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COMPLETENESS OF L_1 SPACES OVER FINITELY ADDITIVE PROBABILITIES

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1. Introduction. Finitely additive measures—though not as technically amenable as their countably additive counterparts—seem to lead to interesting, and at times peculiar, mathematical problems. For an exposition of the theory of finitely additive measures, see Dunford and Schwartz [5] or Bhaskara Rao and Bhaskara Rao [1]. The bold initiative of Dubins and Savage [4] to develop gambling in a finitely additive setup and the beautiful existence theorem of Purves and Sudderth [18] in infinite product spaces paved the way to develop a substantial part of classical probability theory in a finitely additive setup—called the strategic setup.

The strong law of large numbers was treated in Purves and Sudderth [18] and Chen [3]; the law of iterated logarithm in Chen [2]; the central limit theorem in Ramakrishnan [22] and Karandikar [15]; Markov chains and potential theory in Ramakrishnan [20, 21, 23]; random walks in Karandikar [16] and S. Gangopadhyay and Rao [7, 8]; martingales in Dubins and Savage [4], Purves and Sudderth [18]; Komlos type theorems in Halevy and Bhaskara Rao [11]. The zero-one laws of Lévy and Kolmogorov hold as shown in Purves and Sudderth [18, 19] whereas the Hewitt–Savage zero-one law needs modification as shown in Purves and Sudderth [19], Gangopadhyay and Rao [9]. It is interesting to note that an appropriate finitely additive version of this zero-one law already appears in the fundamental paper of Hewitt and Savage [14], but of course, not in the strategic setup.

More recently (as pointed out to us by J. K. Ghosh), Heath and Sudderth [12, 13] and Lane and Sudderth [17] have advocated that it is beneficial to use finitely additive priors in some problems of statistical inference. In fact the prescription of [13] is simple: A Bayesian who seeks to avoid incoherent inferences might be advised to abandon improper countably additive priors and use only finitely additive priors.

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This paper is concerned with the completeness of the space $L_1(\gamma)$ of integrable functions over a finitely additive nonnegative bounded measure γ defined on a σ-field of subsets of a set. In the mathematical literature [1, 5, 14, 26] the domain for such a γ is a field of sets. Our interest is not in finitely additive measures per se, but in the development of probability. The natural domain for γ then is a σ-field and that is what we take.

By the Hewitt–Yosida theorem $\gamma = \gamma_0 + \gamma_1$ where γ_0 is countably additive and γ_1 is purely finitely additive. We show (Theorem 3) that $L_1(\gamma)$ is complete iff $L_1(\gamma_1)$ is complete and γ_0, γ_1 are supported on disjoint sets. By the Sobczyk–Hammer theorem $\gamma_1 = \gamma_2 + \gamma_3$ where γ_2 is discrete and γ_3 is strongly continuous. We show (Theorem 6) that $L_1(\gamma_1)$ is complete iff $L_1(\gamma_2)$, $L_1(\gamma_3)$ are complete and γ_2, γ_3 are supported on disjoint sets. We have $\gamma_2 = \sum_i a_i \delta_i$ where each δ_i is a 0-1 valued measure. We show (Theorem 4) that $L_1(\gamma_2)$ is complete iff the δ_i are uniformly singular.

Next we consider finite strategic products, $\gamma = \bigotimes_{i=1}^k \gamma_i$. We show (Theorem 9) that if $L_1(\gamma_i)$ is complete for each i then $L_1(\gamma)$ is complete. If $L_1(\gamma)$ is complete then $L_1(\gamma_1)$ is complete, but for $i > 1$, $L_1(\gamma_i)$ need not be complete. We have partial results for infinite strategic products.

It should be remarked that the completeness of L_1 spaces is not only interesting in its own right (see [6, 10] and reference therein) but is also related to the existence of Radon–Nikodym derivatives. As noted in [6, 10] the completeness of $L_1(\gamma)$ is equivalent to the completeness of $L_p(\gamma)$ for any p with $1 \le p < \infty$. It is interesting to note [6] that $L_\infty(\gamma)$ is always complete.

2. Preliminaries. Throughout we consider a nonnegative finitely additive bounded set function γ defined on a σ-field $\mathcal{F}$ of subsets of a space Ω. Even though our motivation and interest is only in probabilities, it is convenient to deal with bounded measures. Recall that $[1,5]$ $L_1(\gamma)$ is the collection of $\mathcal{F}$-measurable real-valued functions f on Ω such that $\int |f|\,d\gamma < \infty$, with the usual pseudometric.

This paper can be regarded as a continuation of [6]. Our starting point is the following theorem:

THEOREM 1 (see [6]). *The following are equivalent*:

1. $L_1(\gamma)$ *is complete.*

2. $(\mathcal{F}, d)$ *is complete where* d *is the usual pseudometric on* $\mathcal{F}$ *given by* $d(A,B) = \gamma(A \triangle B)$.

3. *Given any sequence* $\{A_n\}$ *of sets in* $\mathcal{F}$, *there exists a set* $A \in \mathcal{F}$ *such that* $\gamma(A_n \setminus A) = 0$ *for each* n, *and* $\gamma(A) \le \sum \gamma(A_n)$.

