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SOME REMARKS ON THE ALTITUDE INEQUALITY

BY

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Abstract. We introduce and study a new class of ring extensions based on a new formula involving the heights of their primes. We compare them with the classical altitude inequality and altitude formula, and we give another characterization of locally Jaffard domains, and domains satisfying absolutely the altitude inequality (resp., the altitude formula). Then we study the extensions $R \subseteq S$ where R satisfies the corresponding condition with respect to S (Definition 3.1). This leads to a new characterization of integral extensions.

0. Introduction. Throughout this paper, we adopt the conventions that each ring considered is integral, commutative, with unit, and an inclusion (extension) of rings signifies that the smaller ring is a subring of the larger and has the same multiplicative identity. The quotient field of a ring R is denoted by $\mathrm{qf}(R)$, its Krull dimension by $\dim R$, and its integral closure by R'. We denote by $R[n]$ the polynomial ring in n indeterminates over R, and by $\dim_{\mathrm{v}} R$ the *valuative dimension*, that is, the limit of the sequence $(\dim R[n] - n, n \in \mathbb{N})$. If p is a prime ideal of R, we denote by $\mathrm{ht}\, p$ its height, and by $\mathrm{ht}_{\mathrm{v}}\, p$ its *valuative height*, that is, the limit of the sequence $(\mathrm{ht}\, p[n], n \in \mathbb{N})$. A finite-dimensional ring R is said to be *Jaffard* if $\dim_{\mathrm{v}} R = \dim R$, and *locally Jaffard* if R_p is a Jaffard ring for each prime p of R ([1]).

For a ring extension $R \subseteq S$, we denote by $\mathrm{tr.deg}[S : R]$ the transcendence degree of $\mathrm{qf}(S)$ over $\mathrm{qf}(R)$. Recall that an extension $R \subseteq S$ is said to satisfy the *altitude inequality* (resp., the *altitude formula*, the *valuative altitude formula*) if for any prime ideal Q of S over a prime ideal P of R, we have, respectively,

$$\mathrm{ht}\, Q + \mathrm{tr.deg}[S/Q : R/P] \le \mathrm{ht}\, P + \mathrm{tr.deg}[S : R],$$
$$\mathrm{ht}\, Q + \mathrm{tr.deg}[S/Q : R/P] = \mathrm{ht}\, P + \mathrm{tr.deg}[S : R],$$
$$\mathrm{ht}_{\mathrm{v}}\, Q + \mathrm{tr.deg}[S/Q : R/P] = \mathrm{ht}_{\mathrm{v}}\, P + \mathrm{tr.deg}[S : R].$$

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A domain R satisfies the altitude inequality (resp., the altitude formula, the valuative altitude formula) if $R \subseteq S$ satisfies the respective condition for each finite type R-algebra S containing R.

S. Kabbaj established the equivalence between the following statements [16, Lemme 1.4]:

(i) R is a locally Jaffard domain;

(ii) $\mathrm{ht}\, p[n] = \mathrm{ht}\, p$ for each prime ideal p of R and each positive integer n;

(iii) R satisfies the altitude inequality;

(iv) the extension $R \subseteq R[n]$ satisfies the altitude formula for each positive integer n.

A. Ayache and P.-J. Cahen [2] generalized Kabbaj's result as well as A. Bouvier, D. E. Dobbs and M. Fontana's result [7, Proposition 9.3]. They proved that R is a locally Jaffard domain if and only if each extension $R \subseteq S$ satisfies the altitude inequality.

The main purpose of this paper is to study a new class of extensions. We say that an extension $R \subseteq S$ of rings satisfies:

(a) the *restrictive altitude inequality* (RAI) if $\mathrm{ht}\, Q \leq \mathrm{ht}(Q \cap R) + \mathrm{tr.deg}[S : R]$ for each prime ideal Q of S;

(b) the *restrictive altitude formula* (RAF) if $\mathrm{ht}\, Q = \mathrm{ht}(Q \cap R) + \mathrm{tr.deg}[S : R]$ for each prime ideal Q of S;

(c) the *restrictive valuative altitude formula* (RVAF) if $\mathrm{ht_v}\, Q = \mathrm{ht_v}(Q \cap R) + \mathrm{tr.deg}[S : R]$ for each prime ideal Q of S.

