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HARMONIC FUNCTIONS ON THE REAL HYPERBOLIC BALL I: BOUNDARY VALUES AND ATOMIC DECOMPOSITION OF HARDY SPACES

BY

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Abstract. We study harmonic functions for the Laplace–Beltrami operator on the real hyperbolic space $\mathbb{B}_n$. We obtain necessary and sufficient conditions for these functions and their normal derivatives to have a boundary distribution. In doing so, we consider different behaviors of hyperbolic harmonic functions according to the parity of the dimension of the hyperbolic ball $\mathbb{B}_n$. We then study the Hardy spaces $H^p(\mathbb{B}_n)$, $0 < p < \infty$, whose elements appear as the hyperbolic harmonic extensions of distributions belonging to the Hardy spaces $H^p(\mathbb{S}^{n-1})$ of the sphere. In particular, we obtain an atomic decomposition of those spaces.

1. Introduction. In this article, we study the boundary behavior of harmonic functions on the real hyperbolic ball, partly with a view to establishing a theory of Hardy and Hardy–Sobolev spaces of such functions.

While studying Hardy spaces of Euclidean harmonic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{R}^n$, one is often led to consider estimates of these functions on balls with radius smaller than the distance from the center of the ball to the boundary $\mathbb{S}^{n-1}$ of $\mathbb{B}_n$. Thus hyperbolic geometry is implicitly used for the study of Euclidean harmonic functions, in particular when one considers boundary behavior. As Hardy spaces of Euclidean harmonic functions are the spaces of Euclidean harmonic extensions of distributions in the Hardy spaces on the sphere, it is tempting to study these last spaces directly through their hyperbolic harmonic extension.

Another source for this paper is the study of Hardy and Hardy–Sobolev spaces of $\mathcal{M}$-harmonic functions related to the complex hyperbolic metric

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on the unit ball, as presented in [1] and [3]. Our aim is to develop a similar theory in the case of the real hyperbolic ball. In the sequel, n is an integer, $n \geq 3$, and p a real number, $0 < p < \infty$.

Let $\mathrm{SO}(n,1)$ be the Lorentz group. It is well known that $\mathrm{SO}(n,1)$ acts conformally on $\mathbb{B}_n$. The corresponding Laplace–Beltrami operator, invariant for the action being considered, is given by

$$D = (1-|x|^2)^2 \Delta + 2(n-2)(1-|x|^2)N$$

with Δ the Euclidean laplacian and

$$N = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

the normal derivation operator. Functions u that are harmonic for this laplacian will be called *$\mathcal{H}$-harmonic*.

The "hyperbolic" Poisson kernel that solves the Dirichlet problem for D is defined for $x \in \mathbb{B}_n$ and $\xi \in \mathbb{S}^{n-1}$ by

$$\mathbb{P}_{\mathrm{h}}(x,\xi) = \left(\frac{1-|x|^2}{1+|x|^2 - 2\langle x,\xi\rangle}\right)^{n-1}.$$

With the help of this kernel, one can extend distributions on $\mathbb{S}^{n-1}$ to $\mathcal{H}$-harmonic functions on $\mathbb{B}_n$ in the same way as the Euclidean Poisson kernel extends distributions on $\mathbb{S}^{n-1}$ to Euclidean harmonic functions on $\mathbb{B}_n$. Our first concern is to determine which $\mathcal{H}$-harmonic functions are obtained in this way. We then study the boundary behavior of their normal derivatives. In doing so, we observe that, in odd dimensions, the normal derivatives of $\mathcal{H}$-harmonic functions behave similarly to $\mathcal{M}$-harmonic functions whereas they behave like Euclidean harmonic functions in even dimensions.

Finally, define $H^p(\mathbb{S}^{n-1})$ as $L^p(\mathbb{S}^{n-1})$ if $1 < p < \infty$ and as the real analog of Garnett–Latter's atomic H^p space if $p \leq 1$. Let $H^p(\mathbb{B}_n)$ be the space of Euclidean harmonic functions u on $\mathbb{B}_n$ such that $\zeta \mapsto \sup_{0<r<1} |u(r\zeta)| \in L^p(\mathbb{S}^{n-1})$. Garnett–Latter's theorem asserts that this space is exactly that of Euclidean harmonic extensions of distributions in $H^p(\mathbb{S}^{n-1})$. We prove here that the space $\mathcal{H}^p(\mathbb{B}_n)$ of $\mathcal{H}$-harmonic functions u such that $\zeta \mapsto \sup_{0<r<1} |u(r\zeta)| \in L^p(\mathbb{S}^{n-1})$ is the space of $\mathcal{H}$-harmonic extensions of distributions in $H^p(\mathbb{S}^{n-1})$.

This article is organized as follows: in Section 2 we present the setting of the problem and a few preliminary results. Section 3 is devoted to the study of the boundary behavior of $\mathcal{H}$-harmonic functions and concludes with the study of the behavior of their normal derivatives. Finally, in Section 4 we present the atomic decomposition theorem.

2. The setting

2.1. SO$(n,1)$ *and its action on* $\mathbb{B}_n$. Let $\mathrm{SO}(n,1) \subset \mathrm{GL}_{n+1}(\mathbb{R})$ $(n \geq 3)$ be the identity component of the group of matrices $g = (g_{ij})_{0\leq i,j\leq n}$ such that $g_{00} \geq 1$, $\det g = 1$ and which leave invariant the quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$.

Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$, $\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}_n = \{x \in \mathbb{R}^n : |x| = 1\}$. It is well known (cf. [13]) that SO$(n,1)$ acts conformally on $\mathbb{B}_n$. The action is given by $y = g.x$ with

$$y_p = \frac{\frac{1+|x|^2}{2} g_{p0} + \sum_{l=1}^{n} g_{pl} x_l}{\frac{1-|x|^2}{2} + \frac{1+|x|^2}{2} g_{00} + \sum_{l=1}^{n} g_{0l} x_l} \qquad \text{for } p = 1, \ldots, n.$$

The invariant measure on $\mathbb{B}_n$ is given by

$$d\mu = \frac{dx}{(1-|x|^2)^{n-1}} = \frac{r^{n-1} dr d\sigma}{(1-r^2)^{n-1}}$$

where dx is the Lebesgue measure on $\mathbb{B}_n$ and $d\sigma$ is the surface measure on $\mathbb{S}^{n-1}$.

We will need the following fact about this action (see [8]):

FACT 1. *Let* $g \in \mathrm{SO}(n,1)$ *and let* $x_0 = g.0$. *If* $0 < \varepsilon < 1/6$, *then*

$$B\left(x_0, \frac{\sqrt{2}}{8}(1-|x_0|^2)\varepsilon\right) \subset g.B(0,\varepsilon) \subset B(x_0, 6(1-|x_0|^2)\varepsilon).$$

2.2. *The invariant laplacian on* $\mathbb{B}_n$ *and the associated Poisson kernel.* From [13] we know that the invariant laplacian on $\mathbb{B}_n$ for the action being considered can be written as

$$D = (1-r^2)^2 \Delta + 2(n-2)(1-r^2) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and Δ is the Euclidean laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Note that D is given in radial-tangential coordinates by

$$D = \frac{1-r^2}{r^2}[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma]$$

with

$$N = r\frac{d}{dr} = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

and Δ_σ the tangential part of the Euclidean laplacian.

