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INVARIANT OPERATORS ON FUNCTION SPACES ON HOMOGENEOUS TREES

BY

MICHAEL COWLING (SYDNEY, N.S.W.) STEFANO MEDA (MILANO) AND ALBERTO G. SETTI (COMO)

A *homogeneous tree* $\mathfrak{X}$ of degree $q+1$ is a connected graph with no loops in which each vertex is adjacent to $q+1$ others. We assume that $q \geq 2$. The tree $\mathfrak{X}$ has a natural measure, counting measure, and a natural distance d, viz. $d(x,y)$ is the number of edges between vertices x and y. Let o be a fixed but arbitrary reference point in $\mathfrak{X}$, and let G_o be the stabiliser of o in the isometry group G of $\mathfrak{X}$. We write $|x|$ for $d(x,o)$. The map $g \mapsto g\cdot o$ identifies the coset space G/G_o with $\mathfrak{X}$; thus a function f on $\mathfrak{X}$ gives rise to a G_o-invariant function f' on G by the formula $f'(g) = f(g\cdot o)$, and every G_o-invariant function arises in this way. A function f on $\mathfrak{X}$ is said to be *radial* if $f(x)$ depends only on $|x|$, or equivalently, if f is G_o-invariant, or f' is G_o-bi-invariant. We endow the totally disconnected group G with the Haar measure such that the mass of the open subgroup G_o is 1. The reader may find much more on the group G in the book of Figà-Talamanca and Nebbia [FTN].

We denote by $|E|$ the measure of a subset E of a measure space. We write $\mathfrak{S}_n$ for $\{x \in \mathfrak{X} : |x| = n\}$. Clearly, $|\mathfrak{S}_0| = 1$, and $|\mathfrak{S}_n| = (q+1)q^{n-1}$ when $n \in \mathbb{Z}^+$. We pick points $w_0, w_1, w_2, \ldots$ in $\mathfrak{X}$ such that $|w_d| = d$. A radial function f on $\mathfrak{X}$ is determined by its restriction to these points.

It is well known that G-invariant linear operators from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$ correspond to linear operators from $L^p(G/G_o)$ to $L^r(G/G_o)$ given by convolution on the right by G_o-bi-invariant kernels. We denote by $\mathrm{Cv}_p^r(\mathfrak{X})$ the space of radial functions on $\mathfrak{X}$ associated to these G_o-bi-invariant kernels. The norm of an element k of $\mathrm{Cv}_p^r(\mathfrak{X})$ is then defined as the norm of the corresponding operator from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$, and denoted by $|||k|||_{p;r}$. Equipped

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with this norm, $\mathrm{Cv}_p^r(\mathfrak{X})$ is a Banach space. We note that the maps $f \mapsto f'$ and $f' \mapsto \mathcal{E}f'$ given by the formulae

$$f'(g) = f(g \cdot o), \quad \mathcal{E}f'(g \cdot o) = \int_{G_o} f'(gg_1)\, dg_1 \quad \forall g \in G$$

are isometric from $L^p(\mathfrak{X})$ into $L^p(G)$ and norm-decreasing from $L^p(G)$ into $L^p(\mathfrak{X})$, for all p in $[1,\infty]$. It follows that the norm of an element k in $\mathrm{Cv}_p^r(\mathfrak{X})$ is equal to the norm of its G_o-bi-invariant extension k' to G in $\mathrm{Cv}_p^r(G)$, the space of convolution operators from $L^p(G)$ to $L^r(G)$.

For any function space $E(\mathfrak{X})$ on $\mathfrak{X}$, we denote by $E(\mathfrak{X})^\sharp$ the (usually closed) subspace of $E(\mathfrak{X})$ of radial functions. We denote by $L^{p,r}(\mathfrak{X})$ the standard Lorentz space, as in Bergh and Löfström [BL]. Pytlik [Py] proved that, given p and r in $[1,\infty)$, a radial function f belongs to $L^{p,r}(\mathfrak{X})$ if and only if the function $d \mapsto f(w_d)|\mathfrak{S}_d|^{1/p}$ is in $L^r(\mathbb{N})$, and

$$\Big[\sum_{d\in\mathbb{N}} |f(w_d)|^r |\mathfrak{S}_d|^{r/p}\Big]^{1/r} \sim \|f\|_{p,r}. \tag{2}$$

The key to the proof is that $|\mathfrak{S}_d|$ grows exponentially in d. Pytlik used this lemma to show that $L^{p,1}(\mathfrak{X})^\sharp \subseteq \mathrm{Cv}_p^p(\mathfrak{X}) \subseteq L^p(\mathfrak{X})^\sharp$, and that the cone of positive radial convolution operators on $L^p(\mathfrak{X})$ coincides with the cone of positive functions in $L^{p,1}(\mathfrak{X})$.