4. *Given any sequence* $\{A_n\}$ *of sets in* $\mathcal{F}$ *and* $\varepsilon > 0$, *there exists a set* $A \in \mathcal{F}$ *such that*

$$\gamma(A_n - A) = 0 \ \ \textit{for each } n, \quad \textit{and} \ \ \gamma(A) \le \sum \gamma(A_n) + \varepsilon.$$

5. *Given any sequence* $\{A_n\}$ *of sets in* $\mathcal{F}$ *there is a sequence* $\{B_n\}$ *of sets in* $\mathcal{F}$ *such that*

$$B_n \subset A_n, \quad \gamma(A_n - B_n) = 0 \quad \textit{for each } n, \quad \textit{and}$$
$$\gamma\Big(\bigcup B_n\Big) \le \sum \gamma(B_n) = \sum \gamma(A_n).$$

6. *Given a sequence* $\{A_n\}$ *of pairwise disjoint sets in* $\mathcal{F}$, *there exists a sequence* $\{B_n\}$ *in* $\mathcal{F}$ *such that*

$$B_n \subset A_n, \quad \gamma(A_n - B_n) = 0 \quad \textit{for each } n, \ \textit{and}$$
$$\gamma\Big(\bigcup B_n\Big) = \sum \gamma(B_n) = \sum \gamma(A_n).$$

7. *Given an increasing sequence* $\{A_n\}$ *of sets in* $\mathcal{F}$, *there exists a set* $A \in \mathcal{F}$ *such that*

$$\gamma(A_n - A) = 0 \ \ \textit{for each } n, \quad \textit{and} \ \ \gamma(A) = \lim_n \gamma(A_n).$$

This theorem is not stated in this form in [6], but a proof can be based on the proofs of Lemma 2.2, Theorem 2.4 and Remark 2.5 of [6]. See also [10] where a notion of self-separability of γ was introduced and was shown to be equivalent to the completeness of Banach space valued L_1 spaces. Outer measure was used in [6], but the above pleasing form of the result is a consequence of the fact that the domain of γ is now a σ-field.

As a useful consequence of the criteria given above, we have the following

THEOREM 2. (a) *Suppose* $\gamma = \gamma_1 + \gamma_2$.

(i) *If* $L_1(\gamma)$ *is complete then so are* $L_1(\gamma_1)$ *and* $L_1(\gamma_2)$.
(ii) *If* $L_1(\gamma_1)$ *and* $L_1(\gamma_2)$ *are complete then* $L_1(\gamma)$ *is not necessarily complete.*
(iii) *If* $L_1(\gamma_1)$ *and* $L_1(\gamma_2)$ *are complete and* γ_1, γ_2 *are supported on disjoint sets then* $L_1(\gamma)$ *is complete.*

(b) *Suppose* $\Omega_0 \in \mathcal{F}$, $0 < \gamma(\Omega_0) < \gamma(\Omega)$. *Then* $L_1(\gamma)$ *is complete iff* $L_1(\gamma_1)$ *and* $L_1(\gamma_2)$ *are complete where* γ_1, γ_2 *are the restrictions of* γ *to* Ω_0 *and* $\Omega - \Omega_0$ *respectively.*

(c) *Suppose* $L_1(\gamma)$ *is complete. Let* $\mathcal{F}_0$ *be a sub-*σ*-field of* $\mathcal{F}$ *which includes all* γ *null sets that are in* $\mathcal{F}$. *Let* γ_0 *be* γ *restricted to* $\mathcal{F}_0$. *Then* $L_1(\gamma_0)$ *is complete.*

Proof. (c) and (aiii) follow from criterion 6 of Theorem 1. (b) follows from (ai) and (aiii).

To show (aii) take $\Omega = \{1, 2, \ldots\}$, $\mathcal{F} =$ power set of Ω; γ_1 is the countably additive measure $\gamma_1(n) = 1/2^n$, $n = 1, 2, \ldots$; γ_2 is a diffuse 0-1 valued measure on $\mathcal{F}$ giving 0 to singletons; $\gamma = \gamma_1 + \gamma_2$. Then both $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are complete. However, criterion 6 of Theorem 1 fails for $A_n = \{n\}$, showing that $L_1(\gamma)$ is not complete.

Finally, we prove (ai) as follows: Towards verifying criterion 6 of Theorem 1, suppose $\{A_n\}$ is a sequence of disjoint sets in $\mathcal{F}$. Since $L_1(\gamma)$ is complete, get a sequence $\{B_n\}$ as stated there. In particular, $\gamma_1(A_n - B_n) = 0 = \gamma_2(A_n - B_n)$ for each n. Moreover

$$\gamma\Big(\bigcup B_n\Big) = \gamma_1\Big(\bigcup B_n\Big) + \gamma_2\Big(\bigcup B_n\Big),$$
$$\sum \gamma(B_n) = \sum \gamma_1(B_n) + \sum \gamma_2(B_n).$$

By choice of $\{B_n\}$, the left sides of the equations above are the same. So must be the right sides. But $\gamma_1(\bigcup B_n) \geq \sum \gamma_1(B_n)$ and $\gamma_2(\bigcup B_n) \geq \sum \gamma_2(B_n)$ so that equality must hold at both places. In other words, the same sequence $\{B_n\}$ witnesses that criterion 6 of Theorem 1 holds for both γ_1 and γ_2.

REMARK 1. Theorem 2(ai) can equivalently be stated as follows: If $L_1(\gamma)$ is complete and $\gamma_1 \leq \gamma$ (inequality being understood setwise) then $L_1(\gamma_1)$ is also complete.

3. Yosida–Hewitt decomposition. Recall that a finitely additive positive measure μ on $(\Omega, \mathcal{F})$ is said to be *purely finitely additive* if λ being a positive countably additive measure with $\lambda(A) \leq \mu(A)$ for all $A \in \mathcal{F}$ implies that $\lambda \equiv 0$.