Definition. A function u on $\mathbb{B}_n$ is *$\mathcal{H}$-harmonic* if $Du = 0$ on $\mathbb{B}_n$.

The Poisson kernel that solves the Dirichlet problem associated with D is

$$\mathbb{P}_{\mathrm{h}}(r\eta,\xi) = \left(\frac{1-r^2}{1+r^2-2r\langle\eta,\xi\rangle}\right)^{n-1}$$

for $0 \le r < 1$ and $\eta,\xi \in \mathbb{S}^{n-1}$, i.e. for $r\eta \in \mathbb{B}_n$ and $\xi \in \mathbb{S}^{n-1}$.

Recall that the Euclidean Poisson kernel on the ball is given by

$$\mathbb{P}_{\mathrm{e}}(r\eta,\xi) = \frac{1-r^2}{(1+r^2-2r\langle\eta,\xi\rangle)^{n/2}}.$$

Notation. For a distribution φ on $\mathbb{S}^{n-1}$, we define $\mathbb{P}_{\mathrm{e}}[\varphi] : \mathbb{B}_n \to \mathbb{R}$ and $\mathbb{P}_{\mathrm{h}}[\varphi] : \mathbb{B}_n \to \mathbb{R}$ by

$$\mathbb{P}_{\mathrm{e}}[\varphi](r\eta) = \langle\varphi, \mathbb{P}_{\mathrm{e}}(r\eta,\cdot)\rangle, \qquad \mathbb{P}_{\mathrm{h}}[\varphi](r\eta) = \langle\varphi, \mathbb{P}_{\mathrm{h}}(r\eta,\cdot)\rangle$$

We call $\mathbb{P}_{\mathrm{e}}[\varphi]$ the *Poisson integral* of φ, and $\mathbb{P}_{\mathrm{h}}[\varphi]$ the *$\mathcal{H}$-Poisson integral* of φ.

Finally, $\mathcal{H}$-harmonic functions satisfy mean value equalities: let $a \in \mathbb{B}_n$ and $g \in \mathrm{SO}(n,1)$ be such that $g.0 = a$. Then, for every $\mathcal{H}$-harmonic function u,

$$u(a) = \frac{1}{\mu(B(0,r))} \int\limits_{g.B(0,r)} u(x)\,d\mu(x).$$

Thus, with Fact 1 and $d\mu = dx/(1-|x|^2)^{n-1}$, we see that

$$|u(a)| \le \frac{C}{(1-|a|^2)^n} \int\limits_{B(a,6(1-|a|^2)\varepsilon)} |u(x)|\,dx. \tag{2.1}$$

2.3. *Expansion of $\mathcal{H}$-harmonic functions in spherical harmonics*

Notation. For $a \in \mathbb{R}$, write $(a)_k = \Gamma(a+k)/\Gamma(a)$; hence $(a)_0 = 1$ and $(a)_k = a(a+1)\ldots(a+k-1)$ if $k = 1,2,\ldots$ For real parameters a,b,c, ${}_2F_1$ denotes Gauss' *hypergeometric function*

$${}_2F_1(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k.$$

Let $F_l(x) = {}_2F_1(l, 1-n/2, l+n/2; x)$ and $f_l(x) = F_l(x)/F_l(1)$. (See [5] for properties of ${}_2F_1$.)

Remark. If $n > 2$ is even, then $1 - n/2$ is a negative integer, thus ${}_2F_1(l, 1-n/2, l+n/2; r^2)$ is a polynomial in r of degree n.

In [11]–[13], the spherical harmonic expansion of $\mathcal{H}$-harmonic functions has been obtained. Another proof based on [1] can be found in [8]. We have the following:

THEOREM 1. *Let u be an $\mathcal{H}$-harmonic function of class $\mathcal{C}^2$ on $\mathbb{B}_n$. Then the spherical harmonic expansion of u is given by*

$$u(r\zeta) = \sum_l F_l(r^2) u_l(r\zeta),$$

where this series is absolutely convergent and uniformly convergent on every compact subset of $\mathbb{B}_n$.

Moreover, if $\varphi \in \mathcal{C}(\mathbb{S}^{n-1})$, then the Dirichlet problem $Du = 0$ in $\mathbb{B}_n$ and $u = \varphi$ on $\mathbb{S}^{n-1}$ has a unique solution $u \in \mathcal{C}(\overline{\mathbb{B}}_n)$ given by

$$u(z) = \int_{\mathbb{S}^{n-1}} \varphi(\zeta) \mathbb{P}_{\mathrm{h}}(z,\zeta)\, d\sigma(\zeta) = \mathbb{P}_{\mathrm{h}}[\varphi](z),$$

also given by

$$u(r\zeta) = \sum_l f_l(r^2) r^l \varphi_l(\zeta)$$

where $\varphi = \sum_l \varphi_l$ is the spherical harmonic expansion of φ.

3. Boundary values of $\mathcal{H}$-harmonic functions. In this chapter we prove results about the behavior on the boundary of $\mathcal{H}$-harmonic functions and their normal derivatives. For $\mathcal{H}$-harmonic functions, the results are similar to those for Euclidean harmonic functions. On the other hand, for the normal derivatives of $\mathcal{H}$-harmonic functions, the boundary behavior depends on the dimension of the space.

3.1. *Definition of Hardy spaces*

NOTATION. For a function u defined on $\mathbb{B}_n$, define the *radial maximal function* $\mathcal{M}[u] : \mathbb{S}^{n-1} \to \mathbb{R}_+$ by

$$\mathcal{M}[u](\zeta) = \sup_{0<t<1} |u(t\zeta)|.$$

We now study $\mathcal{H}^p$ spaces of $\mathcal{H}$-harmonic functions defined as follows:

DEFINITION. Let $0 < p < \infty$. Let $\mathcal{H}^p$ be the space of $\mathcal{H}$-harmonic functions u such that $\mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})$, endowed with the "norm"

$$\|u\|_{\mathcal{H}^p} = \|\mathcal{M}u\|_{L^p(\mathbb{S}^{n-1})} = \|\sup_{0<t<1} |u(t)|\,\|_{L^p(\mathbb{S}^{n-1})}.$$

We call $\mathcal{H}^p$ the *Hardy space of $\mathcal{H}$-harmonic functions.*

REMARK. If $0 < p < 1$, the map $u \mapsto \|u\|_{\mathcal{H}^p}$ is not a norm; however, the map $u, v \mapsto \|u - v\|_{\mathcal{H}^p}$ defines a metric on $\mathcal{H}^p$. In the sequel, by abuse of language we often call $\|\cdot\|_{\mathcal{H}^p}$ a norm whatever p might be.

DEFINITION. A function u on $\mathbb{B}_n$ is said to have a *distribution boundary value* if for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, the limit

$$\lim_{r\to 1} \int_{\mathbb{S}^{n-1}} u(r\zeta) \Phi(\zeta)\, d\sigma(\zeta)$$

exists. In case u is $\mathcal{H}$-harmonic, this is equivalent to the existence of a distribution f such that $u = \mathbb{P}_{\mathrm{h}}[f]$.

3.2. *Boundary distributions of functions in $\mathcal{H}^p$.* In this section, we are going to characterize boundary values of functions in $\mathcal{H}^p$. The characterizations we obtain are similar to those for harmonic functions on $\mathbb{R}_+^{n+1}$ or for $\mathcal{M}$-harmonic functions. The proofs are inspired by [1] and [6].

The first result concerns functions in $\mathcal{H}^p$, $p \geq 1$.

