INVARIANT OPERATORS ON FUNCTION SPACES
ON HOMOGENEOUS TREES

BY
MICHAEL COWLING (SYDNEY, N.S.W.)
STEFANO MEDA (MILANO)
AND
ALBERTO G. SETTI (COMO)

A homogeneous tree $X$ of degree $q + 1$ is a connected graph with no loops in which each vertex is adjacent to $q + 1$ others. We assume that $q \geq 2$. The tree $X$ has a natural measure, counting measure, and a natural distance $d$, viz. $d(x, y)$ is the number of edges between vertices $x$ and $y$. Let $o$ be a fixed but arbitrary reference point in $X$, and let $G_o$ be the stabiliser of $o$ in the isometry group $G$ of $X$. We write $|x|$ for $d(x, o)$. The map $g \mapsto g \cdot o$ identifies the coset space $G/G_o$ with $X$; thus a function $f$ on $X$ gives rise to a $G_o$-invariant function $f'$ on $G$ by the formula $f'(g) = f(g \cdot o)$, and every $G_o$-invariant function arises in this way. A function $f$ on $X$ is said to be radial if $f(x)$ depends only on $|x|$, or equivalently, if $f$ is $G_o$-invariant, or $f'$ is $G_o$-bi-invariant. We endow the totally disconnected group $G$ with the Haar measure such that the mass of the open subgroup $G_o$ is 1. The reader may find much more on the group $G$ in the book of Figà-Talamanca and Nebbia [FTN].

We denote by $|E|$ the measure of a subset $E$ of a measure space. We write $S_n$ for $\{x \in X : |x| = n\}$. Clearly, $|S_0| = 1$, and $|S_n| = (q + 1)q^{n-1}$ when $n \in \mathbb{Z}^+$. We pick points $w_0, w_1, w_2, \ldots$ in $X$ such that $|w_d| = d$. A radial function $f$ on $X$ is determined by its restriction to these points.

It is well known that $G$-invariant linear operators from $L^p(X)$ to $L^r(X)$ correspond to linear operators from $L^p(G/G_o)$ to $L^r(G/G_o)$ given by convolution on the right by $G_o$-bi-invariant kernels. We denote by $Cv^r_p(X)$ the space of radial functions on $X$ associated to these $G_o$-bi-invariant kernels. The norm of an element $k$ of $Cv^r_p(X)$ is then defined as the norm of the corresponding operator from $L^p(X)$ to $L^r(X)$, and denoted by $\|k\|_{pr}$. Equipped
with this norm, $Cv^r_p(\mathfrak{X})$ is a Banach space. We note that the maps $f \mapsto f'$ and $f' \mapsto \mathcal{E}f'$ given by the formulae

$$f'(g) = f(g \cdot o), \quad \mathcal{E}f'(g \cdot o) = \left\{ f'(gg_1) \right\}_{g_1} \quad \forall g \in G$$

are isometric from $L^p(\mathfrak{X})$ into $L^p(G)$ and norm-decreasing from $L^p(G)$ into $L^p(\mathfrak{X})$, for all $p \in [1, \infty]$. It follows that the norm of an element $k$ in $Cv^r_p(\mathfrak{X})$ is equal to the norm of its $G_o$-bi-invariant extension $k'$ to $G$ in $Cv^r_p(G)$, the space of convolution operators from $L^p(G)$ to $L^r(G)$.

For any function space $E(\mathfrak{X})$ on $\mathfrak{X}$, we denote by $E(\mathfrak{X})^2$ the (usually closed) subspace of $E(\mathfrak{X})$ of radial functions. We denote by $L^{p,r}(\mathfrak{X})$ the standard Lorentz space, as in Bergh and Lőfström [BL]. Pytlik [Py] proved that, given $p$ and $r$ in $[1, \infty)$, a radial function $f$ belongs to $L^{p,r}(\mathfrak{X})$ if and only if the function $d \mapsto f(w_d)|\mathfrak{G}_d|^{1/p}$ is in $L^r(N)$, and

$$\left( \sum_{d \in N} |f(w_d)|^r |\mathfrak{G}_d|^{r/p} \right)^{1/r} \sim \|f\|_{p,r}.$$  