The celebrated decomposition theorem due to Yosida and Hewitt [26] (see also [24]) says that any finitely additive positive measure γ on $(\Omega, \mathcal{F})$ can be decomposed as $\gamma = \gamma_1 + \gamma_2$ where γ_1 is countably additive and γ_2 is purely finitely additive. Moreover such a decomposition is unique.

THEOREM 3. *Let $\gamma = \gamma_1 + \gamma_2$ be the Yosida–Hewitt decomposition of γ. Then $L_1(\gamma)$ is complete iff γ_1, γ_2 are supported on disjoint sets and $L_1(\gamma_2)$ is complete.*

Proof. If γ_1, γ_2 satisfy the conditions of the theorem, then $L_1(\gamma)$ is complete in view of Theorem 2(aiii).

Conversely, assume that $L_1(\gamma)$ is complete. The idea is the following: We shall express $\Omega = A \cup B \cup C$ where $A, B, C \in \mathcal{F}$ are pairwise disjoint, $\gamma_1(A) = 0$, $\gamma_2(B) = 0$, and if $S \in \mathcal{F}$, $S \subset C$ then $\gamma_1(S) > 0$ iff $\gamma_2(S) > 0$.

Assume the decomposition for a moment. We claim that γ_2 (and hence γ_1) is null on C. If not, γ_2 being purely finitely additive we can get $C_n \subset C$ with $C_n \uparrow C$ and $\lim_n \gamma_2(C_n) < \gamma_2(C)$. By criterion 7 of Theorem 1, get $\widetilde{C}$ such that $\gamma(C_n - \widetilde{C}) = 0$ for each n and $\gamma(\widetilde{C}) = \lim_n \gamma(C_n)$. In particular,

$\gamma_1(C_n - \widetilde{C}) = 0$ for each n so that $\gamma(\widetilde{C}) \geq \lim_n \gamma_1(C_n)$. In case $\gamma_2(C - \widetilde{C}) = 0$ we have $\gamma_2(\widetilde{C}) \geq \gamma_2(C) > \lim_n \gamma_2(C_n)$, implying that $\gamma(\widetilde{C}) > \lim_n \gamma(C_n)$, a contradiction. Thus $\gamma_2(C - \widetilde{C}) > 0$. But then $\gamma_1(C - \widetilde{C}) > 0$. Since $C_n - \widetilde{C} \uparrow C - \widetilde{C}$ and γ_1 is countably additive we conclude that $\gamma_1(C - \widetilde{C}) = \lim_n \gamma_1(C_n - \widetilde{C}) = 0$, again a contradiction. Thus γ_2 must be null on C. So must be γ_1. Thus γ_1 and γ_2 are supported on B and A respectively. The proof is completed by using Theorem 2(b).

We now proceed to exhibit the stated decomposition. Consider

$$\mathcal{C} = \{S \in \mathcal{F} : \gamma_1(S) > 0 \text{ and } \gamma_2(S) = 0\},$$
$$\beta = \sup\{\gamma_1(S) : S \in \mathcal{C}\}.$$

Completeness of $L_1(\gamma)$ implies that this supremum is indeed attained. To see this, pick $S_n \in \mathcal{C}$ with $\gamma_1(S_n) \uparrow \beta$. Since $\mathcal{C}$ is closed under finite unions we can assume that S_n increases with n. By criterion 7 of Theorem 1, get B such that $\gamma(B) = \beta$ and $\gamma(S_n - B) = 0$ for each n. There is no loss to assume that $B \subset \bigcup_n S_n$. First observe that $\gamma_1(B) + \gamma_2(B) = \gamma(B) = \beta$. Secondly, $\gamma(S_n - B) = 0$ and hence $\gamma_1(S_n - B) = 0$ for each n, so that $\gamma_1(B) \geq \gamma_1(S_n)$ for all n, implying that $\gamma_1(B) \geq \beta$. These two observations show that $\gamma_1(B) = \beta$ and $\gamma_2(B) = 0$.

In an analogous manner, consider

$$\mathcal{D} = \{S \in \mathcal{F} : \gamma_2(S) > 0 \text{ and } \gamma_1(S) = 0\}.$$

Since γ_1 is countably additive, $\mathcal{D}$ is closed under countable unions and hence there is a set A such that $\gamma_1(A) = 0$ and $\gamma_2(A) = \sup\{\gamma_2(S) : S \in \mathcal{D}\}$. By construction it is clear that $\gamma_1(A \cap B) = 0 = \gamma_2(A \cap B)$. Thus we can assume $A \cap B = \emptyset$. Set $C = \Omega - \{A \cup B\}$ to complete the proof.

REMARK 2. Since γ_1 in the theorem above is countably additive, clearly $L_1(\gamma_1)$ is complete.

4. Discrete measures. Recall that a finitely additive nonnegative measure γ on $(\Omega, \mathcal{F})$ is said to be *discrete* if $\gamma = \sum a_i \delta_i$ where the δ_i are distinct 0-1 valued measures and $a_i > 0$. Since we are considering only bounded measures, clearly $\sum a_i < \infty$.

THEOREM 4. *Let $\gamma = \sum a_i \delta_i$ be discrete. Then $L_1(\gamma)$ is complete iff the δ_i are uniformly singular, that is, there are pairwise disjoint sets $A_i \in \mathcal{F}$ with $\delta_i(A_i) = 1$ for each i.*

Proof. If $\{\delta_i\}$ are uniformly singular then criterion 6 of Theorem 1 applies to show that $L_1(\gamma)$ is complete.