PROPOSITION 2. *Let u be an $\mathcal{H}$-harmonic function.*

(1) *If $1 < p < \infty$, then*

$$\sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |u(r\zeta)|^p \, d\sigma(\zeta) < \infty$$

if and only if there exists $f \in L^p(\mathbb{S}^{n-1})$ such that $u = \mathbb{P}_{\mathrm{h}}[f]$.

(2) *For $p = 1$,*

$$\sup_{0<r<1} \int_{\mathbb{S}^{n-1}} |u(r\zeta)| \, d\sigma(\zeta) < \infty$$

if and only if there exists a measure μ on $\mathbb{S}^{n-1}$ such that $u = \mathbb{P}_{\mathrm{h}}[\mu]$.

Proof. Assume that $u = \mathbb{P}_{\mathrm{h}}[f]$ with $f \in L^p(\mathbb{S}^{n-1})$. As

$$\|\mathbb{P}_{\mathrm{h}}(r\zeta, \cdot)\|_{L^1(\mathbb{S}^{n-1})} = 1,$$

Hölder's inequality gives

$$|u(r\zeta)|^p \leq \int_{\mathbb{S}^{n-1}} \mathbb{P}_{\mathrm{h}}(r\zeta, \xi)|f(\xi)|^p \, d\sigma(\xi) = \int_{\mathbb{S}^{n-1}} \mathbb{P}_{\mathrm{h}}(\zeta, r\xi)|f(\xi)|^p \, d\sigma(\xi),$$

and integration in ζ and Fubini lead to the desired result.

Conversely, if the $L^p(\mathbb{S}^{n-1})$ norms of $\zeta \mapsto u(r\zeta)$ are uniformly bounded, there exists a sequence $r_m \to 1$ and a function $\varphi \in L^p$ such that $u(r_m\zeta) \to \varphi(\zeta)$ $*$-weakly and thus weakly in $L^p(\mathbb{S}^{n-1})$. But then, for $r\zeta \in \mathbb{B}_n$ fixed,

$$\begin{aligned}
\mathbb{P}_{\mathrm{h}}[\varphi](r\zeta) &= \lim_{m\to\infty} \int_{\mathbb{S}^{n-1}} \mathbb{P}_{\mathrm{h}}(r\zeta, \xi) u(r_m\xi) \, d\sigma(\xi) \\
&= \lim_{m\to\infty} \sum_{l\geq 0} \frac{F_l(r_m^2)}{F_l(1)} r_m^l \int_{\mathbb{S}^{n-1}} \mathbb{P}_{\mathrm{h}}(r\zeta, \xi) u_l(\xi) \, d\sigma(\xi) \\
&= \lim_{m\to\infty} \sum_{l\geq 0} \frac{F_l(r_m^2)}{F_l(1)} r_m^l f_l(r) r^l u_l(\zeta) \\
&= \sum_{l\geq 0} f_l(r) r^l u_l(\zeta) = u(r\zeta).
\end{aligned}$$

The proof in the case $p = 1$ is obtained in a similar fashion using the duality $(L^1, \mathcal{M}(\mathbb{S}^{n-1}))$. ■

We are now going to prove that an $\mathcal{H}$-harmonic function has a boundary distribution if and only if it satisfies a certain growth condition. For this, we need the folowing lemma ([1], Lemma 10).

LEMMA 3. *Let* $F \in \mathcal{C}^2([1/2,1])$ *and* $h \in \mathcal{C}^1([1/2,1])$. *Assume that*

$$F''(x) + \frac{h(x)}{1-x} F'(x) = O((1-x)^{-\alpha})$$

as $x \to 1$. *Then*

(1) *If* $\alpha > 2$ *then* $F(x) = O((1-x)^{-\alpha+1})$.
(2) *If* $1 < \alpha < 2$ *then* $\lim_{x\to 1} F(x)$ *exists.*

We are now in a position to prove

THEOREM 4. *Let* u *be an* $\mathcal{H}$*-harmonic function. Then* u *admits a boundary value in the sense of distributions if and only if there exists a constant* A *such that*

$$u(r\zeta) = O((1-r)^{-A}).$$

Proof. Recall that

$$D = \frac{1-r^2}{r^2}[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma]. \tag{3.1}$$

Assume that $Du = 0$ and that $u(r\zeta) = O((1-r)^{-A})$. Let $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and let

$$F(r) = \int_{\mathbb{S}^{n-1}} u(r\zeta)\varphi(\zeta)\,d\sigma(\zeta).$$

Formula (3.1) with $Du = 0$ tells us that

$$(1-r^2)N^2F + (n-2)(1+r^2)NF + (1-r^2)\Delta_\sigma F = 0$$

where

$$\Delta_\sigma F(r) = \int_{\mathbb{S}^{n-1}} \Delta_\sigma u(r\zeta)\varphi(\zeta)\,d\sigma(\zeta) = \int_{\mathbb{S}^{n-1}} u(r\zeta)\Delta_\sigma^*\varphi(\zeta)\,d\sigma(\zeta)$$

with Δ_σ^* the adjoint operator to Δ_σ. Recall that $N = r\frac{d}{dr}$ and thus

$$r^2F''(r) + \frac{(n-1)+(n-3)r^2}{1-r^2} rF'(r) + \Delta_\sigma F = 0. \tag{3.2}$$

Write $\psi = -\Delta_\sigma^*\varphi$ and let T be the differential operator

$$T = r^2\frac{d^2}{dr^2} + \frac{(n-1)+(n-3)r^2}{1-r^2} r\frac{d}{dr}$$

so that equation (3.2) reads

$$TF(r) = \int_{\mathbb{S}^{n-1}} u(r\zeta)\psi(\zeta)\,d\sigma(\zeta).$$

One then immediately deduces the existence for $k=1,2,\dots$ of a function $\psi_k\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ such that

$$T^kF(r)=\int\limits_{\mathbb{S}^{n-1}}u(r\zeta)\psi_k(\zeta)\,d\sigma(\zeta).$$

But we assumed that $u(r\zeta)=O((1-r)^{-A})$. We thus have

$$T^kF(r)=O((1-r)^{-A})$$

and applying Lemma 3 we obtain

$$T^{k-1}F(r)=O((1-r)^{-A+1}).$$

Therefore, starting from T^k with $k=[A]+1$ and iterating the process k times, one deduces that $\lim_{r\to 1}F(r)$ exists.

Conversely, if u admits a boundary distribution f, then $u=\mathbb{P}_{\mathrm{h}}[f]$, i.e. $u(r\zeta)=\langle f,\mathbb{P}_{\mathrm{h}}(r\zeta,\cdot)\rangle$. But as f is a compactly supported distribution, it is of finite order, and thus there exists $k\geq 0$ such that

$$|u(r\zeta)|=|\langle f,\mathbb{P}_{\mathrm{h}}(r\zeta,\cdot)\rangle|\leq C\|\nabla_\xi^k\mathbb{P}_{\mathrm{h}}(r\zeta,\cdot)\|_{L^\infty}\leq\frac{C}{(1-r)^{n-1+k}},$$

which gives the desired estimate. ■

PROPOSITION 5. *Let $0<p<\infty$ and u be an $\mathcal{H}$-harmonic function. Assume that*

$$\sup_{0<r<1}\int\limits_{\mathbb{S}^{n-1}}|u(r\zeta)|^p\,d\sigma(\zeta)<\infty.$$

Then there exists a constant C such that for every $a\in\mathbb{B}_n$,

$$|u(a)|\leq\frac{C}{(1-|a|)^{(n-1)/p}}.$$

In particular, u has a boundary distribution f, i.e. $u=\mathbb{P}_{\mathrm{h}}[f]$.