One shows easily that the extensions satisfying either RAF or RVAF are algebraic.

There are two basic, classical examples of extensions satisfying RAI:

(1) extensions satisfying the altitude inequality;

(2) algebraic extensions satisfying INC.

Our concern in the first section is primarily with the restrictive altitude inequality. Our initial line of inquiry was to compare the classical and restrictive altitude inequalities. This section explores consequences of Theorem 1.2 which states that any finite type extension $R \subseteq S$ satisfying RAI always satisfies the classical altitude inequality. Perhaps the most surprising of these consequences, Theorem 1.9, indicates that if $R \subset S$ is a finite type extension such that R is integrally closed in S, then $R \subset S$ satisfies RAF if and only if it satisfies RVAF. On the other hand, we notice that for a ring R, the extension $R \subset R[n]$ always satisfies the valuative altitude formula [12, Théorème 2.1] for each positive integer n. But if R is not locally Jaffard, then there exists n such that $R \subset R[n]$ does not satisfy the altitude formula.

The second section is concluded with a study of the relationship between locally Jaffard domains and RAI. We show that R is a locally Jaffard domain if and only if each extension $R \subseteq S$ of R satisfies RAI (Proposition 2.1).

Recall that a domain R is said to *satisfy absolutely the altitude inequality* if any overring of R is Jaffard (equivalently, the integral closure of R is a Prüfer domain [2, Théorème 2.6]). R is said to *satisfy absolutely the altitude formula* if $R \subseteq S$ satisfies the altitude formula for each overring S of R. We show in Proposition 2.3 that the following are equivalent:

(i) R satisfies absolutely the altitude formula;

(ii) each ring extension $R \subseteq S$ satisfies RAF;

(iii) each ring extension $R \subseteq S$ satisfies RVAF;

(iv) for each valuation overring V of R, the extension $R \subseteq V$ satisfies RAF.

In the final section, we turn our attention to the extensions $R \subseteq S$ where R satisfies RAF (resp., RVAF) with respect to S (Definition 3.1). We characterize those extensions in terms of residually algebraic pairs [4]. This leads to a new characterization of integral extensions.

Any unexplained terminology is standard, as in [14] and [17].

1. The restrictive altitude inequality. Recall some definitions given in the introduction.

DEFINITION 1.1. An extension $R \subset S$ of domains satisfies the *restrictive altitude inequality* (RAI) if for each prime ideal Q in S,

$$\operatorname{ht} Q \leq \operatorname{ht}(Q \cap R) + \operatorname{tr.deg}[S : R]. \tag{$\sharp$}$$

$R \subset S$ satisfies the *restrictive altitude formula* (RAF) if equality holds in ($\sharp$).

In this section, we collect more information on this kind of extensions, especially in order to enlighten their relationship with those satisfying the altitude formula (or inequality).

We start with the following result.

THEOREM 1.2. *Let $R \subset S$ be an extension of domains such that S is a finitely generated domain over R. Then $R \subset S$ satisfies RAI if and only if it satisfies the altitude inequality.*

Proof. It is obvious that an extension of domains which satisfies the altitude inequality always satisfies RAI. Conversely, let Q be a prime ideal of S, and set $P = Q \cap R$. We need to prove that

$$\operatorname{ht} Q + \operatorname{tr.deg}[S/Q : R/P] \leq \operatorname{ht} P + \operatorname{tr.deg}[S : R].$$

There exists a maximal ideal M_P of S_P such that $Q_P \subseteq M_P$. Since R_P is local with maximal ideal PR_P, we get $M_P \cap R_P = PR_P$. We have

$$\operatorname{ht} Q + \operatorname{ht}(M/Q) \le \operatorname{ht} M. \tag{1}$$

As R_P/PR_P is a field and S_P/Q_P is finitely generated over R_P/PR_P, the extension $R_P/PR_P \subset S_P/Q_P$ satisfies the altitude formula. Therefore

$$\operatorname{ht}(M/Q) + \operatorname{tr.deg}[S/M : R/P] = \operatorname{tr.deg}[S/Q : R/P]. \tag{2}$$