The key to the proof is that $|\mathfrak{G}_d|$ grows exponentially in $d$. Pytlik used this lemma to show that $L^{p,1}(\mathfrak{X})^2 \subseteq Cv^r_p(\mathfrak{X}) \subseteq L^p(\mathfrak{X})^2$, and that the cone of positive radial convolution operators on $L^p(\mathfrak{X})$ coincides with the cone of positive functions in $L^{p,1}(\mathfrak{X})$.

In this paper, we first outline “spherical harmonic analysis” on $G$, and then prove some general theorems on $Cv^r_p(\mathfrak{X})$. In particular, we generalise results of Pytlik [Py] and of C. Nebbia [N, Thm. 2].

1. Notation and preliminaries. We write $\tau$ for $2\pi/\log q$, and define $T$ to be the torus $\mathbb{R}/\tau\mathbb{Z}$, usually identified with the interval $[-\tau/2, \tau/2]$. We denote by $\mathcal{F}$ the Fourier transformation on $\mathbb{Z}$, given by

$$\mathcal{F}f(s) = \sum_{d \in \mathbb{Z}} F(d) q^{-ids} \quad \forall s \in T.$$  

Clearly, $\mathcal{F}f(s + \tau) = \mathcal{F}f(s)$. A distribution $m$ on $T$ is said to be in $M^r_p(T)$ if convolution with $\mathcal{F}^{-1}m$ defines a bounded operator from $L^p(\mathbb{Z})$ to $L^r(\mathbb{Z})$. We define $\mathcal{F}L^r_p(T)$ to be $\{\mathcal{F}F : F \in L^r_p(\mathbb{Z})\}$, and note that $\mathcal{F}L^r_p(T)$ is continuously included in $L^r_p(T)$, by the classical Hausdorff–Young inequality, for $r$ in $[1, 2]$.

For $p$ in $[1, \infty]$, let $p'$, $\delta(p)$, $\mathfrak{S}_p$ and $\mathfrak{S}_p$ denote $p/(p - 1)$, $1/p - 1/2$,

$$\{z \in \mathbb{C} : |\Im(z)| < |\delta(p)|\} \quad \text{and} \quad \{z \in \mathbb{C} : |\Im(z)| \leq |\delta(p)|\}.  $$

If $f$ is holomorphic in $\mathfrak{S}_p$, then $f_{i\delta(p)}$ and $f_{-i\delta(p)}$ denote its boundary functions $f(i\delta(p) + \cdot)$ and $f(-i\delta(p) + \cdot)$, when these exist distributionally. The letter $C$, sometimes with subscripts or superscripts, denotes a positive constant which may vary from place to place; it may depend on any factor quantified
(implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. Given functions $A$ and $B$, defined on a set $\mathbb{D}$, we say that $A \sim B$ in $\mathbb{D}$ if there exist $C$ and $C'$ such that

$$CA(t) \leq B(t) \leq C'A(t) \quad \forall t \in \mathbb{D}.$$  

We conclude this section by summarising some features of spherical analysis on $\mathfrak{X}$. The theory parallels that of spherical analysis on a noncompact symmetric space of rank one. The Gel’fand pair $(G, G_o)$ has associated spherical functions $\phi_z$, parametrised by the complex number $z$. We refer to [CMS1] for explicit formulae, noting that our parametrisation differs from that used by some authors (e.g., [FTP] and [FTN]; our $\phi_z$ corresponds to their $\phi_{1/2+iz}$). The spherical Fourier transform $\tilde{f}$ of $f$ in $L^1(\mathfrak{X})^\#$ is defined by

$$\tilde{f}(z) = \sum_{x \in \mathfrak{X}} f(x) \phi_z(x) \quad \forall z \in S_1.$$  

Since $\phi_{z+\tau} = \phi_z$ and $\phi_z = \phi_{-z}$, $\tilde{f}$ is even and $\tau$-periodic in $S_1$. We say that a holomorphic function in a strip $\mathcal{S}_p$ is Weyl-invariant if it satisfies these conditions in $\mathcal{S}_p$.