In this paper, we first outline "spherical harmonic analysis" on G, and then prove some general theorems on $\mathrm{Cv}_p^r(\mathfrak{X})$. In particular, we generalise results of Pytlik [Py] and of C. Nebbia [N, Thm. 2].

1. Notation and preliminaries. We write τ for $2\pi/\log q$, and define $\mathbb{T}$ to be the torus $\mathbb{R}/\tau\mathbb{Z}$, usually identified with the interval $[-\tau/2,\tau/2)$. We denote by $\mathcal{F}$ the Fourier transformation on $\mathbb{Z}$, given by

$$\mathcal{F}F(s) = \sum_{d\in\mathbb{Z}} F(d)\, q^{-ids} \quad \forall s \in \mathbb{T}.$$

Clearly, $\mathcal{F}F(s+\tau) = \mathcal{F}F(s)$. A distribution m on $\mathbb{T}$ is said to be in $M_p^r(\mathbb{T})$ if convolution with $\mathcal{F}^{-1}m$ defines a bounded operator from $L^p(\mathbb{Z})$ to $L^r(\mathbb{Z})$. We define $\mathcal{F}L^r(\mathbb{T})$ to be $\{\mathcal{F}F : F \in L^r(\mathbb{Z})\}$, and note that $\mathcal{F}L^r(\mathbb{T})$ is continuously included in $L^{r'}(\mathbb{T})$, by the classical Hausdorff–Young inequality, for r in $[1,2]$.

For p in $[1,\infty]$, let p', $\delta(p)$, $\mathbb{S}_p$ and $\overline{\mathbb{S}}_p$ denote $p/(p-1)$, $1/p-1/2$,

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| < |\delta(p)|\} \quad \text{and} \quad \{z \in \mathbb{C} : |\mathrm{Im}(z)| \le |\delta(p)|\}.$$

If f is holomorphic in $\mathbb{S}_p$, then $f_{\delta(p)}$ and $f_{-\delta(p)}$ denote its boundary functions $f(i\delta(p)+\cdot)$ and $f(-i\delta(p)+\cdot)$, when these exist distributionally. The letter C, sometimes with subscripts or superscripts, denotes a positive constant which may vary from place to place; it may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. Given functions A and B, defined on a set $\mathbb{D}$, we say that $A \sim B$ in $\mathbb{D}$ if there exist C and C' such that

$$CA(t) \leq B(t) \leq C'A(t) \qquad \forall t \in \mathbb{D}.$$

We conclude this section by summarising some features of spherical analysis on $\mathfrak{X}$. The theory parallels that of spherical analysis on a noncompact symmetric space of rank one. The Gel'fand pair (G, G_o) has associated spherical functions ϕ_z, parametrised by the complex number z. We refer to [CMS1] for explicit formulae, noting that our parametrisation differs from that used by some authors (e.g., [FTP] and [FTN]; our ϕ_z corresponds to their $\phi_{1/2+iz}$). The spherical Fourier transform $\widetilde{f}$ of f in $L^1(\mathfrak{X})^\sharp$ is defined by

$$\widetilde{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\, \phi_z(x) \qquad \forall z \in \overline{\mathbb{S}}_1.$$

Since $\phi_{z+\tau} = \phi_z$ and $\phi_z = \phi_{-z}$, $\widetilde{f}$ is even and τ-periodic in $\overline{\mathbb{S}}_1$. We say that a holomorphic function in a strip $\mathbb{S}_p$ is *Weyl-invariant* if it satisfies these conditions in $\mathbb{S}_p$.