On the other hand, $\operatorname{tr.deg}[S/M : R/P] = 0$, because the field S_P/M_P is finitely generated over R_P/PR_P [17, Theorem 22]. Hence by (2), we get

$$\operatorname{ht}(M/Q) = \operatorname{tr.deg}[S/Q : R/P]. \tag{3}$$

(1) and (3) give

$$\operatorname{ht} Q + \operatorname{tr.deg}[S/Q : R/P] \le \operatorname{ht} M.$$

But by hypothesis we have

$$\operatorname{ht} M \le \operatorname{ht} P + \operatorname{tr.deg}[S : R],$$

which gives

$$\operatorname{ht} Q + \operatorname{tr.deg}[S/Q : R/P] \le \operatorname{ht} P + \operatorname{tr.deg}[S : R]. \ \blacksquare$$

It was shown in [16] that a domain R is locally Jaffard if and only if $R \subset R[n]$ satisfies the altitude formula for each positive integer n. By the previous theorem, R is a locally Jaffard domain if and only if $R \subset R[n]$ satisfies RAI for each n.

Remark 1.3. Notice that, in general, if S is not a finitely generated domain over R, the extension $R \subset S$ may satisfy RAI and even RAF without satisfying the altitude inequality (see Example 4.1 below).

Recall that an extension $R \subset S$ of domains is said to be *residually algebraic* if $R/Q \cap R \subset S/Q$ is algebraic for each prime ideal Q of S ([4] and [13]). In the next result we show that any finite type extension satisfying RAF is residually algebraic.

Proposition 1.4. *Let $R \subset S$ be an extension of domains such that S is a finitely generated domain over R. If $R \subset S$ satisfies RAF, then*:

(i) *$R \subset S$ is a residually algebraic extension*;
(ii) *$R \subset S$ satisfies the altitude formula.*

Proof. (i) According to Theorem 1.2, $R \subset S$ satisfies the altitude inequality. Thus, for each prime ideal Q of S and $P = Q \cap R$, we get

$$\operatorname{ht} Q + \operatorname{tr.deg}[S/Q : R/P] \le \operatorname{ht} P + \operatorname{tr.deg}[S : R].$$

Since $\operatorname{ht} Q = \operatorname{ht} P + \operatorname{tr.deg}[S : R]$, we obtain $\operatorname{tr.deg}[S/Q : R/P] = 0$. Hence, $R \subset S$ is a residually algebraic extension.

(ii) It is straightforward to check that a residually algebraic extension satisfying RAF always satisfies the altitude formula. ∎

For a locally Jaffard domain R, the extension $R \subset R[n]$ (where $n \geq 1$) satisfies the altitude formula but does not satisfy RAF because it is not algebraic. In Example 4.2, we give an algebraic extension $R \subset S$ of domains where S is finitely generated over R, $R \subset S$ satisfies the altitude formula and it is not incomparable (hence not residually algebraic). Thus, by the previous proposition, $R \subset S$ does not satisfy RAF.

REMARK 1.5. We claim that there exists an extension $R \subset S$ of domains such that S is not finitely generated over R, $R \subset S$ satisfies RAF and is not residually algebraic (see Example 4.1).

As mentioned in the introduction, A. Ayache and P.-J. Cahen proved that a domain R is locally Jaffard if and only if each domain extension $R \subset S$ satisfies the altitude inequality. They also proved that any extension $R \subset S$ of domains satisfies the valuative altitude inequality, that is, for each prime ideal Q of S, we have

$$\mathrm{ht_v}\, Q + \mathrm{tr.deg}[S/Q : R/Q \cap R] \leq \mathrm{ht_v}(Q \cap R) + \mathrm{tr.deg}[S : R],$$

and that this inequality may not be an equality [2, Exemple 5.1]. Hence, O. Echi in [12] introduced the following definition:

An extension $R \subset S$ of domains is said to *satisfy the valuative altitude formula* if for each prime ideal Q of S, we have

$$\mathrm{ht_v}\, Q + \mathrm{tr.deg}[S/Q : R/Q \cap R] = \mathrm{ht_v}(Q \cap R) + \mathrm{tr.deg}[S : R].$$

There are many examples of ring extensions satisfying the altitude formula without satisfying the valuative altitude formula or conversely (see for instance [2, Exemple 5.3]). In this vein, we make the following definition.