Proof. The mean value inequality implies that

$$|u(a)|^p\leq\frac{C}{(1-|a|)^n}\int\limits_{B(a,(1-|a|)\varepsilon)}|u(x)|^p\,dx$$

for ε small enough. But $B(a,(1-|a|)\varepsilon)\subset\{r\zeta:(1-\varepsilon)(1-|a|)\leq 1-r\leq(1+\varepsilon)(1-|a|)\}$ and thus

$$\begin{aligned}|u(a)|^p&\leq\frac{C}{(1-|a|)^n}\int\limits_{1-(1+\varepsilon)(1-|a|)}^{1-(1-\varepsilon)(1-|a|)}\int\limits_{\mathbb{S}^{n-1}}|u(r\zeta)|^p\,d\sigma(\zeta)\,r^{n-1}\,dr\\&\leq\frac{C}{(1-|a|)^{n-1}}.\end{aligned}$$ ■

REMARK.Theorem 4 is well known. It has been proved by J. B. Lewis [10] in the case of symmetric spaces of rank 1 and eigenvectors of the Laplace–Beltrami operator (for arbitrary eigenvalues), and further generalized by E. P. van den Ban and H. Schlichtkrull [2].

3.3. *Distribution boundary values of $\mathcal{H}$-harmonic functions*

NOTATION. For $1 \le i, j \le n$, $i \ne j$, let

$$\mathcal{L}_{i,j} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.$$

Then the $\mathcal{L}_{i,j}$'s commute and commute with N. Further, if u is $\mathcal{H}$-harmonic, then $\mathcal{L}_{i,j}$ is also $\mathcal{H}$-harmonic. Finally, N and $\{\mathcal{L}_{i,j}\}_{1\le i\ne j\le n}$ generate ∇^k outside a neighborhood of the origin.

Recall that $Du = 0$ if and only if

$$(1 - r^2)N^2u + (n-2)(1 + r^2)Nu + (1 - r^2)\Delta_\sigma u = 0. \tag{3.3}$$

Apply N^{k-1} on both sides of this equality and isolate terms of order $k+1$ and k:

$$\begin{aligned}(1 - r^2)N^{k+1}u &- 2(k-1)r^2N^ku + (n-2)(1 + r^2)N^ku \\ &= r^2\sum_{j=0}^{k-3}\binom{k-1}{j}2^{k-j-1}N^{j+2}u + r^2\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}N^j\Delta_\sigma u \\ &\quad - (n-2)r^2\sum_{j=0}^{k-2}\binom{k-1}{j}2^{k-j-1}N^{j+1}u - (1 - r^2)N^{k-1}\Delta_\sigma u.\end{aligned} \tag{3.4}$$

We are now in a position to prove the following lemma:

LEMMA 6. *Let u be an $\mathcal{H}$-harmonic function with a boundary distribution. Let $\mathbb{Y}$ be a product of operators of the form $\mathcal{L}_{i,j}$ and let $\mathbb{X} = N^k\mathbb{Y}$. If $k \le n-2$, then $\mathbb{X}u$ has a distribution boundary value in the sense that*

$$\lim_{r\to 1}\int_{\mathbb{S}^{n-1}} \mathbb{X}u(r\zeta)\Phi(\zeta)\,d\sigma(\zeta)$$

exists for every function $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. If $k = n-1$, the previous integral is $O(\log 1/(1-r))$; in particular,

$$\lim_{r\to 1}(1 - r^2)\int_{\mathbb{S}^{n-1}} \mathbb{X}u(r\zeta)\Phi(\zeta)\,d\sigma(\zeta) = 0.$$

REMARK 1. If u has a boundary distribution, then so does $\mathcal{L}_{i,j}u$.

REMARK 2. As ∇^k is generated outside a neighborhood of the origin by operators of the form $N^l\mathbb{Y}$ where $\mathbb{Y}$ is a product of at most $k - l$ operators of the form $\mathcal{L}_{i,j}$, we deduce from the lemma that if $k \le n-2$, then ∇^k has a boundary distribution, whereas

$$\int_{\mathbb{S}^{n-1}} \nabla^{n-1} u(r\zeta)\Phi(\zeta)\,d\sigma(\zeta)$$

has *a priori* logarithmic growth.

Proof of Lemma 6. Proceed by induction on k. Fix $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and let $\mathbb{Y}$ be a product of operators of the form $\mathcal{L}_{i,j}$. Let

$$\psi_k(r) = \int_{\mathbb{S}^{n-1}} N^k \mathbb{Y} u(r\zeta)\Phi(\zeta)\,d\sigma(\zeta), \qquad 0 < r < 1.$$

Applying $\mathbb{Y}$ to formula (3.4) and noticing that $\mathbb{Y}$ and N commute, the induction hypothesis implies that the function

$$g(r) = (1-r^2)N\psi_k(r) - 2(k-1)r^2\psi_k(r) + (n-2)(1+r^2)\psi_k(r) \tag{3.5}$$

has a limit L as $r \to 1$.

But solving the differential equation (3.5) (recall that $N = rd/dr$), we get

$$\psi_k(r) = \lambda \frac{(1-r^2)^{n-k-1}}{r^{n-2}} + \frac{1}{r^{n-2}}(1-r^2)^{n-k-1} \int_0^r \frac{g(s)s^{n-3}}{(1+s)^{n-k}}(1-s)^{-(n-k-1)-1}\,ds.$$

Thus, if $k < n-1$, we deduce that $\psi_k(r)$ has limit $L/(2(n-k-1))$, whereas if $k = n-1$, then $\psi_k(r)$ has logarithmic growth. ∎

REMARK. We will show at the end of this section that if n is even, then the growth of $N^{n-1}u$ can be better than logarithmic, whereas if n is odd, only constant functions have growth better than logarithmic.

COROLLARY 7. *Let P_k be the sequence of polynomials defined by $P_0 = 2(n-1)$, $P_1 = 0$ and, for $2 \le k \le n$,*

$$P_k(X) = 2^{k-1}(k-1)! \sum_{j=2}^{k-2} \frac{n(j-1)-(n-2)k}{2^j(n-j-1)(k-j+1)!(j-1)!} P_j(X) + 2^{k-2}(k-1)! \sum_{j=2}^{k-3} \frac{1}{2^j(n-j-1)(k-j-1)!j!} X P_j(X) + 2^{k-1}X.$$

Then, for every $\mathcal{H}$-harmonic function u having a distribution boundary value, and for every $1 \le k \le n-2$, we have $N^k u = \frac{1}{2(n-k-1)} P_k(\Delta_\sigma)u$ as boundary distributions, i.e. for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$\lim_{r\to 1} \int_{\mathbb{S}^{n-1}} \left(N^k u(r\zeta) - \frac{1}{2(n-k-1)} P_k(\Delta_\sigma) u(r\zeta) \right) \Phi(\zeta)\,d\sigma(\zeta) = 0.$$

Proof. For convenience, write

$$Q_k = \frac{1}{2(n-k-1)} P_k.$$

As $n \geq 3$, for an $\mathcal{H}$-harmonic u having a boundary distribution, formula (3.3) and Lemma 6 imply that $Nu = 0$ on the boundary, hence the result for $k = 1$.