We denote by μ the Plancherel measure on $\mathbb{T}$ [CMS1, (1.2)]. We note that the relation $\overline{\mathbf{c}(z)} = \mathbf{c}(-\overline{z})$ and the symmetry properties of spherical functions imply that

$$\phi_s(x) \frac{d\mu(s)}{ds} = c_G\, \mathbf{c}(-s)^{-1}\, q^{(is-1/2)|x|} + c_G\, \mathbf{c}(s)^{-1}\, q^{(-is-1/2)|x|},$$

for all x in $\mathfrak{X}$ and s in $\mathbb{T}$. Therefore, if $m : \mathbb{R} \to \mathbb{C}$ is even and τ-periodic, then

$$\begin{aligned}\int_{\mathbb{T}} m(s)\, \phi_s(x)\, d\mu(s) &= c_G \int_{\mathbb{T}} m(s)\, \mathbf{c}(-s)^{-1} q^{(is-1/2)|x|}\, ds \\ &\quad + c_G \int_{\mathbb{T}} m(s)\, \mathbf{c}(s)^{-1} q^{(-is-1/2)|x|}\, ds,\end{aligned}$$

and by changing the variable s to $-s$, we see that the two integrals on the right hand side are equal. In particular, if we set $\check{\mathbf{c}}(s) = \mathbf{c}(-s)$, we have

$$\begin{aligned}f(x) &= 2\, c_G \int_{\mathbb{T}} \widetilde{f}(s)\, \mathbf{c}(s)^{-1} q^{(-is-1/2)|x|}\, ds \\ &= 2\, c_G \int_{\mathbb{T}} \widetilde{f}(s)\, \check{\mathbf{c}}(s)^{-1} q^{(is-1/2)|x|}\, ds.\end{aligned} \tag{3}$$

In the following theorem, we use the results of [CMS2] on the range of the radial Abel transformation to characterise the spherical Fourier transforms of the radial functions in the Lorentz spaces $L^{p,r}(\mathfrak{X})$, and derive a version of the Hausdorff–Young inequality. For related results in the setting of noncompact symmetric spaces see [CGM].

THEOREM 1.1. *Suppose that* $1 \le p < 2$. *If* f *is in* $L^{p,r}(\mathfrak{X})^\sharp$, *then* $\widetilde{f}$ *extends to a Weyl-invariant holomorphic function in* $\mathbb{S}_p$, *with boundary functions* $\widetilde{f}_{\delta(p)}$ *and* $\widetilde{f}_{-\delta(p)}$ *in* $\mathcal{F}L^r(\mathbb{T})$. *If also* $1 \le r \le 2$, *then the map* $z \mapsto \widetilde{f}(z+\cdot)$ *is continuous from* $\overline{\mathbb{S}}_p$ *into* $L^{r'}(\mathbb{T})$, *and*

$$\Big[\int_{\mathbb{T}} |\widetilde{f}(z+s)|^{r'}\, ds\Big]^{1/r'} \le C\, \|f\|_{p,r} \qquad \forall z \in \overline{\mathbb{S}}_p.$$

Conversely, *if* f *is radial and* $\widetilde{f}$ *extends to a Weyl-invariant holomorphic function in* $\mathbb{S}_p$, *the map* $z \mapsto \widetilde{f}(z+\cdot)$ *is continuous from* $\overline{\mathbb{S}}_p$ *into the space of distributions on* $\mathbb{T}$, *and the boundary functions* $\widetilde{f}_{\delta(p)}$ *and* $\widetilde{f}_{-\delta(p)}$ *are in* $\mathcal{F}L^r(\mathbb{T})$, *then* f *is in* $L^{p,r}(\mathfrak{X})^\sharp$, *and*

$$\|f\|_{p,r} \le C\, \|\mathcal{F}^{-1}\widetilde{f}_{\delta(p)}\|_r.$$

P r o o f. Let $\mathcal{A}$ denote the Abel transformation on $\mathfrak{X}$; see [CMS2] for notation and discussion. We recall that, for sufficiently nice radial functions on $\mathfrak{X}$, the spherical Fourier transformation factors as $\widetilde{f} = \mathcal{F}(\mathcal{A}f)$. Further, by [CMS2, Thm. 2.5], $\mathcal{A}$ is a bicontinuous isomorphism of $L^{p,r}(\mathfrak{X})^\sharp$ onto the space $q^{-\delta(p)|\cdot|}\, L^r(\mathbb{Z})$, for any p in $[1,2)$ and r in $[1,+\infty)$. Thus, if f is in $L^{p,r}(\mathfrak{X})^\sharp$, it follows from the definition of $\mathcal{F}$ that $\widetilde{f}$ extends to a holomorphic function on the strip $\mathbb{S}_p$ with the required continuity properties, and with boundary functions in $\mathcal{F}L^r(\mathbb{T})$. Moreover, from the classical Hausdorff–Young inequality,