DEFINITION 1.6. An extension $R \subset S$ of domains satisfies the *restrictive valuative altitude formula* (RVAF) if for each prime ideal Q of S,

$$\mathrm{ht_v}\, Q = \mathrm{ht_v}(Q \cap R) + \mathrm{tr.deg}[S : R]. \tag{$\aleph$}$$

In Theorem 1.9, we prove that if $R \subset S$ is an extension of domains where S is finitely generated over R and integrally closed in S, then $R \subset S$ satisfies RAF if and only if it satisfies RVAF. However, in the general case, there are extensions satisfying only one of those formulae (Example 4.3).

First of all, we establish the following result:

PROPOSITION 1.7. *Let $R \subset S$ be an extension of domains satisfying RVAF. Then*:

(i) *$R[n] \subset S[n]$ is a residually algebraic extension for each n*;
(ii) *$R[n] \subset S[n]$ satisfies the valuative altitude formula for each n.*

We need the following lemma.

LEMMA 1.8. *Let $R \subset S$ be an extension of domains. Then the following statements are equivalent*:

(i) *$R \subset S$ satisfies RVAF*;
(ii) *$R[n] \subset S[n]$ satisfies RVAF for each n*;
(iii) *$R[1] \subset S[1]$ satisfies RVAF*;
(iv) *there exists a positive integer n such that $R[n] \subset S[n]$ satisfies RVAF.*

Proof. (i)⇒(ii). For each prime ideal Q of $S[n]$, we set $P = Q \cap R[n]$, $q = Q \cap S$ and $p = q \cap R$. By the special valuative chain theorem [12], we have $\mathrm{ht_v}\, Q = \mathrm{ht_v}\, q + {}^*Q$ and $\mathrm{ht_v}\, P = \mathrm{ht_v}\, p + {}^*P$, where ${}^*Q = \mathrm{ht}(Q/q[n])$ and ${}^*P = \mathrm{ht}(P/p[n])$. Since $R \subset S$ satisfies RVAF, we have $\mathrm{ht_v}\, q = \mathrm{ht_v}\, p + \mathrm{tr.deg}[S : R]$. Hence $\mathrm{ht_v}\, Q = \mathrm{ht_v}\, P + ({}^*Q - {}^*P) + \mathrm{tr.deg}[S : R]$. But by [3, Lemme A], we have ${}^*Q - {}^*P \geq 0$, which gives $\mathrm{ht_v}\, Q \geq \mathrm{ht_v}\, P + \mathrm{tr.deg}[S : R]$. The opposite inequality is always true, because $R[n] \subset S[n]$ satisfies the valuative altitude inequality [2, Théorème 1.3]. Therefore, $\mathrm{ht_v}\, Q = \mathrm{ht_v}\, P + \mathrm{tr.deg}[S[n] : R[n]]$.

(ii)⇒(iii)⇒(iv) are trivial.

(iv)⇒(i). Let q be a prime ideal of S and $p = q \cap R$. Then $\mathrm{ht_v}\, q[n] = \mathrm{ht_v}\, p[n] + \mathrm{tr.deg}[S[n] : R[n]]$. Thus, $\mathrm{ht_v}\, q = \mathrm{ht_v}\, p + \mathrm{tr.deg}[S : R]$. ■

Proof of Proposition 1.7. (i) It will suffice to prove that if $R \subset S$ satisfies RVAF, then it is residually algebraic; then we conclude by the previous lemma. For each prime ideal Q of S, we set $P = Q \cap R$. We have $\mathrm{ht_v}\, Q + \mathrm{tr.deg}[S/Q : R/P] \leq \mathrm{ht_v}\, P + \mathrm{tr.deg}[S : R]$ [2, Théorème 1.3]. But $\mathrm{ht_v}\, Q = \mathrm{ht_v}\, P + \mathrm{tr.deg}[S : R]$. Hence, $\mathrm{tr.deg}[S/Q : R/P] = 0$. Therefore, $R \subset S$ is residually algebraic.