To prove the converse, assume that $\{\delta_i\}$ are not uniformly singular. Let us say that δ_i *can be separated* if there is a set A_i in $\mathcal{F}$ such that $\delta_i(A_i) = 1$ and $\delta_j(A_i) = 0$ for $j \neq i$. If each δ_i can be separated, witnessed by say A_i, then setting $B_n = A_n - \bigcup_{i<n} A_i$, we observe that $\delta_i(B_i) = 1$ for each i, showing that δ_i are uniformly singular. Thus there is a δ_i, say δ_1, which cannot be separated. We shall construct a sequence of pairwise disjoint sets $(A_n)_{n\geq 1}$ such that (i) if $j > 1$ then $\delta_j(A_i) = 1$ for some i and (ii) $\delta_1(A_i) = 0$ for each i. If this is done, then we claim that criterion 6 of Theorem 1 fails for this sequence. Indeed, suppose we have sets $B_i \subset A_i$ with $\gamma(A_i - B_i) = 0$ for each i. By properties (i) and (ii) we have $\sum_{i\geq 1} \gamma(B_i) = \sum_{i>1} a_i$. If $\delta_1(\bigcup B_i) = 0$ then property (i) implies that δ_1 can be separated, which is not the case. Thus $\delta_1(\bigcup B_i) = 1$, implying that $\gamma(\bigcup B_i) = \sum_{i\geq 1} a_i > \sum \gamma(B_i)$.

We now proceed to exhibit sets A_i as stated. Pick B_0 such that $\delta_1(B_0) = 1$. Set $A_1 = B_0^{\mathrm{c}}$ and $S_1 = \{j : \delta_j(B_0) = 1\}$. Then S_1 is infinite because δ_1 cannot be separated. Pick the first integer $j_1 \in S_1$ and write $B_0 = B_1 \cup A_2$, a disjoint union with $\delta_1(B_1) = 1$ and $\delta_{j_1}(A_2) = 1$. To do this, just note that the δ_i, being 0-1 valued, are pairwise singular. Then $S_2 = \{j : \delta_j(B_1) = 1\}$ is again infinite. Proceed inductively by picking the first j in S_n at the nth stage. This completes the proof of the theorem.

The following theorem, a slight extension of Theorem 4, will be needed later. Theorem 4 corresponds to the case when γ_0 is absent.

THEOREM 5. *Suppose $\gamma = \sum_{i\geq 0} a_i\gamma_i$ with $a_i > 0$ and $\sum a_i < \infty$. Assume that γ_i, $i \geq 1$, are* 0-1 *valued. If $L_1(\gamma)$ is complete then γ_i, $i \geq 1$, are uniformly singular.*

Proof. Apply Theorems 2 and 4.

5. Sobczyk–Hammer decomposition. Recall that a finitely additive nonnegative measure γ on $(\Omega, \mathcal{F})$ is said to be *strongly continuous* if given $\varepsilon > 0$, there is a finite decomposition $\Omega = \bigcup A_i$ with $\gamma(A_i) < \varepsilon$ for each i. The well known decomposition theorem due to Sobczyk and Hammer [25] says that any finitely additive positive measure γ on $(\Omega, \mathcal{F})$ can be decomposed as $\gamma = \gamma_1 + \gamma_2$ where γ_1 is discrete and γ_2 is strongly continuous. Further such a decomposition is unique.

THEOREM 6. *Let $\gamma = \gamma_1 + \gamma_2$ be the Sobczyk–Hammer decomposition of γ. Then $L_1(\gamma)$ is complete iff $L_1(\gamma_1), L_1(\gamma_2)$ are complete and γ_1, γ_2 are supported on disjoint sets.*

Proof. The “if” part is a consequence of Theorem 2. To prove the converse, assume that $L_1(\gamma)$ is complete. By Theorem 2 again, $L_1(\gamma_1)$ and $L_1(\gamma_2)$ are also complete. We only need to show now that γ_1, γ_2 are supported on disjoint sets.

First assume that γ_1 is 0-1 valued. By using the strong continuity of γ_2, one can obtain, for each n, a set A_n such that $\gamma_1(A_n^{\mathrm{c}}) = 1$ and $\gamma_2(A_n^{\mathrm{c}}) < 1/2^n$. We can also assume that A_n increases with n. We have $\lim_n \gamma(A_n) =$

$\lim_n \gamma_2(A_n) = \gamma_2(\Omega)$. By Theorem 1(7) get $B \subset \bigcup A_n$ with $\gamma(B) = \gamma_2(\Omega)$ and $\gamma(A_n - B) = 0$ for each n. In particular $\gamma_2(A_n - B) = 0$ for each n so that $\gamma_2(B) \geq \gamma_2(A_n \cap B) = \gamma_2(A_n)$, which increases to $\gamma_2(\Omega)$. Thus $\gamma_2(B) = \gamma_2(\Omega)$. But since $\gamma(B) = \gamma_2(\Omega)$ we conclude that $\gamma_1(B) = 0$. In other words γ_1, γ_2 are supported on B^c and B respectively.