$$\begin{aligned}\Big[\int_{\mathbb{T}} |\mathcal{F}(\mathcal{A}f)(z+s)|^{r'}\, ds\Big]^{1/r'} &\le C\|q^{\mathrm{Im}(z)(\cdot)}\mathcal{A}f\|_{L^r(\mathbb{Z})}\\ &\le C\|q^{\delta(p)|\cdot|}\mathcal{A}f\|_{L^r(\mathbb{Z})} \le C\|f\|_{p,r} \qquad \forall z \in \overline{\mathbb{S}}_p.\end{aligned}$$

Conversely, assume that $\widetilde{f}$ has the stated properties. By Cauchy's Theorem,

$$\mathcal{A}f(h) = \mathcal{F}^{-1}(\widetilde{f}(h)) = \frac{1}{\tau}\int_{\mathbb{T}} \widetilde{f}(s+i\delta(p))\, q^{i(s+i\delta(p))h}\, ds = q^{-\delta(p)h}\mathcal{F}^{-1}(\widetilde{f}_{\delta(p)}).$$

Since $\mathcal{F}^{-1}(\widetilde{f}_{\delta(p)})$ is in $L^r(\mathbb{Z})$ by assumption, and $\mathcal{A}f$ is even, $\mathcal{A}f$ is in $q^{-\delta(p)|\cdot|}\, L^r(\mathbb{Z})$; the required norm inequality follows from (2). ∎

2. On radial convolutors. Recall that $\mathrm{Cv}_p^r(\mathfrak{X})$ denotes the space of radial kernels which convolve $L^p(\mathfrak{X})$ into $L^r(\mathfrak{X})$. In this section, we apply the results of the previous section to study these spaces.

The spherical Fourier transforms of the elements of the space $\mathrm{Cv}_p^r(\mathfrak{X})$ are called *spherical* L^p-L^r *Fourier multipliers*, or L^p *Fourier multipliers* if $p = r$. It is easy to see that the Clerc–Stein condition [CS] for spherical L^p multipliers on noncompact symmetric spaces holds in the present situation.

Thus a spherical L^p Fourier multiplier extends to a bounded holomorphic function on $\mathbb{S}_p$ [CMS1, Thm. 1.3], and

$$\sup_{z\in\mathbb{S}_p} |\widetilde{k}(z)| \le |||k|||_p \quad \forall k \in \mathrm{Cv}_p^p(\mathfrak{X}).$$

The symmetry properties of spherical functions imply the Weyl-invariance of spherical L^p-L^r multipliers in their strip of holomorphy. The following theorem, which may be proved using Theorem 1.1, generalises the Clerc–Stein condition.

THEOREM 2.1. *Suppose that $1 \le p < 2$ and $1 \le r \le s \le \infty$, and that k is a radial function on $\mathfrak{X}$. The following conditions are equivalent*:

(i) *$\widetilde{k}$ extends to a holomorphic function on $\mathbb{S}_p$, and the map $z \mapsto \widetilde{k}(z+\cdot)$ extends to a continuous map from $\overline{\mathbb{S}}_p$ into the space of distributions on $\mathbb{T}$, and $\widetilde{k}_{\delta(p)}$ is in $M_r^s(\mathbb{T})$*;

(ii) *the operator of right convolution with k is bounded from $L^{p,r}(\mathfrak{X})^\sharp$ to $L^{p,s}(\mathfrak{X})^\sharp$.*

In particular, if k is in $\mathrm{Cv}_p^p(\mathfrak{X})$ then $\widetilde{k}_{\delta(p)}$ is in $M_p^p(\mathbb{T})$.

We omit the proof, since it is also an immediate corollary of [CMS2, Prop. 2.7]. Using Theorem 1.1 we moreover obtain the following.