(ii) One can easily check that a residually algebraic extension satisfying RVAF always satisfies the valuative altitude formula. ■

THEOREM 1.9. *Let $R \subset S$ be an extension of domains such that S is finitely generated over R and R is integrally closed in S. Then the following statements are equivalent*:

(i) *$R \subset S$ satisfies RAF*;
(ii) *$R \subset S$ satisfies RVAF*;
(iii) *for each $Q \in \mathrm{Spec}(S)$, we have $S_Q = R_{Q \cap R}$.*

To prove this theorem, we need the following lemma.

LEMMA 1.10. *Let $R \subset S$ be an extension of domains such that S is finitely generated over R. Then the following statements are equivalent.*

(i) *$R \subset S$ is a residually algebraic extension*;
(ii) *for each $Q \in \mathrm{Spec}(S)$, we have $S_Q = R^*_{Q \cap R^*}$. (R^* denotes the integral closure of R in S).*

P r o o f. (i)$\Rightarrow$(ii). Without loss of generality, we can assume that $S = R[u]$ where u is algebraic over R. Let $Q \in \mathrm{Spec}(S)$ and set $P = Q \cap R$. By localization, we can suppose that R is local with P as maximal ideal. Denote by $\Phi_u : R[X] \to R[u]$ the epimorphism of R-algebras that sends u to X. Let $I = \mathrm{Ker}\,\Phi_u$. We have $R[X]/I \simeq R[u]$. Since $R \subset R[u]$ is residually algebraic we have $I \not\subseteq P[X]$. On the other hand, let $\varphi_u : R^*[X] \to R^*[u] = R[u]$, $X \mapsto u$, and $J = \mathrm{Ker}\,\varphi_u$. Since $I \not\subseteq P[X]$, it follows that $J \not\subseteq (Q \cap R^*)[X]$. Consider the multiplicative subset $N = R^* \setminus (Q \cap R^*)$ of R^*. We have $N^{-1}J \subseteq N^{-1}(Q \cap R^*)[X]$. The ring $R^*_{Q\cap R^*}$ is local and integrally closed in $N^{-1}(R[u]) = (R^*_{Q\cap R^*})[u]$ and since $N^{-1}(Q \cap R^*)$ is the unique maximal ideal of $R^*_{Q\cap R^*}$, using the u-u^{-1} Lemma [17, Exercise 31, pp. 43–44] we obtain $u \in R^*_{Q\cap R^*}$ or $u^{-1} \in R^*_{Q\cap R^*}$.

Now we show that $u \in R^*_{Q\cap R^*}$. For if $u \notin R^*_{Q\cap R^*}$, then u^{-1} is not invertible in $R^*_{Q\cap R^*}$. Hence $u^{-1} \in N^{-1}(Q\cap R^*)$. Therefore there is no prime ideal of $(R^*_{Q\cap R^*})[u]$ lying over $N^{-1}(Q \cap R^*)$. However, $Q \in \mathrm{Spec}(R[u]) = \mathrm{Spec}(R^*[u]), N^{-1}Q \in \mathrm{Spec}(N^{-1}(R^*[u])) = \mathrm{Spec}((N^{-1}R^*)[u])$ and $N^{-1}Q \cap N^{-1}R^* = N^{-1}(Q \cap R^*)$. This contradiction yields $u \in R^*_{Q\cap R^*}$.

Thus $N^{-1}R^* = (N^{-1}R^*)[u] = N^{-1}(R[u])$, and $N^{-1}Q = N^{-1}Q \cap N^{-1}(R[u]) = N^{-1}Q \cap N^{-1}R^* = N^{-1}(Q\cap R^*)$. Hence Q is the unique ideal of $R[u]$ maximal for the property of not meeting S. This yields $N^{-1}(R[u]) = (R[u])_Q = R^*_{Q\cap R^*}$.