We denote by $\mu$ the Plancherel measure on $\mathbb{T}$ [CMS1, (1.2)]. We note that the relation $\overline{c(z)} = c(-\overline{z})$ and the symmetry properties of spherical functions imply that

$$\phi_s(x) \frac{d\mu(s)}{ds} = c_G c(-s)^{-1} q^{(is-1/2)|x|} + c_G c(s)^{-1} q^{(-is-1/2)|x|},$$  

for all $x$ in $\mathfrak{X}$ and $s$ in $\mathbb{T}$. Therefore, if $m : \mathbb{R} \to \mathbb{C}$ is even and $\tau$-periodic, then

$$\int_{\mathbb{T}} m(s) \phi_s(x) d\mu(s) = c_G \int_{\mathbb{T}} m(s) c(-s)^{-1} q^{(is-1/2)|x|} ds$$

$$+ c_G \int_{\mathbb{T}} m(s) c(s)^{-1} q^{(-is-1/2)|x|} ds,$$

and by changing the variable $s$ to $-s$, we see that the two integrals on the right hand side are equal. In particular, if we set $\tilde{c}(s) = c(-s)$, we have

$$f(x) = 2 c_G \int_{\mathbb{T}} \tilde{f}(s) \tilde{c}(s)^{-1} q^{(-is-1/2)|x|} ds$$

$$= 2 c_G \int_{\mathbb{T}} \tilde{f}(s) \tilde{c}(s)^{-1} q^{(is-1/2)|x|} ds.$$  

In the following theorem, we use the results of [CMS2] on the range of the radial Abel transformation to characterise the spherical Fourier transforms of the radial functions in the Lorentz spaces $L^{p,r}(\mathfrak{X})$, and derive a version of the Hausdorff–Young inequality. For related results in the setting of noncompact symmetric spaces see [CGM].
Theorem 1.1. Suppose that $1 \leq p < 2$. If $f$ is in $L^{p,r}(\mathcal{X})^2$, then $\tilde{f}$ extends to a Weyl-invariant holomorphic function in $\mathcal{S}_p$, with boundary functions $\tilde{f}_{\delta}(p)$ and $\tilde{f}_{-\delta}(p)$ in $\mathcal{F}L'(\mathcal{T})$. If also $1 \leq r \leq 2$, then the map $z \mapsto \tilde{f}(z + \cdot)$ is continuous from $\mathcal{S}_p$ into $L^r(\mathcal{T})$, and

$$\left[ \int_{\mathcal{T}} |\tilde{f}(z + s)|^{r'} ds \right]^{1/r'} \leq C \|f\|_{p,r} \quad \forall z \in \mathcal{S}_p.$$  

Conversely, if $\tilde{f}$ is radial and $\tilde{f}$ extends to a Weyl-invariant holomorphic function in $\mathcal{S}_p$, the map $z \mapsto \tilde{f}(z + \cdot)$ is continuous from $\mathcal{S}_p$ into the space of distributions on $\mathcal{T}$, and the boundary functions $\tilde{f}_{\delta}(p)$ and $\tilde{f}_{-\delta}(p)$ are in $\mathcal{F}L'(\mathcal{T})$, then $f$ is in $L^{p,r}(\mathcal{X})^2$, and

$$\|f\|_{p,r} \leq C \|\mathcal{F}^{-1}\tilde{f}_{\delta}(p)\|_{r'}.$$  

Proof. Let $\mathcal{A}$ denote the Abel transformation on $\mathcal{X}$; see [CMS2] for notation and discussion. We recall that, for sufficiently nice radial functions on $\mathcal{X}$, the spherical Fourier transformation factors as $\tilde{f} = \mathcal{F}(\mathcal{A}f)$. Further, by [CMS2, Thm. 2.5], $\mathcal{A}$ is a bicontinuous isomorphism of $L^{p,r}(\mathcal{X})^2$ onto the space $q^{-\delta(p)|\cdot|}L'(\mathcal{Z})$, for any $p$ in $[1,2)$ and $r$ in $[1,\infty)$. Thus, if $f$ is in $L^{p,r}(\mathcal{X})^2$, it follows from the definition of $\mathcal{F}$ that $\tilde{f}$ extends to a holomorphic function on the strip $\mathcal{S}_p$ with the required continuity properties, and with boundary functions in $\mathcal{F}L'(\mathcal{T})$. Moreover, from the classical Hausdorff–Young inequality,