Next, notice that $N^k u = Q_k(\Delta_\sigma)u$ on the boundary implies $\Delta_\sigma N^k u = \Delta_\sigma Q_k(\Delta_\sigma)u$ on the boundary.

Assume now that $N^j u = Q_j(\Delta_\sigma)u$ on the boundary for $j \leq k-1$. If $k \leq n-2$, Lemma 6 tells us that $(1-r^2)N^{k+1}u = 0$ on the boundary and that $(1-r^2)N^{k-1}\Delta_\sigma u = 0$ on the boundary. Formula (3.4) then gives, as $r \to 1$,

$$\begin{aligned}(-2(k-1) + 2(n-2))N^k u = \sum_{j=0}^{k-3} \binom{k-1}{j} 2^{k-j-1} N^{j+2} u \\ + \sum_{j=0}^{k-3} \binom{k-1}{j} 2^{k-j-1} N^j \Delta_\sigma u \\ - (n-2) \sum_{j=0}^{k-2} \binom{k-1}{j} 2^{k-j-1} N^{j+1} u.\end{aligned}$$

But, by the induction hypothesis, $N^j u = Q_j(\Delta_\sigma)u$ and with the previous remark $N^j \Delta_\sigma u = \Delta_\sigma N^j u = \Delta_\sigma Q_j(\Delta_\sigma)u$. Therefore

$$\begin{aligned}(-2(k-1) + 2(n-2))N^k u = \sum_{j=0}^{k-3} \binom{k-1}{j} 2^{k-j-1} Q_{j+2}(\Delta_\sigma) u \\ + \sum_{j=0}^{k-3} \binom{k-1}{j} 2^{k-j-1} \Delta_\sigma Q_j(\Delta_\sigma) u \\ - (n-2) \sum_{j=0}^{k-2} \binom{k-1}{j} 2^{k-j-1} Q_{j+1}(\Delta_\sigma) u.\end{aligned}$$

Finally, using $Q_0 = 1$ and $Q_1 = 0$ and grouping terms, we get the desired result. ■

Remark 1. One easily sees that P_k is a polynomial of degree $[k/2]$ and that for $k \geq 2$, P_k has no constant term.

Remark 2. According to Corollary 7, $Nu = 0$ on the boundary. On the other hand, an easy computation leads to $DNu = -4(n-2)Nu$, i.e. Nu is an eigenvector of D for an eigenvalue of the form $(s^2-1)(n-1)^2$ (with $s = (n-3)/(n-1)$), thus $(s+1)(n-1)/2 = n-2 \in \mathbb{N}^*$. This is precisely the case where it is impossible to reconstruct Nu with the help of a convolution by a power of the Poisson kernel (see [13]).

REMARK 3. The fact that for every $\mathcal{H}$-harmonic function u, $Nu = 0$ on the boundary is in strong contrast with Euclidean harmonic functions. Actually, if v is a Euclidean harmonic function on $\mathbb{B}_n$, and if $Nv = 0$ on the boundary, then v is a constant.

3.4. *Boundary distribution of the* $(n-1)$*th derivative.* In this section we prove that, in odd dimensions, normal derivatives of $\mathcal{H}$-harmonic functions have a boundary behavior similar to the complex case of $\mathcal{M}$-harmonic functions as exhibited in [3] (with pluriharmonic functions playing the role of constant functions), whereas, in even dimensions, the behavior is similar to the Euclidean harmonic case.

THEOREM 8. *Assume n is odd. Let u be an $\mathcal{H}$-harmonic function having a boundary distribution. The following assertions are equivalent*:

(1) *u is a constant*,

(2) *$N^{n-1}u$ has a boundary distribution*,

(3) $\int_{\mathbb{S}^{n-1}} N^{n-1}u(r\zeta)\Phi(\zeta)\,d\sigma(\zeta) = o(\log 1/(1-r))$ *for all* $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$.

Assume now n is even. If $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, then $\mathbb{P}_{\mathrm{h}}[\varphi] \in \mathcal{C}^\infty(\overline{\mathbb{B}}_n)$. In particular, *if u is $\mathcal{H}$-harmonic with a boundary distribution*, *then for every $k \geq 0$, $N^k u$ has a boundary distribution.*

Proof. Assume first n is odd. The implications (1)⇒(2) and (2)⇒(3) being obvious, let us prove (3)⇒(1). Theorem 1 tells us that an $\mathcal{H}$-harmonic function u admits an expansion in spherical harmonics

$$u(r\zeta) = \sum_{l\geq 0} f_l(r^2) r^l u_l(\zeta) \tag{3.6}$$

where u_l is a spherical harmonic of degree l and f_l is the hypergeometric function

$$\begin{aligned} f_l(x) &= \frac{{}_2F_l(l, 1-n/2, l+n/2; x)}{{}_2F_l(l, 1-n/2, l+n/2; 1)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(l+k)\Gamma(1-n/2+k)\Gamma(l+n/2)\Gamma(1)}{\Gamma(l)\Gamma(1-n/2)\Gamma(l+n/2+k)\Gamma(1+k)} x^k. \end{aligned}$$

Moreover the sum (3.6) converges uniformly on compact subsets of $\mathbb{B}_n$; in particular,

$$\|u_l\|_{L^2(\mathbb{S}^{n-1})} f_l(r^2) r^l = \int_{\mathbb{S}^{n-1}} u(r\zeta) u_l(\zeta)\,d\sigma(\zeta).$$

On the other hand, if $l \neq 0$ then as n is odd,

$$\frac{\Gamma(l+k)\Gamma(1-n/2+k)\Gamma(l+n/2)\Gamma(1)}{\Gamma(l)\Gamma(1-n/2)\Gamma(l+n/2+k)\Gamma(1+k)} = \frac{\Gamma(l+n/2)\Gamma(1)}{\Gamma(l)\Gamma(1-n/2)}\frac{1}{k^n}\left[1+O\left(\frac{1}{k}\right)\right].$$

Thus the first $n-2$ derivatives of F_l have a limit as $x \to 1$, whereas the $(n-1)$st derivative grows like $\log(1-x)$ as $x \to 1$, so (3) implies that $u_l = 0$ for $l \neq 0$, that is, u is constant.

Assume now n is even and write $n = 2p$. If $\varphi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, then φ admits a decomposition in spherical harmonics $\varphi = \sum_{l=0}^\infty \varphi_l$ with $\|\varphi_l\|_\infty = O(l^{-\alpha})$ for every $\alpha > 0$ ([14], Appendix C). But then

$$\mathbb{P}_{\mathrm{h}}[\varphi](r\zeta) = \sum_{l=0}^\infty f_l(r) r^l \varphi_l(\zeta)$$

with

$$f_l(r) r^l = \frac{{}_2F_1(l, 1-p, l+p; r^2)}{{}_2F_1(l, 1-p, l+p; 1)} r^l = \frac{\Gamma(l+2p-1)\Gamma(p)}{\Gamma(l+p)\Gamma(2p-1)} \sum_{j=0}^p \frac{(l)_j(1-p)_j}{(l+p)_j j!} r^{2j+l}.$$

But, for every $k \geq 0$,

$$N^k\left(\sum_{j=0}^p \frac{(l)_j(1-p)_j}{(l+p)_j j!} r^{2j+l}\right) = \sum_{j=0}^p \frac{(l)_j(1-p)_j}{(l+p)_j j!}(2j+l)^k 2^k r^{2j+l}.$$

Therefore $N^k(f_l r^l)(1) = O(l^{k+p-1})$. But $\|\varphi_l\|_\infty = O(l^{-(k+p+1)})$ and thus $\sum_{l=0}^\infty N^k f_l(r)\varphi_l(\zeta)$ converges uniformly on $\overline{\mathbb{B}}_n$ and $\mathbb{P}_{\mathrm{h}}[\varphi] \in \mathcal{C}^\infty(\overline{\mathbb{B}}_n)$.