(ii)$\Rightarrow$(i). Let $Q\in \mathrm{Spec}(S)$. We have $\mathrm{tr.deg}[S/Q : R/Q\cap R] = \mathrm{tr.deg}[S/Q : R^*/Q \cap R^*] + \mathrm{tr.deg}[R^*/Q \cap R^* : R/Q \cap R] = \mathrm{tr.deg}[S/Q : R^*/Q \cap R^*] = 0$, since $S_Q = R^*_{Q\cap R^*}$. ∎

Proof of Theorem 1.9. (i)$\Rightarrow$(ii). Assume that $R \subset S$ satisfies RAF. By Proposition 1.4, $R \subset S$ is residually algebraic and by Lemma 1.10, $S_Q = R_{Q\cap R}$ for each $Q \in \mathrm{Spec}(S)$. Hence $\mathrm{ht_v}\, Q = \mathrm{ht_v}(Q \cap R)$. Thus $R \subset S$ satisfies RVAF.

(ii)$\Rightarrow$(iii). By Proposition 1.7, $R \subset S$ is residually algebraic. Using Lemma 1.10 we have $S_Q = R_{Q\cap R}$ for each $Q \in \mathrm{Spec}(S)$.

(iii)$\Rightarrow$(i). This is straightforward. ∎

Notice that the assumption that R is integrally closed in S is essential in Theorem 1.9. To see this, consider an integral ring extension of integral domains $R \subset S$ which satisfies RAF but not RVAF (see for instance [2, Exemple 5.3]).

2. RAI and Jaffard domains. The main purpose of this brief section is to give a new characterization of locally Jaffard domains in terms of RAI. We start with the following result.

Proposition 2.1. *Let R be a domain. Then the following statements are equivalent*:

(i) *R is a locally Jaffard domain*;

(ii) *for each domain S containing R, the extension $R \subset S$ satisfies RAI*;

(iii) *for each finitely generated domain S over R, the extension $R \subset S$ satisfies RAI*;

(iv) *for each finitely generated overring S of R, the extension $R \subset S$ satisfies RAI*;

(v) *for each overring S of R, the extension $R \subset S$ satisfies RAI*;

(vi) *for each valuation overring V of R, the extension $R \subset V$ satisfies RAI.*

Proof. The implications (i)⇒(ii)⇒(iii)⇒(iv) and (v)⇒(vi) are straightforward.

(iv)⇒(v). We assume that S is an overring of R containing a prime ideal Q such that $\operatorname{ht} Q > \operatorname{ht}(Q \cap R)$. Let $s = \operatorname{ht} Q$ and consider a maximal chain of $Q : (0) = Q_0 \subset Q_1 \subset \ldots \subset Q_s = Q$. For each $i \in \{1, \ldots, s\}$, there exists $x_i \in Q_i - Q_{i-1}$. If we set $T = R[x_1, \ldots, x_s]$ and $Q' = Q \cap T$, we have $\operatorname{ht} Q' \geq s > ht(Q' \cap R)$. Hence, the finite type extension $R \subset T$ does not satisfy RAI, contrary to hypothesis.

(vi)⇒(i). Let P be a prime ideal of R and V a valuation overring of R_P with M as maximal ideal. If we set $P' = M \cap R$, we get $\dim V = \operatorname{ht} M \leq \operatorname{ht} P' \leq \operatorname{ht} P = \dim R_P$. Therefore $\dim_{\mathrm{v}} R_P \leq \dim R_P$. Since the inverse inequality $\dim R_P \leq \dim_{\mathrm{v}} R_P$ always holds, we obtain $\dim R_P = \dim_{\mathrm{v}} R_P$. Hence, R is a locally Jaffard domain. ∎

Remark 2.2. By Proposition 2.1, a domain R satisfies absolutely the altitude inequality if and only if for each overring S of R, and each overring T of S, the extension $S \subset T$ satisfies RAI.

As was promised, the next result points up the connection between satisfying absolutely the altitude formula and RAF.