$$\left[ \int_{\mathcal{T}} |\mathcal{F}(\mathcal{A}f)(z + s)|^{r'} ds \right]^{1/r'} \leq C\|q^{\text{Im}(z)(\cdot)}\mathcal{A}f\|_{L^r(\mathcal{Z})}$$

$$\leq C\|q^{\delta(p)|\cdot|}\mathcal{A}f\|_{L^r(\mathcal{Z})} \leq C\|f\|_{p,r} \quad \forall z \in \mathcal{S}_p.$$  

Conversely, assume that $\tilde{f}$ has the stated properties. By Cauchy’s Theorem,

$$\mathcal{A}f(h) = \mathcal{F}^{-1}(\tilde{f}(h)) = \frac{1}{\tau} \int_{\mathcal{T}} \tilde{f}(s + i\delta(p)) q^{(s + i\delta(p))h} ds = q^{-\delta(p)h} \mathcal{F}^{-1}(\tilde{f}_{\delta}(p)).$$

Since $\mathcal{F}^{-1}(\tilde{f}_{\delta}(p))$ is in $L'(\mathcal{Z})$ by assumption, and $\mathcal{A}f$ is even, $\mathcal{A}f$ is in $q^{-\delta(p)|\cdot|}L'(\mathcal{Z})$; the required norm inequality follows from (2). $lacksquare$

2. On radial convolutors. Recall that $\text{CV}_p^r(\mathcal{X})$ denotes the space of radial kernels which convolve $L^p(\mathcal{X})$ into $L^r(\mathcal{X})$. In this section, we apply the results of the previous section to study these spaces.

The spherical Fourier transforms of the elements of the space $\text{CV}_p^r(\mathcal{X})$ are called spherical $L^p$-$L^r$ Fourier multipliers, or $L^p$ Fourier multipliers if $p = r$. It is easy to see that the Clerc–Stein condition [CS] for spherical $L^p$ multipliers on noncompact symmetric spaces holds in the present situation.
Thus a spherical $L^p$ Fourier multiplier extends to a bounded holomorphic function on $S_p$ [CMS1, Thm. 1.3], and
\[
\sup_{z \in S_p} |\tilde{k}(z)| \leq \|k\|_p \quad \forall k \in \text{Cv}_p(X).
\]
The symmetry properties of spherical functions imply the Weyl-invariance of spherical $L^p$-$L^r$ multipliers in their strip of holomorphy. The following theorem, which may be proved using Theorem 1.1, generalises the Clerc–Stein condition.

\textbf{Theorem 2.1.} Suppose that $1 \leq p < 2$ and $1 \leq r \leq s \leq \infty$, and that $k$ is a radial function on $X$. The following conditions are equivalent:

(i) $\tilde{k}$ extends to a holomorphic function on $S_p$, and the map $z \mapsto \tilde{k}(z + \cdot)$ extends to a continuous map from $S_p$ into the space of distributions on $T$, and $\tilde{k}_{\delta(p)}$ is in $M^r_s(T)$;

(ii) the operator of right convolution with $k$ is bounded from $L^p,r(X)^\diamond$ to $L^p,s(X)^\diamond$.

In particular, if $k$ is in $\text{Cv}_p(X)$ then $\tilde{k}_{\delta(p)}$ is in $M^r_p(T)$.

We omit the proof, since it is also an immediate corollary of [CMS2, Prop. 2.7]. Using Theorem 1.1 we moreover obtain the following.