The fact that for an $\mathcal{H}$-harmonic u with a boundary distribution, $N^k u$ also has a boundary distribution, then results from the symmetry of the Poisson kernel: $\mathbb{P}_{\mathrm{h}}(r\zeta, \xi) = \mathbb{P}_{\mathrm{h}}(r\xi, \zeta)$. ∎

Remark 1. Normal derivatives of $\mathcal{H}$-harmonic functions have two opposite behaviors depending on the dimension of $\mathbb{B}_n$. In odd dimensions, the behavior is similar to the complex case (see [3]; in this case, the analog of constant functions is pluriharmonic functions). On the other hand, in even dimensions, the behavior is similar to that of Euclidean harmonic functions.

Remark 2. The similarity to the Euclidean case can also be seen in a different way. In [13], the following link between Euclidean harmonic functions and $\mathcal{H}$-harmonic functions has been proved:

Lemma 9. *For every $\mathcal{H}$-harmonic function u, there exists a unique Euclidean harmonic function v such that $v(0) = 0$ and*

$$u(r\zeta) = u(0) + \int_0^1 v(rt\zeta)[(1-t)(1-tr^2)]^{n/2-1} \frac{dt}{t}$$

for every $0 \le r < 1$ and every $\zeta \in \mathbb{S}^{n-1}$.

Moreover, let $f = \sum_l u_l$ be the spherical harmonics expansion of $f \in L^2(\mathbb{S}^{n-1})$. If

$$g = \sum_l \frac{\Gamma(l+n-1)}{\Gamma(n-1)\Gamma(l)} u_l,$$

then Lemma 9 links $u = \mathbb{P}_{\mathrm{h}}[f]$ to $v = \mathbb{P}_{\mathrm{e}}[g]$.

But, if $f = \sum_l u_l \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and g is as above, then, as $\|u_l\|_\infty = O(l^{-\alpha})$ for every $\alpha > 0$, we have $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and thus $v = \mathbb{P}_{\mathrm{e}}[g] \in \mathcal{C}^\infty(\overline{\mathbb{B}}_n)$.

Moreover, if n is even then $(1-tr^2)^{n/2-1}$ is a polynomial and is therefore $\mathcal{C}^\infty$, hence we find again that $u \in \mathcal{C}^\infty(\overline{\mathbb{B}}_n)$.

On the other hand, if n is odd, we find again the $n-1$ obstacle since the highest order term of $(1-t)^{n/2-1}N^k(1-tr^2)^{n/2-1}$ is

$$(1-t)^{n/2-1}(1-tr^2)^{n/2-1-k} \simeq (1-t)^{n-2-k}$$

as $r \to 1$, and since $(1-t)^{n-2-k}$ is not integrable for $k \ge n-1$.

4. Atomic decomposition of $\mathcal{H}^p$ spaces. In this section we prove that $\mathcal{H}^p$ spaces admit an atomic decomposition. In 4.2 we define $\mathcal{H}^p_{\mathrm{at}}$ and show that this space is included in $\mathcal{H}^p$. Conversely, we have seen in the previous section that $\mathcal{H}$-harmonic functions in $\mathcal{H}^p$ are obtained by $\mathcal{H}$-Poisson integration of distributions on $\mathbb{S}^{n-1}$, hence they are extensions of distributions from $\mathbb{S}^{n-1}$ to $\mathbb{B}_n$. Another means to extend a distribution on $\mathbb{S}^{n-1}$ to $\mathbb{B}_n$ is integration with respect to the Euclidean Poisson kernel. In 4.1 we study the links between these two extensions, which allows us in 4.3 to deduce the inclusion $\mathcal{H}^p \subset \mathcal{H}^p_{\mathrm{at}}$ from the atomic decomposition of H^p spaces of Euclidean harmonic functions.

4.1. *Links between Euclidean harmonic functions and $\mathcal{H}$-harmonic functions.* We now prove a "converse" to Lemma 9.

Lemma 10. *There exists a function $\eta : [0,1] \times [0,1] \to \mathbb{R}^+$ such that*

(i) $\mathbb{P}_{\mathrm{e}}(r\zeta, \xi) = \int_0^1 \eta(r,\varrho)\mathbb{P}_{\mathrm{h}}(\varrho r\zeta, \xi)\, d\varrho$,

(ii) *there exists a constant C such that $\int_0^1 \eta(r,\varrho)\, d\varrho \le C$ for every $r \in [0,1]$.*

Proof. Note that

$$\frac{1}{(x+y)^{n/2}} = c_n \int_0^\infty \frac{z^{n/2-2}}{(x+y+z)^{n-1}}\,dz.$$

Writing $X = 2(1-\langle\zeta,\xi\rangle)$, with an obvious abuse of language, we then get

$$\begin{aligned}\mathbb{P}_{\rm e}(r,X) &= \frac{1-r^2}{((1-r)^2+rX)^{n/2}} = \frac{1-r^2}{r^{n/2}}\frac{1}{[(1-r)^2/r+X]^{n/2}}\\ &= \frac{1-r^2}{r^{n/2}}c_n\int_0^\infty \frac{z^{n/2-2}}{[X+(1-r)^2/r+z]^{n-1}}\,dz.\end{aligned}$$

The change of variable

$$z = \frac{(1-\varrho)^2}{\varrho} - \frac{(1-r)^2}{r} = \frac{(r-\varrho)(1-\varrho r)}{\varrho r}$$

leads to

$$\begin{aligned}&\mathbb{P}_{\rm e}(r,X)\\ &\quad= \frac{1-r^2}{r^{n/2}}c_n\int_0^r \frac{[(r-\varrho)(1-\varrho r)]^{n/2-2}}{[X+(1-\varrho)^2/\varrho]^{n-1}(\varrho r)^{n/2-2}}\frac{1-\varrho^2}{\varrho^2}\,d\varrho\\ &\quad= \frac{1-r^2}{r^{n-2}}c_n\int_0^r \frac{[(r-\varrho)(1-\varrho r)]^{n/2-2}(1-\varrho^2)}{[\varrho X+(1-\varrho^2)]^{n-1}\varrho^{1-n/2}}\,d\varrho\\ &\quad= \frac{1-r^2}{r^{n-2}}c_n\int_0^r \mathbb{P}_{\rm h}(\varrho,X)(1-\varrho^2)^{2-n}[(r-\varrho)(1-\varrho r)]^{n/2-2}\varrho^{n/2-1}\,d\varrho\\ &\quad= c_n(1-r^2)\int_0^1 \mathbb{P}_{\rm h}(rs,X)(1-r^2s^2)^{2-n}[(1-s)(1-sr^2)]^{n/2-2}s^{n/2-1}\,ds.\end{aligned}$$

We thus obtain (i) with

$$\eta(r,s) = c_n(1-r^2)(1-r^2s^2)^{2-n}[(1-s)(1-sr^2)]^{n/2-2}s^{n/2-1}.$$

Of course, $\eta \ge 0$ and one easily checks that $\int_0^1 \eta(r,s)\,ds \le C$, since $n \ge 3$. ∎

Corollary 11. *Let η be the function defined in Lemma* 10. *Let f be a distribution on $\mathbb{S}^{n-1}$ and let $u = \mathbb{P}_{\rm h}[f]$ and $v = \mathbb{P}_{\rm e}[f]$. Then u and v are linked by*

$$v(r\zeta) = \int_0^1 \eta(r,s)u(rs\zeta)\,ds.$$

In particular, if $u \in \mathcal{H}^p$, then $v \in H^p(\mathbb{B}_n)$ and $\|v\|_{H^p(\mathbb{B}_n)} \le C\|u\|_{\mathcal{H}^p}$.