Proposition 2.3. *Let R be a domain. Then the following statements are equivalent*:

(i) *R satisfies absolutely the altitude formula*;

(ii) *each ring extension $R \subset S$ satisfies RAF*;

(iii) *each ring extension $R \subset S$ satisfies RVAF*;

(iv) *for each valuation overring V of R, the extension $R \subset V$ satisfies RAF*;

(v) *for each valuation overring V of R, the extension $R \subset V$ satisfies RVAF.*

P r o o f. (i)$\Rightarrow$(ii). Since R satisfies absolutely the altitude formula, each ring extension $R \subset S$ is residually algebraic and satisfies the altitude formula [2, Théorèmes 2.6 and 3.8]. Thus, it satisfies RAF.

(ii)$\Rightarrow$(iii). For each overring S of R, and each overring T of S, the extension $S \subset T$ satisfies RAF. Hence, S is a locally Jaffard domain (Proposition 2.1).

(iii)$\Rightarrow$(i) (resp., (v)$\Rightarrow$(i)). By Proposition 1.7, for each overring (resp. valuation overring) S of R, the extension $R \subset S$ is residually algebraic. Hence R satisfies absolutely the altitude inequality (otherwise R' is a Prüfer domain) [2, Théorème 2.6 and Remarque]. Therefore, each overring of R is a locally Jaffard domain. Hence, for each overring (resp., valuation overring) S of R, the extension $R \subset S$ satisfies the altitude formula. Thus R satisfies absolutely the altitude formula [2, Théorème 3.3].

(ii)$\Rightarrow$(iv). Trivial.

(iv)$\Rightarrow$(v). This is straightforward, since R is a locally Jaffard domain (Proposition 2.1). ∎

3. RAF and residually algebraic pairs. Recall that a pair (R, S) is said to be *residually algebraic* if for any ring T between R and S, the extension $R \subset T$ is residually algebraic [4, Definition 2.1]. Our work's principal motivation in this section arises from the following characterization of residually algebraic pairs in terms of RAF.

By analogy to [5], we introduce the following.

DEFINITION 3.1. Let $R \subset S$ be an extension of domains. We say that R satisfies RAF (resp., RVAF) *with respect to* S if for any ring T between R and S, the extension $R \subset T$ satisfies RAF (resp., RVAF).

PROPOSITION 3.2. *Let $R \subset S$ be an extension of domains. Then R satisfies RAF* (*resp.*, *RVAF*) *with respect to S if and only if* (R, S) *is a residually algebraic pair and the extension $R \subseteq R^*$ satisfies RAF* (*resp.*, *RVAF*). (R^* *denotes the integral closure of R in S.*)

P r o o f. Assume that (R, S) is a residually algebraic pair. Then so is the pair (R^*, S). Thus by [4, Theorem 2.10], for any ring T between R^* and S and any prime ideal Q of T, we have $T_Q = R^*_{Q \cap R^*}$. Thus $\mathrm{ht}_T\, Q = \mathrm{ht}_{R^*}(Q \cap R^*)$ and $\mathrm{ht}_{\mathrm{v}T}\, Q = \mathrm{ht}_{\mathrm{v}R^*}(Q \cap R^*)$. Hence, R^* satisfies RAF (resp., RVAF) with respect to S. Since $R \subseteq R^*$ satisfies RAF (resp., RVAF), R satisfies RAF (resp., RVAF) with respect to S.

Conversely, let T be a ring between R and S. Our task is to show that $R \subset T$ is an incomparable extension. This is immediate since an extension satisfying RAF or RVAF is incomparable. ∎

As an immediate consequence, we have:

COROLLARY 3.3. *Let $R \subset S$ be an extension of domains such that R is integrally closed in S. Then the following statements are equivalent*:

(i) *R satisfies RAF with respect to S*;
(ii) *R satisfies RVAF with respect to S*;
(iii) *(R, S) is a residually algebraic pair.*

COROLLARY 3.4. *Let R be a domain. The following statements are equivalent*:

(i) *R satisfies RAF with respect to* $\mathrm{qf}(R)$;
(ii) *R' is a Prüfer domain and $R \subset R'$ satisfies RAF*;
(iii) *R' is a Prüfer domain and $R \subset R'$ satisfies the altitude formula*;
(iv) *R satisfies RVAF with respect to* $\mathrm{qf}(R)$;
(v) *R satisfies absolutely the altitude formula.*

P r o o f. Note that the pair $(R, \mathrm{qf}(R))$ is residually algebraic if and only if R' is a Prüfer domain [4, Corollary 2.8]. ■

After proving 3.5, a number of corollaries are given.