\textbf{Theorem 2.2.} Suppose that $p$ is in $[1, 2)$ and that $k$ is a radial function on $X$ whose Fourier transform $\tilde{k}$ is holomorphic on $S_p$ and such that the map $z \mapsto \tilde{k}(z + \cdot)$ is a continuous distribution-valued map on $S_p$.

(i) If $p > 1$ and $\tilde{k}_{\delta(p)}$ is in $FL^r(T)$, then right convolution with $k$ is a bounded operator from $L^{p,s}(X)$ into $L^{p,t}(X)$, where $1/t = 1/r + 1/s - 1$. In particular, if $\tilde{k}$ is in $H^\infty(S_p)$, then right convolution with $k$ is of weak type $(p, p)$.

(ii) If $p > 1$ and $\tilde{k}_{\delta(p)}$ is bounded and smooth in $\C \setminus \pi \Z$, and satisfies
\[
\left| \frac{d}{ds} \tilde{k}_{\delta(p)}(s) \right| \leq C |s|^{-1} \quad \forall s \in T,
\]
then right convolution with $k$ maps $L^{p,s}(X)$ continuously into $L^{p,t}(X)$ whenever $t > s$.

(iii) If $\tilde{k}$ is in $H^\infty(S_1)$, then right convolution with $k$ is of weak type $(1, 1)$, and of strong type $(p, p)$ for every $p$ in $(1, \infty)$.

\textbf{Proof.} We claim that $L^{p,s}(X) \ast L^{p,r}(X)^\diamond \subseteq L^{p,t}(X)$ when $1 \leq p < 2$, $1 \leq r, s, t < \infty$, and $1 + 1/t = 1/r + 1/s$. Indeed, $L^1(X) \ast L^1(X)^\diamond \subseteq L^1(X)$, and Pytlik [Py] showed that if $p$ is in $(1, 2)$, then $L^p(X) \ast L^{p,1}(X)^\diamond \subseteq L^p(X)$ (see also Theorem 2.4 below). The claim then follows by multilinear interpolation [BL, 3.13.5, p. 76].
Assume now that $\tilde{k}_{\delta(p)}$ is in $FL^r(T)$. By Theorem 1.1, $k$ is in $L^{p,r}(X)$, and the first statement in (i) follows from the claim above.

If $k$ is in $H^\infty(S_p)$, then $\tilde{k}_{\delta(p)}$ is in $L^\infty(T)$ and a fortiori in $FL^2(T)$. The second statement in (i) follows from the first.

Under hypothesis (ii), $\tilde{k}_{\delta(p)}$ is in $L^{\infty}(T)$ and a fortiori in $F_{L^2}(T)$. The second statement in (i) follows from the first.

Finally, assume that $\tilde{k}$ is in $H^\infty(S_1)$. By (i) and interpolation and duality, it suffices to prove that convolution with $k$ is of weak type $(1,1)$. By (3), we see that $k(x) = 2c_G \int_T \tilde{f}(s) c(s)^{-1} q^{(-i\delta-1/2)|x|} ds$; by changing the contour of integration and inserting the value of $c_G$, we deduce that $k(x) = \frac{q \log q}{2\pi (q+1)} q^{-|x|} \int_T \tilde{f}(s-i/2) c(s-i/2)^{-1} q^{-i|x|} ds$.

We may therefore estimate $|k(x)| \leq \frac{q}{q+1} q^{-|x|} \sup_{s \in T} |\tilde{f}(s-i/2) c(s-i/2)^{-1}| \leq \frac{q}{q-1} \|\tilde{f}\|_\infty q^{-|x|}$.

Now, according to R. Rochberg and M. Taibleson [RT], Green’s operator (the inverse of the Laplacian) for a strongly reversible random walk on a tree of bounded degree is of weak type $(1,1)$. It is easily verified that the convolution kernel of Green’s operator on a homogeneous tree of degree $q+1$ is given by $k(x) = \frac{q}{q-1} q^{-|x|}$, and the required conclusion follows.