To treat the general case, assume that the discrete part is $\gamma_1 = \sum_{i\geq 1} a_i\delta_i$, where $a_i > 0$, $\sum a_i < \infty$ and the δ_i are distinct 0-1 valued measures. By Theorem 5, the δ_i are uniformly singular so that Ω can be written as a disjoint union $\bigcup_{i\geq 1} A_i$ with $\delta_i(A_i) = 1$ for each i. By Theorem 2, $L_1(a_i\delta_i + \gamma_2)$ is complete for each i and hence by earlier para we can get $B_i \subset A_i$ such that $\delta_i(B_i) = 1$ and $\gamma_2(B_i) = 0$. Since $L_1(\gamma)$ is complete, by Theorem 1(6) we can get $C_i \subset B_i$ with $\gamma(B_i - C_i) = 0$ for each i and $\gamma(\bigcup C_i) = \sum \gamma(C_i)$. In particular for each i, $\delta_i(B_i - C_i) = 0$ so that $\delta_i(C_i) = \delta_i(B_i) = 1$. Since $\gamma_2(C_i) = 0$ and $\gamma_1(C_i) = a_i$ for each i, we have

$$\sum \gamma(C_i) = \sum \gamma_1(C_i) + \sum \gamma_2(C_i) = \sum a_i,$$

$$\gamma\Big(\bigcup C_i\Big) = \gamma_1\Big(\bigcup C_i\Big) + \gamma_2\Big(\bigcup C_i\Big) \geq \sum a_i + \gamma_2\Big(\bigcup C_i\Big).$$

Since the left sides are equal we conclude that $\gamma_2(\bigcup C_i) = 0$. In other words γ_1 is supported on $\bigcup C_i$ and γ_2 is supported on its complement, as claimed.

Combining Theorems 3–6 we obtain the following two versions of the main characterization theorem.

THEOREM 7. *Let γ be a finitely additive probability on $(\Omega, \mathcal{F})$. Then $L_1(\gamma)$ is complete iff Ω has a decomposition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ such that* (i) *γ restricted to Ω_0 is countably additive*, (ii) *γ restricted to Ω_1 is discrete and is a combination of uniformly singular* 0-1 *probabilities and* (iii) *γ restricted to Ω_2 is strongly continuous and its L_1 space is complete.*

THEOREM 8. *Let γ be a finitely additive probability on $(\Omega, \mathcal{F})$. Then $L_1(\gamma)$ is complete iff Ω has a decomposition $\Omega = \bigcup_{i\geq 0} \Omega_i$ such that* (i) *γ restricted to Ω_0 is countably additive*, (ii) *γ restricted to each Ω_i, $i \geq 2$, is at most two-valued*, (iii) *γ restricted to Ω_1 is strongly continuous and its L_1 space is complete and* (iv) *for each $A \in \mathcal{F}$, $\gamma(A) = \sum_{i=0}^{\infty} \gamma(\Omega_i \cap A)$.*

In Theorem 7 we had a finite decomposition so that the last condition of Theorem 8 was not imposed. We conclude this section with a few remarks.

REMARK 3. Here is an example of a sequence of 0-1 measures which are not uniformly singular. Set $\Omega = \{0, 1, 2, \cdots\}$. For $i \geq 2$, let γ_i be a diffuse 0-1 measure concentrated on the powers of the ith prime. Let

$$\mathcal{C} = \{A : \gamma_i(A) = 1 \text{ for all but finitely many } i \geq 2\}.$$

Extend $\mathcal{C}$ to an ultrafilter and denote by γ_1 the corresponding 0-1 measure. Then $\{\gamma_i : i \geq 1\}$ are not uniformly singular. Note that all these γ_i are purely finitely additive. If we did not want this, we could have taken point masses and any diffuse 0-1 measure. Also note that all these γ_i are defined on the power set of Ω. The same construction can be carried out on any $(\Omega, \mathcal{F})$ provided $\mathcal{F}$ is infinite.

REMARK 4. Given any sequence $\{\gamma_i\}$ of 0-1 measures on $(\Omega, \mathcal{F})$ there exists an infinite subsequence which is uniformly singular. We inductively construct a sequence of disjoint sets $A_i \geq 1$ and indices n_i, $i \geq 1$, such that $\gamma_{n_i}(A_i) = 1$ for all i. Just make sure that at the kth stage infinitely many γ_i are concentrated on the complement of $\bigcup_{i \leq k} A_i$.

REMARK 5. If $\gamma = \sum 2^{-i}\gamma_i$ where γ_i are as in Remark 3, then $L_1(\gamma)$ is not complete—though γ is defined on the power set of Ω.

REMARK 6. Given $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is infinite it is always possible to obtain strongly continuous γ on $\mathcal{F}$ such that $L_1(\gamma)$ is not complete. $\mathcal{F}$ being infinite, the general case can be reduced to $\Omega = \mathbb{N}$ and $\mathcal{F}$ the power set. This is what we treat. Let μ be any extension of the density charge defined on arithmetic progressions. Then μ is clearly strongly continuous. Fix a decomposition of $\mathbb{N}$ into disjoint sets A^n, $n \geq 1$, with $\mu(A^n) > 0$ for each n. Let

$$\mathcal{F} = \{B : \mu(B \cap A^n) = \mu(A^n) \text{ for all but finitely many } n\}.$$

For each k, $1 \leq k \leq n$, and each sequence $(\varepsilon_1, \ldots, \varepsilon_k)$ of 0's and 1's fix a subset $A^n_{\varepsilon_1 \ldots \varepsilon_k}$ of A^n with positive μ measure such that $A^n_{\varepsilon_1 \ldots \varepsilon_k}$ is the disjoint union of $A^n_{\varepsilon_1 \ldots \varepsilon_k 0}$ and $A^n_{\varepsilon_1 \ldots \varepsilon_k 1}$, and A^n is the disjoint union of A^n_0 and A^n_1. This can be done by the strong continuity of μ. For $k \geq 1$ define $B_{\varepsilon_1 \ldots \varepsilon_k} = \bigcup_{n \geq k} A^n_{\varepsilon_1 \ldots \varepsilon_k}$. Extend $\mathcal{F}$ restricted to $B_{\varepsilon_1 \cdots \varepsilon_k}$ to an ultrafilter on $\bigcup_{n \geq k} A^n$. Let $\eta_{\varepsilon_1 \ldots \varepsilon_k}$ be the associated 0-1 measure supported on $\bigcup_{n \geq k} A^n$, defined on the power set of $\mathbb{N}$. For $k \geq 1$, let η_k be the average of the 2^k measures $\eta_{\varepsilon_1 \ldots \varepsilon_k}$. Fix any Banach limit ℓ and set $\eta(A) = \ell\{\eta_k(A) : k \geq 1\}$ for $A \subset \mathbb{N}$. Let $\gamma = \frac{1}{2}\eta + \frac{1}{2}\mu$. It is not difficult to see that γ is strongly continuous. Since $\eta_k(A_n) = 0$ for $k > n$, it follows that $\eta(A_n) = 0$ for any n. Using this, one can argue that condition 6 of Theorem 1 fails for the sequence $\{A_n\}$ so that $L_1(\gamma)$ is not complete.