THEOREM 2.2. *Suppose that p is in $[1, 2)$ and that k is a radial function on $\mathfrak{X}$ whose Fourier transform $\widetilde{k}$ is holomorphic on $\mathbb{S}_p$ and such that the map $z \mapsto \widetilde{k}(z+\cdot)$ is a continuous distribution-valued map on $\overline{\mathbb{S}}_p$.*

(i) *If $p > 1$ and $\widetilde{k}_{\delta(p)}$ is in $\mathcal{F}L^r(\mathbb{T})$, then right convolution with k is a bounded operator from $L^{p,s}(\mathfrak{X})$ into $L^{p,t}(\mathfrak{X})$, where $1/t = 1/r + 1/s - 1$. In particular, if $\widetilde{k}$ is in $H^\infty(\mathbb{S}_p)$, then right convolution with k is of weak type (p, p).*

(ii) *If $p > 1$ and $\widetilde{k}_{\delta(p)}$ is bounded and smooth in $\mathbb{C} \setminus \tau\mathbb{Z}$, and satisfies*

$$\left|\frac{d}{ds}\widetilde{k}_{\delta(p)}(s)\right| \le C\,|s|^{-1} \quad \forall s \in \mathbb{T},$$

then right convolution with k maps $L^{p,s}(\mathfrak{X})$ continuously into $L^{p,t}(\mathfrak{X})$ whenever $t > s$.

(iii) *If $\widetilde{k}$ is in $H^\infty(\mathbb{S}_1)$, then right convolution with k is of weak type $(1, 1)$, and of strong type (p, p) for every p in $(1, \infty)$.*

Proof. We claim that $L^{p,s}(\mathfrak{X}) * L^{p,r}(\mathfrak{X})^\sharp \subseteq L^{p,t}(\mathfrak{X})$ when $1 \le p < 2$, $1 \le r, s, t < \infty$, and $1+1/t = 1/r+1/s$. Indeed, $L^1(\mathfrak{X})*L^1(\mathfrak{X})^\sharp \subseteq L^1(\mathfrak{X})$, and Pytlik [Py] showed that if p is in $(1, 2)$, then $L^p(\mathfrak{X}) * L^{p,1}(\mathfrak{X})^\sharp \subseteq L^p(\mathfrak{X})$ (see also Theorem 2.4 below). The claim then follows by multilinear interpolation [BL, 3.13.5, p. 76].

Assume now that $\widetilde{k}_{\delta(p)}$ is in $\mathcal{F}L^r(\mathbb{T})$. By Theorem 1.1, k is in $L^{p,r}(\mathfrak{X})$, and the first statement in (i) follows from the claim above.

If $\widetilde{k}$ is in $H^\infty(\mathbb{S}_p)$, then $\widetilde{k}_{\delta(p)}$ is in $L^\infty(\mathbb{T})$ and *a fortiori* in $\mathcal{F}L^2(\mathbb{T})$. The second statement in (i) follows from the first.

Under hypothesis (ii), $\widetilde{k}_{\delta(p)}$ is in $\mathcal{F}L^r(\mathbb{T})$ when $r > 1$, and the result follows from (i).

Finally, assume that $\widetilde{k}$ is in $H^\infty(\mathbb{S}_1)$. By (i) and interpolation and duality, it suffices to prove that convolution with k is of weak type $(1,1)$. By (3), we see that

$$k(x) = 2c_G \int_{\mathbb{T}} \widetilde{f}(s)\, \mathbf{c}(s)^{-1} q^{(-is-1/2)|x|}\, ds;$$

by changing the contour of integration and inserting the value of c_G, we deduce that

$$k(x) = \frac{q \log q}{2\pi(q+1)}\, q^{-|x|} \int_{\mathbb{T}} \widetilde{f}(s-i/2)\mathbf{c}(s-i/2)^{-1} q^{-is|x|}\, ds.$$

We may therefore estimate

$$|k(x)| \le \frac{q}{q+1}\, q^{-|x|} \sup_{s\in\mathbb{T}} |\widetilde{f}(s-i/2)\, \mathbf{c}(s-i/2)^{-1}| \le \frac{q}{q-1}\, \|\widetilde{f}\|_\infty\, q^{-|x|}.$$

Now, according to R. Rochberg and M. Taibleson [RT], Green's operator (the inverse of the Laplacian) for a strongly reversible random walk on a tree of bounded degree is of weak type $(1,1)$. It is easily verified that the convolution kernel of Green's operator on a homogeneous tree of degree $q+1$ is given by

$$k(x) = \frac{q}{q-1}\, q^{-|x|},$$

and the required conclusion follows. ∎

We now focus on the Banach space $\mathrm{Cv}_p^r(\mathfrak{X})$ of radial convolutors from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$. First, we state the analogue of Herz's *principe de majoration* on trees. This is known, and may be found in a more general setting, for instance, in [Lo].