We now focus on the Banach space $Cv_p^r(X)$ of radial convolutors from $L^p(X)$ to $L^r(X)$. First, we state the analogue of Herz’s principe de majoration on trees. This is known, and may be found in a more general setting, for instance, in [Lo].

**Proposition 2.3.** Suppose that $1 \leq p \leq 2$, and that $k$ belongs to $Cv_p^r(X)$. Then

$$\|k\|_p \leq \|k\|_p = |k|(|\delta(p)|),$$

and equality holds if $k$ is nonnegative.

Denote by $Y(X)$ the Banach space of functions $f$ on $X$ such that $\|f\|_Y < \infty$, where

$$\|f\|_Y = \sum_{d \in \mathbb{N}} (d+1) \left( \sum_{x \in \mathbb{E}_d} |f(u_d)|^2 \right)^{1/2}.$$

Observe that $Y(X)^{\sharp} \subset L^{2,1}(X)^{\sharp}$; the inclusion is proper, from (2).
Theorem 2.4. Suppose that $1 \leq p, r \leq \infty$. Then $Cv^r_p(\mathcal{X}) = Cv^r_r(\mathcal{X})$.

Further,

(i) if $1 < p < 2$, then $L^{p,1}(\mathcal{X})^♯ \subseteq Cv^p_p(\mathcal{X}) \subseteq L^p(\mathcal{X})^♯$, and if $k \geq 0$ and $k$ is in $Cv^p_p(\mathcal{X})$, then $k$ belongs to $L^{p,1}(\mathcal{X})^♯$;

(ii) if $p = 2$, then $Y(\mathcal{X})^♯ \subseteq Cv^2_2(\mathcal{X}) \subseteq L^2(\mathcal{X})^♯$, and if $k \geq 0$ and $k$ is in $Cv^2_2(\mathcal{X})$, then $k$ belongs to $Y(\mathcal{X})^♯$;

(iii) if $1 \leq p < r \leq 2$, then $Cv^p_p(\mathcal{X}) = L^r(\mathcal{X})^♯$;

(iv) if $1 \leq p \leq 2 \leq r \leq \infty$, and $r \neq p'$, then $Cv^p_r(\mathcal{X}) = L^{\min(p',r)}(\mathcal{X})^♯$;

(v) if $1 < p < 2$, then $L^{p',2'}(\mathcal{X})^♯ \subseteq Cv^p_r(\mathcal{X}) \subseteq L^{p'}(\mathcal{X})^♯$.

Remarks. Both inclusions in (i) and the right hand inclusion in (v) are strict. This follows from the study of the $L^p$-$L^r$ mapping properties of the resolvent operator of the Laplacian $\Delta$. In addition, both inclusions in (ii) are strict. Indeed, the image of the space $Y(\mathcal{X})^♯$ under the spherical Fourier transform is contained in the space of absolutely convergent Fourier series on $\mathbb{T}$, while the images of $Cv^2_2(\mathcal{X})$ and $L^2(\mathcal{X})^♯$ coincide with $L^\infty(\mathbb{T})$ and $L^2(\mathbb{T},\mu)$ respectively. Finally, by considering nonnegative elements of $L^{2,1}(\mathcal{X})^♯$ which are not in $Y(\mathcal{X})^♯$, it may be seen that $L^{2,1}(\mathcal{X})^♯$ is not contained in $Cv^2_2(\mathcal{X})$.

Proof (of Theorem 2.4). Observe that $Cv^r_p(\mathcal{X}) \subseteq L^r(\mathcal{X})^♯$ since the point mass at $o$ is in $L^p(\mathcal{X})^♯$ for all $p$ in $[1, \infty]$. Moreover, $Cv^r_p(\mathcal{X}) = Cv^p_r(\mathcal{X})$, with norm equality, by duality, and since $\mathcal{X}$ is noncompact, $Cv^r_p(\mathcal{X})$ is nontrivial if and only if $p \leq r$, by a theorem of Hörlmander [Hö].