REMARK 7. Later we shall see examples of strongly continuous γ for which $L_1(\gamma)$ is complete. However, we are unable to decide if there are extensions of density charges for which the L_1 space is complete. (The referee has kindly informed us that he has examples of such extensions.)

6. Finite strategic products. When dealing with finitely additive measures, products are not in general well defined on product σ-fields. However, there is one situation where the product measures are well defined by successive integration as was done by Dubins and Savage [4]. For this we need to consider measures defined on power sets. For $1 \le i \le k$, let γ_i be a finitely additive probability defined on the power set of Ω_i. Let $\Omega = \bigotimes_{i=1}^{k} \Omega_i$. On the power set of Ω define

$$\gamma(A) = \int \ldots \int 1_A(x_1, \ldots, x_k)\, d\gamma_k(x_k) \ldots d\gamma_1(x_1).$$

This γ is called the *strategic product* of the γ_i, $1 \le i \le k$. Carefully note the order of integration. For more on this, see [4]. The reader should note that even though the probabilities are defined on power sets, their L_1 spaces may still be incomplete (see Remarks 5 and 6 of the previous section).

THEOREM 9. *Let γ be the strategic product of γ_i, $1 \le i \le k$.*

(a) *If $L_1(\gamma_i)$ is complete for each i, then so is $L_1(\gamma)$.*

(b) *If $L_1(\gamma)$ is complete, then so is $L_1(\gamma_1)$.*

(c) *$L_1(\eta^k)$ is complete iff $L_1(\eta)$ is complete. Here η^k is the k-fold strategic product of η.*

Proof. (c) is immediate from (a) and (b). For simplicity we assume that $k = 2$ in what follows.

To prove (b) we verify condition 7 of Theorem 1 for the measure γ_1. So let $\{A_n\}$ be an increasing sequence of subsets of Ω_1. Set $B_n = A_n \times \Omega_2$. Using the fact that $L_1(\gamma)$ is complete and criterion 7 of Theorem 1 get $B \subset \Omega_1 \times \Omega_2$ such that $\gamma(B_n \setminus B) = 0$ for each n and $\lim_{n\to\infty} \gamma(B_n) = \gamma(B)$. Set

$$A = \{x \in \Omega_1 : \gamma_2(B_x) > 1/2\}.$$

Firstly, for each n,

$$\gamma_1(A_n \setminus A) \le 2 \int_{A_n \setminus A} \gamma_2(B_n \setminus B)_x \, d\gamma_1(x) \le 2\gamma(B_n \setminus B) = 0.$$

Secondly, a similar computation gives $\gamma_1(A \setminus A_n) \le \gamma(B \setminus B_n)$, so that $\lim_{n\to\infty} \gamma_1(A_n) \ge \gamma_1(A)$, which together with the first observation implies that $\lim_{n\to\infty} \gamma_1(A_n) = \gamma_1(A)$.

We now prove (a). Using ideas from [6] we verify condition 7 of Theorem 1 for γ. Let $\{A_n\}$ be an increasing sequence of subsets of $\Omega_1 \times \Omega_2$. Passing to a subsequence if necessary we can and do assume that $\gamma(A_{n+1} \setminus A_n) < 1/2^{2n}$ for each $n \ge 1$. This of course implies that for each $n \ge 1$,

$$\gamma_1\{x_1 : \gamma_2((A_{n+1})_{x_1}) > \gamma_2((A_n)_{x_1}) + 1/2^n\} < 1/2^n.$$

Let the set in braces be denoted by B_n. Fix $k \ge 1$. By completeness of $L_1(\gamma_1)$ and condition 3 of Theorem 1 applied to the sequence $\{B_n : n \ge k\}$ we obtain $C_k \subset \Omega_1$ such that

$$\gamma_1(C_k) \le 1/2^{k-1} \quad \text{and} \quad \gamma_1(B_n \setminus C_k) = 0 \quad \forall n \ge k.$$

By taking successive intersections we can assume that C_k decreases with k. Set $C_\infty = \bigcap_k C_k$. Clearly $\gamma_1(C_\infty) = 0$. If $x_1 \notin C_\infty$ then let $k(x_1)$ be the first integer k such that $x_1 \notin C_k$. Let

$$D_1 = \Big\{x_1 : x_1 \notin C_\infty,\ x_1 \in \bigcup_{n \geq k(x_1)} B_n\Big\},$$
$$D_2 = \Big\{x_1 : x_1 \notin C_\infty,\ x_1 \notin \bigcup_{n \geq k(x_1)} B_n\Big\}.$$

For $x_1 \in D_1$ let $n(x_1)$ be the first integer $n \geq k(x_1)$ such that $x_1 \in B_n$. For each $x_1 \in \Omega_1$ by completeness of $L_1(\gamma_2)$ get a set $A(x_1) \subset \Omega_2$ such that

$$\gamma_2((A_n)_{x_1} \setminus A(x_1)) = 0 \ \text{ for each } n \ \text{ and } \ \lim_n \gamma_2((A_n)_{x_1}) = \gamma_2(A(x_1)).$$

Define

$$A = \bigcup_{x_1 \in D_1} (\{x_1\} \times (A_{n(x_1)})_{x_1}) \cup \bigcup_{x_1 \in D_2} (\{x_1\} \times A(x_1)).$$

Fix any integer $n \geq 1$.