4.2. *The inclusion* $\mathcal{H}^p_{\rm at}\subset\mathcal{H}^p$

DEFINITION. A function a on $\mathbb{S}^{n-1}$ is called a *p-atom* on $\mathbb{S}^{n-1}$ if either a is a constant or a is supported in a ball $\widetilde{B}(\xi_0,r_0)$ and

(1) $|a(\xi)|\le\sigma[\widetilde{B}(\xi_0,r_0)]^{-1/p}$ for almost every $\xi\in\mathbb{S}^{n-1}$,
(2) for every function $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$\Big|\int\limits_{\mathbb{S}^{n-1}}a(\xi)\Phi(\xi)\,d\sigma(\xi)\Big|\le\|\nabla^{k(p)}\Phi\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}r_0^{k(p)}\sigma[\widetilde{B}(\xi_0,r_0)]^{1-1/p}$$

with $k(p)$ an integer strictly greater than $(n-1)(1/p-1)$.

PROPOSITION 12. *There exists a constant* C_p *such that, for every p-atom* a *on* $\mathbb{S}^{n-1}$, $A=\mathbb{P}_{\rm h}[a]$ *satisfies*

$$\|A\|_{\mathcal{H}^p(\mathbb{B}_n)}\le C_p.$$

Proof. Let a be a p-atom on $\mathbb{S}^{n-1}$, with support in $\widetilde{B}(\xi_0,r_0)$. We estimate

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}}\sup_{t\in|0,1|}\Big|\int\limits_{\widetilde{B}(\xi_0,r_0)}&\mathbb{P}_{\rm h}(t\zeta,\xi)a(\xi)\,d\sigma(\xi)\Big|^p d\sigma(\zeta)\\
&=\int\limits_{\widetilde{B}(\xi_0,cr_0)}\sup_{t\in[0,1]}\Big|\int\limits_{\widetilde{B}(\xi_0,r_0)}\mathbb{P}_{\rm h}(t\zeta,\xi)a(\xi)\,d\sigma(\xi)\Big|^p d\sigma(\zeta)\\
&\quad+\int\limits_{\mathbb{S}^{n-1}\setminus\widetilde{B}(\xi_0,cr_0)}\sup_{t\in[0,1]}\Big|\int\limits_{\widetilde{B}(\xi_0,r_0)}\mathbb{P}_{\rm h}(t\zeta,\xi)a(\xi)\,d\sigma(\xi)\Big|^p d\sigma(\zeta)\\
&=I_1+I_2\end{aligned}$$

with $c>1$ a constant. But, by Hölder's inequality,

$$\begin{aligned}I_1&=\int\limits_{\widetilde{B}(\xi_0,cr_0)}\sup_{t\in[0,1]}|\mathbb{P}_{\rm h}[a](t\zeta)|^p\,d\sigma(\zeta)\\
&\le c\sigma(\widetilde{B}(\xi_0,cr_0))^{1-p/2}\Big[\int\limits_{\widetilde{B}(\xi_0,cr_0)}\sup_{t\in[0,1]}|\mathbb{P}_{\rm h}[a](t\zeta)|^2\,d\sigma(\zeta)\Big]^{p/2}\\
&\le c\sigma(\widetilde{B}(\xi_0,cr_0))^{1-p/2}\|\mathbb{P}_{\rm h}[a]\|^p_{\mathcal{H}^2(\mathbb{B}_n)}\le c\sigma(\widetilde{B}(\xi_0,cr_0))^{1-p/2}\|a\|^p_{L^2(\mathbb{S}^{n-1})}\end{aligned}$$

since $\mathbb{P}_{\rm h}$ is bounded $L^2(\mathbb{S}^{n-1})\to\mathcal{H}^2(\mathbb{B}_n)$. Using property (1) of atoms, we see that

$$I_1\le C\bigg(\frac{\sigma(\widetilde{B}(\xi_0,cr_0))}{\sigma(\widetilde{B}(\xi_0,r_0))}\bigg)^{1-p/2}\le C_p.$$

Now we estimate I_2. Using property (2) of atoms, we have, for $\zeta\in\mathbb{S}^{n-1}\setminus\widetilde{B}(\xi_0,cr_0)$,

$$\Big|\int_{\widetilde{B}(\xi_0,r_0)} \mathbb{P}_{\rm h}(t\zeta,\xi)a(\xi)\,d\sigma(\xi)\Big|^p \le r_0^{pk(p)}\|\nabla_\xi^{k(p)}\mathbb{P}_{\rm h}(t\zeta,\xi)\|_{L^\infty}^p\sigma(\widetilde{B}(\xi_0,r_0))^{p-1}$$
$$\le C_p r_0^{pk(p)}(1-t^2)^{n-1} \times \sup_{\xi\in\widetilde{B}(\xi_0,r_0)}\frac{1}{d(\zeta,\xi)^{p(n+k(p)-1)}}\sigma(\widetilde{B}(\xi_0,r_0))^{p-1}$$

and thus

$$I_2 \le C_p r_0^{pk(p)}\sigma(\widetilde{B}(\xi_0,r_0))^{p-1}\int_{\mathbb{S}^{n-1}\setminus\widetilde{B}(\xi_0,cr_0)}\sup_{\xi\in\widetilde{B}(\xi_0,r_0)}\frac{1}{d(\zeta,\xi)^{p(n+k(p)-1)}}\,d\sigma(\zeta)$$
$$\le C_p\frac{r_0^{pk(p)}r_0^{(n-1)(p-1)}}{r_0^{[p(1+k(p)/n-1)-1](n-1)}},$$

since $p(n+k(p)-1) > n-1$, i.e. $k(p) > (n-1)(1/p-1)$. Hence $I_2 \le C_p$. ∎

Remark 1. Condition (2) implies with $\Phi = 1$ that

$$\int_{\mathbb{S}^{n-1}} a(\xi)\,d\sigma(\xi) = 0.$$

Remark 2. Condition (2) is equivalent to the *a priori* weaker condition:

(2′) For every spherical harmonic P of degree $\le k(p)$,

$$\Big|\int_{\mathbb{S}^{n-1}} a(\xi)P(\xi)\,d\sigma(\xi)\Big| \le \|\nabla^{k(p)}P\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}r_0^{k(p)}\sigma[\widetilde{B}(\xi_0,r_0)]^{1-1/p}.$$

Indeed, assume this condition is satisfied and let $\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$. There exists P, a linear combination of spherical harmonics of degree $\le k(p)$, and $R\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ such that

(1) $\Phi = P + R$,

(2) $\|R\|_{L^\infty(\widetilde{B}(\xi_0,r_0))} \le C_p r_0^{k(p)}\|\nabla^{k(p)}\Phi\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}$.