We first prove (i). As stated above, the left hand inclusion in (i) was proved in [Py]. We give a shorter proof. Since $k$ is in $Cv^p_p(\mathcal{X})$ if $|k|$ is, it suffices to take $k$ nonnegative. For these $k$, Herz’s principe shows that

$$\|k\|_p = \tilde{k}(i\delta(p)) = \sum_{d \in \mathbb{N}} k(w_d) \phi_{i\delta(p)}(w_d) \sim \sum_{d \in \mathbb{N}} k(w_d) |\mathcal{G}_d|^{1/p} \sim \|k\|_{p,1},$$

as required. This completes the proof of (i). To prove (ii), we argue in a similar fashion.

Now we prove (iii). We have already observed that $Cv^r_p(\mathcal{X}) \subseteq L^r(\mathcal{X})^♯$, so it suffices to show the reverse inclusion. For this, it suffices to prove that if $k$ is in $L^r(\mathcal{X})^♯$, then the map $f \mapsto f * k$ is bounded from $L^p(\mathcal{X})$ to $L^r(\mathcal{X})$; this follows from the radial form of the Kunze–Stein phenomenon on $\mathcal{X}$ (see [N]).

We now prove (iv). Suppose that $k$ is in $Cv^p_p(\mathcal{X})$; then it also belongs to $L^r(\mathcal{X})$. Since $Cv^r_p(\mathcal{X}) = Cv^p_r(\mathcal{X})$, a similar argument shows that $k$ is also in $L^{p'}(\mathcal{X})^♯$, and hence in $L^{\min(p,r')}(\mathcal{X})^♯$, showing that $Cv^r_p(\mathcal{X}) \subseteq L^{\min(p,r')}(\mathcal{X})$.

To prove the converse, we consider two cases separately. Suppose first that $p < r'$, so that $L^r(\mathcal{X}) = L^{\min(p,r')}(\mathcal{X})$. Assume that $k$ is in $L^r(\mathcal{X})$. Let
$f$ be in $L^p(\mathcal{X})$ and $h$ be in $L^{r'}(\mathcal{X})$; denote by $f'$, $h'$, and $k'$ respectively the $G_0$-right-invariant and $G_0$-bi-invariant extensions to $G$ of $f$, $h$, and $k$. Then
\[
\langle f * h, g \rangle_{\mathcal{X}} = \langle f' * k', h' \rangle_G = \langle k', (f')^* * h' \rangle_G,
\]
where $(f')^*(g) = (f')(g^{-1})$. Since $G$ has the Kunze–Stein property [N] and $1 \leq p < r' < 2$,
\[
\|(f')^* * h'\|_{r'} \leq C_{p,r'}\|f'\|_p\|h'\|_{r'} = C_{p,r'}\|f\|^p_p\|h\|^r_{r'}.
\]
Thus, by Hölder’s inequality,
\[
\sup\{|\langle f * k, h \rangle_{\mathcal{X}}| : \|f\|_p = 1, \|h\|_{r'} = 1\} \leq C_{p,r'}\|k\|_{r'},
\]
so that $k$ is in $C^v_{p,r}(\mathcal{X})$, and
\[
\|k\|_{p,r} \leq C_{p,r'}\|h\|_{r'},
\]
as required. The case where $r' < p$ is treated similarly.

Finally we prove (v). As before, the right inclusion is obvious. The left inclusion follows from the result [CMS2] that, if $1 < p < 2$, then $L^p(G) * L^p(G) \subseteq L^{p,r}(G)$, where $r = p/(2-p)$, much as (iv) follows from the Kunze–Stein phenomenon. The dual form of this sharp inclusion is the inclusion $L^p(G) * L^{p',r'}(G) \subseteq L^{p'}(G)$, where $r' = p'/2$; the desired result follows by specialising to functions with the appropriate invariance properties.

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School of Mathematics  
University of New South Wales  
Sydney, NSW 2052, Australia  
E-mail: m.cowling@unsw.edu.au

Facoltà di Scienze  
Università dell’Insubria–Polo di Como  
via Lucini 3  
I-22100 Como, Italy  
E-mail: setti@fis.unico.it

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