Claim. $\gamma(A_n \setminus A) = 0$.

Firstly, $N = C_\infty \cup (\bigcup_{1 \leq k \leq m < n} (B_m \setminus C_k))$ is γ_1 null. Secondly, if $x_1 \in D_1$ and $n(x_1) \leq n$ then $k(x_1) \leq n(x_1) < n$ and so $x_1 \in N$; whereas if $n(x_1) \geq n$ then clearly $(A_n \setminus A)_{x_1} = \emptyset$. Thirdly, if $x_1 \in D_2$ then by choice of $A(x_1)$ we have $\gamma_2(A_n \setminus A)_{x_1} = 0$. These three observations prove the claim.

Claim. $\lim_n \gamma(A_n) = \gamma(A)$ (or equivalently, in view of the earlier claim, $\lim_n \gamma(A \setminus A_n) = 0$.)

To see this, fix any $k \geq 1$. Set $E_k = \bigcup_{i=1}^k \bigcup_{n=i}^k (B_n \setminus C_i) \cup C_k$. Then firstly, $\gamma_1(E_k) = \gamma_1(C_k) \leq 1/2^{k-1}$. Secondly, suppose $x_1 \notin E_k$. Then $k(x_1) \leq k$ and $n(x_1) > k$.

If $x_1 \in D_1$ then using the fact that $x_1 \notin B_i$ for $k \leq i < n(x_1)$ we get

$$\begin{aligned} \gamma_2(A \setminus A_k)_{x_1} &= \gamma_2((A_{n(x_1)})_{x_1} \setminus (A_k)_{x_1}) \\ &\leq \sum_{k \leq i < n(x_1)} \gamma_2((A_{i+1})_{x_1} \setminus (A_i)_{x_1}) < \frac{1}{2^{k-1}}. \end{aligned}$$

If $x_1 \in D_2$ then pick $l > k$ (depending on x_1) so that $\gamma_2(A(x_1) \setminus (A_l)_{x_1}) < 1/2^{k-1}$. Using the fact that $x_1 \notin B_i$ for all $i \geq k(x_1)$ we conclude that

$$\gamma_2(A \setminus A_k)_{x_1} \leq \sum_{k \leq i < l} \gamma_2((A_{i+1})_{x_1} \setminus (A_i)_{x_1}) + \gamma_2(A(x_1) \setminus (A_l)_{x_1}) < \frac{1}{2^{k-2}}.$$

These two observations show that $\gamma(A \setminus A_k) \leq 1/2^{k-3}$, proving the claim.
This completes the proof of the theorem.

Remark 8. If $L_1(\gamma_1 \times \gamma_2)$ is complete then $L_1(\gamma_2)$ need not be complete. In fact if γ_1 is 0-1 valued and not countably additive then $L_1(\gamma_1 \times \gamma_2)$ is complete for any γ_2. To see this, we verify condition 4 of Theorem 1.

Let $\{A_n\}$ be a sequence of subsets of $\Omega_1 \times \Omega_2$ and $\varepsilon > 0$. Fix a partition $N_1, N_2, \cdots$ of Ω_1 such that $\gamma_1(N_i) = 0$ for each i. Fix $n \geq 1$. Observe that $\gamma_1\{x : \gamma_2(A_n)_x < \gamma(A_n) + \varepsilon/2^{n+1}\} = 1$ where $\gamma = \gamma_1 \times \gamma_2$. If the set in braces is denoted by C_n then take

$$B_n = \Big\{(x, y) : x \in C_n,\ x \notin \bigcup_{k \leq n} N_k\Big\} \cap A_n.$$

We show that these sets satisfy condition 4 of Theorem 1. Since $A_n \setminus B_n \subset C_n^c \cup (\bigcup_{k\leq n} N_k) \times \Omega_2$, we have $\gamma(A_n \setminus B_n) = 0$. To see $\gamma(\bigcup B_n) \leq \sum \gamma(B_n)$ we first observe that for any fixed x there is exactly one k such that $x \in N_k$. So if $n \geq k$, then $(B_n)_x$ is empty. Therefore, for all x there exists a k, depending on x, such that $\gamma_2(\bigcup_{n=1}^{\infty} B_n)_x = \gamma_2(\bigcup_{n=1}^{k} B_n)_x \leq \sum_{n=1}^{k} \gamma_2(B_n)_x$. Hence,

$$\begin{aligned} \gamma(\cup_{n=1}^{\infty} B_n) &= \int \gamma_2 \Big(\bigcup_{n=1}^{\infty} B_n \Big)_x d\gamma_1(x) \\ &\leq \int \sum_{n=1}^{\infty} \gamma_2(B_n)_x \, d\gamma_1(x) \leq \int \sum_{n=1}^{\infty} \gamma_2(A_n)_x 1_{C_n}(x) \, d\gamma_1(x) \\ &< \int \sum_{n=1}^{\infty} (\gamma(A_n) + \varepsilon/2^{n+1}) \, d\gamma_1 = \sum_{n=1}^{\infty} \gamma(A_n) + \varepsilon/2. \end{aligned}$$

This proves condition 4 of Theorem 1 and we are done.

