\Delta^{\mathrm{op}}, (n, x) \mapsto (-n + 1, x)$.

Proof. That ϕ and ψ are injective translation quiver morphisms is shown as in [10, Chapter VIII, 1.15]. Moreover, if C_R is an indecomposable preprojective module and $n \in \mathbb{N}_0$ with $\tau^n C \in \mathcal{P}_0$, then it is easy to check that $\tau^{-n}C^+ \cong (\tau^n C)^+ \cong \tau^{-n+1}\,\mathrm{Tr}\,C$. This proves the last two statements. ■

Acknowledgements. Part of these results were developed in our doctoral dissertations. We would like to thank our thesis advisor Professor W. Zimmermann for many suggestions and discussions. Further, we thank the referee for several useful comments, in particular, for having suggested Corollaries 8.5 and 8.6. The first author also acknowledges an HSPIII-grant.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, 2nd ed., Springer, New York, 1992.
[2] L. Angeleri Hügel, $\mathcal{P}_1$*-hereditary artin algebras*, J. Algebra 176 (1995), 460–479.
[3] —, *Almost split sequences arising from the preprojective partition*, ibid. 194 (1997), 1–13.
[4] —, *Finitely cotilting modules*, Comm. Algebra, to appear.
[5] —, *On some precovers and preenvelopes*, preprint, 1998.

[6] M. Auslander, *Large modules over artin algebras*, in: Algebra, Topology, Category Theory, Academic Press, 1976, 1–17.

[7] —, *Functors and morphisms determined by objects*, in: Lecture Notes in Pure and Appl. Math. 37, Marcel Dekker, 1978, 1–244.

[8] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. 94 (1969).

[9] M. Auslander and I. Reiten, *Representation theory of artin algebras III. Almost split sequences*, Comm. Algebra 3 (1975), 239–294.

[10] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[11] M. Auslander and S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra 66 (1980), 61–122.

[12] R. R. Colby and K. R. Fuller, *Tilting, cotilting, and serially tilted rings*, Comm. Algebra 18 (1990), 1585–1615.

[13] R. Colpi, *Some remarks on equivalences between categories of modules*, ibid. 18 (1990), 1935–1951.

[14] —, *Tilting modules and ∗-modules*, ibid. 21 (1993), 1095–1102.

[15] R. Colpi, G. D'Este and A. Tonolo, *Quasi-tilting modules and counter equivalences*, J. Algebra 191 (1997), 461–494.

[16] N. V. Dung, *Preinjective modules and finite representation type of artinian rings*, Comm. Algebra, to appear.

[17] P. Gabriel and A. V. Roiter, *Algebra VIII: Representations of Finite-Dimensional Algebras*, Encyclopedia Math. Sci. 73, Springer, 1992.

[18] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[19] I. Herzog, *A test for finite representation type*, J. Pure Appl. Algebra 95 (1994), 151–182.

[20] M. Hoshino, *Tilting modules and torsion theories*, Bull. London Math. Soc. 14 (1982), 334–336.

[21] —, *On splitting torsion theories induced by tilting modules*, Comm. Algebra 11 (1983), 427–439.

[22] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[23] M. Schmidmeier, *A dichotomy for finite length modules induced by local duality*, Comm. Algebra 25 (1997), 1933–1944.

[24] —, *The local duality for homomorphisms and an application to pure semisimple PI-rings*, Colloq. Math. 77 (1998), 121–132.

[25] D. Simson, *Pure semisimple categories and rings of finite representation type*, J. Algebra 48 (1977), 290–296; Corrigendum, ibid. 67 (1980), 254–256.

[26] —, *On pure-semisimple Grothendieck categories*, *I*, Fund. Math. 100 (1978), 211–222.

[27] —, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra 71 (1981), 195–218.

[28] —, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon and Breach, 1992.

[29] —, *An Artin problem for division ring extensions and the pure semisimplicity conjecture I*, Arch. Math. (Basel) 66 (1996), 114–122.

[30] —, *A class of potential counter-examples to the pure semisimplicity conjecture*, in: Adv. Algebra Model Theory 9, Gordon and Breach, 1997, 345–373.

[31] H. Valenta, *Existence criteria and construction methods for certain classes of tilting modules*, Comm. Algebra 22 (1994), 6047–6072.
[32] W. Zimmermann, *Existenz von Auslander–Reiten-Folgen*, Arch. Math. (Basel) 40 (1983), 40–49.
[33] B. Zimmermann-Huisgen, *Strong preinjective partitions and representation type of artinian rings*, Proc. Amer. Math. Soc. 109 (1990), 309–322.
[34] B. Zimmermann-Huisgen and W. Zimmermann, *On the sparsity of representations of rings of pure global dimension zero*, Trans. Amer. Math. Soc. 320 (1990), 695–711.

Mathematisches Institut der Universität
Theresienstraße 39
D-80333 München, Germany
E-mail: angeleri@rz.mathematik.uni-muenchen.de

Received 11 August 1998;
revised 30 December 1998