PROPOSITION 2.3. *Suppose that* $1 \le p \le 2$, *and that* k *belongs to* $\mathrm{Cv}_p^p(\mathfrak{X})$. *Then*

$$|||k|||_p \le |||\,|k|\,|||_p = |k|\widehat{\ }(i\delta(p)),$$

and equality holds if k *is nonnegative.*

Denote by $Y(\mathfrak{X})$ the Banach space of functions f on $\mathfrak{X}$ such that $\|f\|_Y < \infty$, where

$$\|f\|_Y = \sum_{d\in\mathbb{N}} (d+1) \Big(\sum_{x\in\mathfrak{S}_d} |f(w_d)|^2 \Big)^{1/2}.$$

Observe that $Y(\mathfrak{X})^\sharp \subset L^{2,1}(\mathfrak{X})^\sharp$; the inclusion is proper, from (2).

THEOREM 2.4. *Suppose that* $1 \le p, r \le \infty$. *Then* $\mathrm{Cv}_p^r(\mathfrak{X}) = \mathrm{Cv}_{r'}^{p'}(\mathfrak{X})$. *Further*,

(i) *if* $1 < p < 2$, *then* $L^{p,1}(\mathfrak{X})^\sharp \subseteq \mathrm{Cv}_p^p(\mathfrak{X}) \subseteq L^p(\mathfrak{X})^\sharp$, *and if* $k \ge 0$ *and* k *is in* $\mathrm{Cv}_p^p(\mathfrak{X})$, *then* k *belongs to* $L^{p,1}(\mathfrak{X})^\sharp$;

(ii) *if* $p = 2$, *then* $Y(\mathfrak{X})^\sharp \subseteq \mathrm{Cv}_2^2(\mathfrak{X}) \subseteq L^2(\mathfrak{X})^\sharp$, *and if* $k \ge 0$ *and* k *is in* $\mathrm{Cv}_2^2(\mathfrak{X})$, *then* k *belongs to* $Y(\mathfrak{X})^\sharp$;

(iii) *if* $1 \le p < r \le 2$, *then* $\mathrm{Cv}_p^r(\mathfrak{X}) = L^r(\mathfrak{X})^\sharp$;

(iv) *if* $1 \le p \le 2 \le r \le \infty$, *and* $r \ne p'$, *then* $\mathrm{Cv}_p^r(\mathfrak{X}) = L^{\min(p',r)}(\mathfrak{X})^\sharp$;

(v) *if* $1 < p < 2$, *then* $L^{p',p'/2}(\mathfrak{X})^\sharp \subseteq \mathrm{Cv}_p^{p'}(\mathfrak{X}) \subseteq L^{p'}(\mathfrak{X})^\sharp$.

REMARKS. Both inclusions in (i) and the right hand inclusion in (v) are strict. This follows from the study of the L^p-L^r mapping properties of the resolvent operator of the Laplacian [CMS1]. In addition, both inclusions in (ii) are strict. Indeed, the image of the space $Y(\mathfrak{X})^\sharp$ under the spherical Fourier transform is contained in the space of absolutely convergent Fourier series on $\mathbb{T}$, while the images of $\mathrm{Cv}_2^2(\mathfrak{X})$ and $L^2(\mathfrak{X})^\sharp$ coincide with $L^\infty(\mathbb{T})$ and $L^2(\mathbb{T},\mu)$ respectively. Finally, by considering nonnegative elements of $L^{2,1}(\mathfrak{X})^\sharp$ which are not in $Y(\mathfrak{X})^\sharp$, it may be seen that $L^{2,1}(\mathfrak{X})^\sharp$ is not contained in $\mathrm{Cv}_2^2(\mathfrak{X})$.

Proof (of Theorem 2.4). Observe that $\mathrm{Cv}_p^r(\mathfrak{X}) \subseteq L^r(\mathfrak{X})^\sharp$ since the point mass at o is in $L^p(\mathfrak{X})^\sharp$ for all p in $[1,\infty]$. Moreover, $\mathrm{Cv}_p^r(\mathfrak{X}) = \mathrm{Cv}_{r'}^{p'}(\mathfrak{X})$, with norm equality, by duality, and since $\mathfrak{X}$ is noncompact, $\mathrm{Cv}_p^r(\mathfrak{X})$ is nontrivial if and only if $p \le r$, by a theorem of Hörmander [Hö].