The same proof works even if γ_1 is a purely finitely additive probability which is a combination of uniformly singular 0-1 valued measures. The same argument shows that if γ is the product $\bigotimes_{i=1}^{k} \gamma_i$ and $L_1(\gamma)$ is complete then $L_1(\gamma_i)$ need not be complete for any $i > 1$.

7. Infinite strategic products. For each $n \geq 1$, let γ_n be a finitely additive probability defined on the power set of Ω. Let $H = \Omega^{\infty}$. Then there is a unique—subject to certain regularity conditions—finitely additive probability σ on the Borel σ-field of H (each coordinate space Ω has discrete topology) $\sigma = \bigotimes_{n\geq 1} \gamma_n$. For definition and properties see [18]. This σ is called the *strategic product* of the γ_n's.

For this setup our results are only fragmentary. Following the arguments of Remark 8 one could show that if $L_1(\sigma)$ is complete then $L_1(\gamma_n)$ for $n > 1$ need not be complete. In fact $L_1(\sigma)$ is complete as soon as γ_1 is a combination of 0-1 valued uniformly singular purely finitely additive probabilities. Of course if $L_1(\sigma)$ is complete then $L_1(\gamma_1)$ is necessarily complete as in Theorem 9(b).

Suppose each γ_n is a fixed finitely additive probability γ, and $\sigma = \gamma^{\infty}$. Then as a consequence of what was said earlier, if $L_1(\sigma)$ is complete then $L_1(\gamma)$ is complete. We could not show that the completeness of $L_1(\gamma)$ implies that of $L_1(\sigma)$. However, we could establish this in special cases. For instance if $\Omega = \mathbb{Z}$ is the set of integers and γ is a combination of a countably additive probability and a sequence of uniformly singular 0-1 measures then $L_1(\sigma)$ is indeed complete. The argument depends on a certain identification of σ developed in [9]. We shall not give the details here. In particular, let δ_∞ (resp. $\delta_{-\infty}$) be a 0-1 purely finitely additive probability concentrated on the set of positive (resp. negative) integers and let $\gamma = \frac{1}{2}\delta_\infty + \frac{1}{2}\delta_{-\infty}$. Then $L_1(\sigma)$ is complete. This gives an example of a strongly continuous probability whose L_1 space is complete.

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REFERENCES

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges—a Study of Finitely Additive Measures*, Academic Press, 1983.

[2] R. Chen, *A finitely additive version of Kolmogorov's law of iterated logarithm*, Israel J. Math. 23 (1976), 209–220.

[3] —, *Some finitely additive versions of the strong law of large numbers*, ibid. 24 (1976), 244–259.

[4] L. E. Dubins and L. J. Savage, *How to Gamble if You Must*: *Inequalities for Stochastic Processes*, McGraw-Hill, 1965.

[5] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Interscience, 1958.

[6] S. Gangopadhyay, *On the completeness of $\mathcal{L}_p$-spaces over a charge*, Colloq. Math. 58 (1990), 291–300.

[7] S. Gangopadhyay and B. V. Rao, *Some finitely additive probability*: *random walks*, J. Theoret. Probab. 10 (1997), 643–657.

[8] —, —, *Strategic purely nonatomic random walks*, ibid. 11 (1998), 409–415.

[9] —, —, *On the Hewitt–Savage zero one law in the strategic setup*, preprint.

[10] J. W. Hagood, *A Radon–Nikodym theorem and L_p completeness for finitely additive vector measures*, J. Math. Anal. App. 113 (1986), 266–279.

[11] A. Halevy and M. Bhaskara Rao, *On an analogue of Komlos' theorem for strategies*, Ann. Probab. 7 (1979), 1073–1077.

[12] D. Heath and W. D. Sudderth, *On finitely additive priors*, *coherence*, *and extended admissibility*, Ann. Statist. 6 (1978), 333–345.

[13] —, —, *Coherent inference from improper priors and from finitely additive priors*, ibid. 17 (1989), 907–919.

[14] E. Hewitt and L. J. Savage, *Symmetric measures on Cartesian products*, Trans. Amer. Math. Soc. 80 (1955), 470–501.

[15] R. L. Karandikar, *A general principle for limit theorems in finitely additive probability*, ibid. 273 (1982), 541–550.

[16] —, *A general principle for limit theorems in finitely additive probability*: *the dependent case*, J. Multivariate Anal. 24 (1988), 189–206.

[17] D. A. Lane and W. D. Sudderth, *Diffuse models for sampling and predictive inference*, Ann. Statist. 6 (1978), 1318–1336.

[18] R. A. Purves and W. D. Sudderth, *Some finitely additive probability*, Ann. Probab. 4 (1976), 259–276.
[19] —, —, *Finitely additive zero-one laws*, Sankhyā Ser. A 45 (1983), 32–37.
[20] S. Ramakrishnan, *Finitely additive Markov chains*, Ph.D. thesis, Indian Statistical Institute, 1980.
[21] —, *Finitely additive Markov chains*, Trans. Amer. Math. Soc. 265 (1981), 247–272.
[22] —, *Central limit theorem in a finitely additive setting*, Illinois J. Math. 28 (1984), 139–161.
[23] —, *Potential theory for finitely additive Markov chains*, Stochastic Process. Appl. 16 (1984), 287–303.
[24] R. Ranga Rao, *A note on finitely additive measures*, Sankhyā Ser. A 18 (1958), 27–28.
[25] A. Sobczyk and P. C. Hammer, *A decomposition of additive set functions*, Duke Math. J. 11 (1944), 839–846.
[26] K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. 72 (1952), 46–66.

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