Then

$$\Big|\int_{\mathbb{S}^{n-1}} a(\xi)\Phi(\xi)\,d\sigma(\xi)\Big| \le \Big|\int_{\mathbb{S}^{n-1}} a(\xi)P(\xi)\,d\sigma(\xi)\Big| + \Big|\int_{\mathbb{S}^{n-1}} a(\xi)R(\xi)\,d\sigma(\xi)\Big|$$
$$\le C_p\|\nabla^{k(p)}P\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}r_0^{k(p)}[\sigma(\widetilde{B}(\xi_0,r_0))]^{1-1/p}$$
$$+\|a\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}\|R\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}\sigma(\widetilde{B}(\xi_0,r_0))$$
$$\le C\|\nabla^{k(p)}\Phi\|_{L^\infty(\widetilde{B}(\xi_0,r_0))}r_0^{k(p)}[\sigma(\widetilde{B}(\xi_0,r_0))]^{1-1/p}.$$

We could also impose the following weaker condition:

(3) For every spherical harmonic P of degree $\leq k(p)$,

$$\Big|\int_{\mathbb{S}^{n-1}} a(\xi)P(\xi)\,d\sigma(\xi)\Big| = 0.$$

We would then obtain a stronger atomic decomposition theorem. However, this version is sufficient for our needs. It is also more intrinsic, the estimates we impose are directly those that are needed in the proof and finally it allows us to stay close to the proof in [9].

DEFINITION. A function A on $\mathbb{B}_n$ is called an $\mathcal{H}^p$-*atom* on $\mathbb{B}_n$ if there exists a p-atom a on $\mathbb{S}^{n-1}$ such that $A = \mathbb{P}_{\mathrm{h}}[a]$. We define $\mathcal{H}^p_{\mathrm{at}}(\mathbb{B}_n)$ as the space of distributions u on $\mathbb{B}_n$ such that there exists:

(1) a sequence $(A_j)_{j=1}^{\infty}$ of $\mathcal{H}_p$-atoms on $\mathbb{B}_n$,
(2) a sequence $(\lambda_j)_{j=1}^{\infty} \in \ell^p$ such that

$$u = \sum_{j=1}^{\infty} \lambda_j A_j, \tag{4.1}$$

with uniform convergence on compact subsets of $\mathbb{B}_n$. We write

$$\|u\|_{\mathcal{H}^p_{\mathrm{at}}} = \inf\Big\{\Big(\sum_{i=1}^{\infty} |\lambda_j|^p\Big)^{1/p}\Big\}$$

where the infimum is taken over all decompositions of u of the form (4.1).

PROPOSITION 13. *For* $0 < p \leq 1$, $\mathcal{H}^p_{\mathrm{at}}(\mathbb{B}_n) \subset \mathcal{H}^p(\mathbb{B}_n)$ *and there exists a constant* C_p *such that for every* $u \in \mathcal{H}^p_{\mathrm{at}}(\mathbb{B}_n)$,

$$\|u\|_{\mathcal{H}^p} \leq C_p \|u\|_{\mathcal{H}^p_{\mathrm{at}}}.$$

Proof. It is *mutatis mutandis* the proof of Theorem 2.2 of [9].

Let $\varepsilon > 0$, let $u = \sum_{j=1}^{\infty} \lambda_j A_j$ be a function in $\mathcal{H}^p_{\mathrm{at}}$ and take an atomic decomposition such that $\sum_{i=1}^{\infty} |\lambda_j|^p \leq (1+\varepsilon)\|u\|^p_{\mathcal{H}^p_{\mathrm{at}}}$.

Property (2) of atoms implies that

$$\begin{aligned}|\nabla^k A_j(x)| &= |\nabla^k \mathbb{P}_{\mathrm{h}}[a_j]| \\ &\leq \|\nabla_\xi^{k(p)} \nabla_x^k \mathbb{P}_{\mathrm{h}}(x,\cdot)\|_{L^\infty(B(\xi_0,r_0))} r_0^{k(p)} \sigma(B(\xi_0,r_0))^{1-1/p} \\ &\leq \frac{C_{p,k}}{(1-|x|)^{k_{p,l}}}.\end{aligned}$$

The series $\sum_{j=1}^{\infty} \lambda_j \nabla^k A_j(x)$ converges uniformly on every compact subset of $\mathbb{B}_n$, thus $\sum_{j=1}^{\infty} \lambda_j A_j(x)$ defines an $\mathcal{H}$-harmonic function on $\mathbb{B}_n$.

Moreover,

$$\Big|\sum_{j=1}^{\infty} \lambda_j A_j(x)\Big|^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p |A_j(x)|^p.$$

Therefore

$$\int_{\mathbb{S}^{n-1}} \sup_{0<r<1} \Big| \sum_{j=1}^{\infty} \lambda_j A_j(r\zeta) \Big|^p d\sigma(\zeta) \le \int_{\mathbb{S}^{n-1}} \sup_{0<r<1} \sum_{j=1}^{\infty} |\lambda_j|^p |A_j(r\zeta)|^p \, d\sigma(\zeta)$$

$$\le C_p^p \sum_{j=1}^{\infty} |\lambda_j|^p \le (1+\varepsilon)^p C_p^p \|u\|^p_{\mathcal{H}^p_{\text{at}}},$$

which means that $\|u\|_{\mathcal{H}^p} \le C\|u\|^p_{\mathcal{H}^p_{\text{at}}}$. ■

4.3. *The inclusion* $\mathcal{H}^p \subset \mathcal{H}^p_{\text{at}}$. We use here the fact that the space $H^p(\mathbb{B}_n)$ of Euclidean harmonic functions v such that $\mathcal{M}[v] \in L^p(\mathbb{S}^{n-1})$ admits an atomic decomposition, i.e. for every function $v \in H^p$, there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^p$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of p-atoms on $\mathbb{S}^{n-1}$ such that

$$v(r\zeta) = \sum_{k\in\mathbb{N}} \lambda_k \mathbb{P}_{\text{e}}[a_k](r\zeta). \tag{4.2}$$

and moreover,

$$\|v\|_{H^p} \simeq \Big(\sum_{k\in\mathbb{N}} |\lambda_k|^p \Big)^{1/p}.$$

This result is well known, however it seems difficult to find an adequate reference. One may for instance adapt the proof of Garnett and Latter [7] as outlined in [4].

Let $u \in \mathcal{H}^p$. Then u admits a boundary distribution f and $u = \mathbb{P}_{\text{h}}[f]$. Then let $v = \mathbb{P}_{\text{e}}[f]$. By Lemma 10, $v \in H^p(\mathbb{B}_n)$. Thus v admits an atomic decomposition, i.e. there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \in \ell^p$ and a sequence $(a_k)_{k\in\mathbb{N}}$ of p-atoms on $\mathbb{S}^{n-1}$ such that v is given by (4.2). So

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

in the sense of distributions. Therefore $u = \mathbb{P}_{\text{h}}[\sum \lambda_k a_k] = \sum \lambda_k \mathbb{P}_{\text{h}}[a_k]$, the series being convergent in $\mathcal{H}^p$ by Proposition 13.

We have thus proved the following theorem:

THEOREM 14. *For every* $0 < p \le 1$, $\mathcal{H}^p = \mathcal{H}^p_{\text{at}}$ *and the norms are equivalent.*

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