We first prove (i). As stated above, the left hand inclusion in (i) was proved in [Py]. We give a shorter proof. Since k is in $\mathrm{Cv}_p^p(\mathfrak{X})$ if $|k|$ is, it suffices to take k nonnegative. For these k, Herz's *principe* shows that

$$|\!|\!|k|\!|\!|_p = \widetilde{k}(i\delta(p)) = \sum_{d\in\mathbb{N}} k(w_d)\,\phi_{i\delta(p)}(w_d) \sim \sum_{d\in\mathbb{N}} k(w_d)\,|\mathfrak{S}_d|^{1/p} \sim \|k\|_{p,1},$$

as required. This completes the proof of (i). To prove (ii), we argue in a similar fashion.

Now we prove (iii). We have already observed that $\mathrm{Cv}_p^r(\mathfrak{X}) \subseteq L^r(\mathfrak{X})^\sharp$, so it suffices to show the reverse inclusion. For this, it suffices to prove that if k is in $L^r(\mathfrak{X})^\sharp$, then the map $f \mapsto f * k$ is bounded from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$; this follows from the radial form of the Kunze–Stein phenomenon on $\mathfrak{X}$ (see [N]).

We now prove (iv). Suppose that k is in $\mathrm{Cv}_p^r(\mathfrak{X})$; then it also belongs to $L^r(\mathfrak{X})$. Since $\mathrm{Cv}_p^r(\mathfrak{X}) = \mathrm{Cv}_{r'}^{p'}(\mathfrak{X})$, a similar argument shows that k is also in $L^{p'}(\mathfrak{X})^\sharp$, and hence in $L^{\min(p',q)}(\mathfrak{X})^\sharp$, showing that $\mathrm{Cv}_p^r(\mathfrak{X}) \subseteq L^{\min(p,r')}(\mathfrak{X})$.

To prove the converse, we consider two cases separately. Suppose first that $p < r'$, so that $L^r(\mathfrak{X}) = L^{\min(p',r)}(\mathfrak{X})$. Assume that k is in $L^r(\mathfrak{X})$. Let

f be in $L^p(\mathfrak{X})$ and h be in $L^{r'}(\mathfrak{X})$; denote by f', h', and k' respectively the G_o-right-invariant and G_o-bi-invariant extensions to G of f, h, and k. Then

$$\langle f * h, g\rangle_{\mathfrak{X}} = \langle f' * k', h'\rangle_G = \langle k', (f')^\star * h'\rangle_G,$$

where $(f')^\star(g) = \overline{(f')}(g^{-1})$. Since G has the Kunze–Stein property [N] and $1 \le p < r' < 2$,

$$\|(f')^\star * h'\|_{r'} \le C_{p,r'}\|(f')^\star\|_p\|h'\|_{r'} = C_{p,r'}\|f'\|_p\|h'\|_{r'} = C_{p,r'}\|f\|_p\|h\|_{r'}.$$

Thus, by Hölder's inequality,

$$\sup\{|\langle f * k, h\rangle_{\mathfrak{X}}| : \|f\|_p = 1,\ \|h\|_{r'} = 1\} \le C_{p,r'}\|k\|_r,$$

so that k is in $\mathrm{Cv}_p^r(\mathfrak{X})$, and

$$|\!|\!|k|\!|\!|_{p;r} \le C_{p,r'}\|h\|_r,$$

as required. The case where $r' < p$ is treated similarly.

Finally we prove (v). As before, the right inclusion is obvious. The left inclusion follows from the result [CMS2] that, if $1 < p < 2$, then $L^p(G) * L^p(G) \subseteq L^{p,r}(G)$, where $r = p/(2-p)$, much as (iv) follows from the Kunze–Stein phenomenon. The dual form of this sharp inclusion is the inclusion $L^p(G) * L^{p',r'}(G) \subseteq L^{p'}(G)$, where $r' = p'/2$; the desired result follows by specialising to functions with the appropriate invariance properties. ∎

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School of Mathematics
University of New South Wales
Sydney, NSW 2052, Australia
E-mail: m.cowling@unsw.edu.au

Facoltà di Scienze
Università dell'Insubria–Polo di Como
via Lucini 3
I-22100 Como, Italy
E-mail: setti@fis.unico.it

Dipartimento di Statistica
Università "Bicocca"
Edificio U7, II piano
viale Sarca 202
I-20100 Milano, Italy
E-mail: stemed@mate.polimi.it

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