RESULT 3.5 [5*, Proposition 2.10]. *An extension $R \subseteq S$ of domains is integral if and only if $(R[X], S[X])$ is a residually algebraic pair.*

P r o o f. Of course the "only if" part is immediate, since $R[X] \subseteq S[X]$ is an integral extension. For the "if" part, we can assume that R is local and integrally closed in S. Then, by [14, Theorem 10.7], $R[X]$ is integrally closed in $S[X]$. Consider the ring $T = R + XS[X]$; we have $R[X] \subseteq T \subseteq S[X]$. Denote by M the maximal ideal of R; then $Q = M + XS[X]$ is a prime ideal of T. Let $P = Q \cap R[X]$. We have $P = M + XR[X]$. Pick $a \in S \setminus R$. Then $aX \in T_Q$, but $aX \notin R[X]_P$. Indeed, if not, there exist $f \in R[X]$ and $g \in R[X] \setminus P$ such that $f/g = aX$. Write $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$. The equality $f = aXg$ shows that $n = m+1$ and $a_1 = ab_0$. But $b_0 \in R \setminus M$. Hence b_0 is invertible in R. Therefore $a = a_1 b_0^{-1} \in R$, a contradiction. Thus $T_Q \neq R[X]_P$ and by [4, Theorem 2.10], $(R[X], S[X])$ is not a residually algebraic pair. ■

COROLLARY 3.6. *Let $R \subset S$ be an extension of domains. The following statements are equivalent*:

(i) *$R \subset S$ is an integral extension and $R[X] \subset S[X]$ satisfies the altitude formula*;
(ii) *$R \subset S$ is an integral extension and $R[X] \subset S[X]$ satisfies RAF*;
(iii) *$R[X]$ satisfies RAF with respect to $S[X]$.*

Proof. (ii)$\Rightarrow$(iii). This follows immediately from Proposition 3.2.

(iii)$\Rightarrow$(ii). By Proposition 3.2, the pair $(R[X], S[X])$ is residually algebraic, and the extension $R[X] \subset R^*[X]$ satisfies RAF. Thus by Result 3.5, $R \subset S$ is an integral extension. Hence $R^* = S$, and $R[X] \subset S[X]$ satisfies RAF.

(i)$\Leftrightarrow$(ii). This is straightforward. ■

We pause to repeat that the techniques used in the preceding proof permit one to prove the following corollary.

COROLLARY 3.7. *Let $R \subset S$ be an extension of domains. The following statements are equivalent*:

(i) *$R \subset S$ is an integral extension satisfying RVAF*;

(ii) *$R \subset S$ is an integral extension satisfying the valuative altitude formula*;

(iii) *$R[X]$ satisfies RVAF with respect to $S[X]$.*

According to [6, Théorème 3, p. 56], if $R \subset S$ is an integral extension such that R is integrally closed then it has the incomparability and going-down properties. Hence it satisfies both RVAF and RAF. Therefore we get easily the following.

COROLLARY 3.8. *Let $R \subset S$ be a ring extension such that R is integrally closed. The following statements are equivalent*:

(i) *$R \subset S$ is an integral extension*;

(ii) *$R[X]$ satisfies RAF with respect to $S[X]$*;

(iii) *$R[X]$ satisfies RVAF with respect to $S[X]$.*

4. Examples and counterexamples. This section is concerned with examples showing limits of the results established in the previous sections. First, recall some terminology from [1], [8] and [9]. Specifically, let S be an integral domain, I a nonzero ideal of S, $\varphi : S \to S/I$ the natural epimorphism, D a subring of S/I and $R = \varphi^{-1}(D)$ the pullback of the following diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ S \arrow[r] & S/I \end{tikzcd}$$

We say that R is the ring of the (S, I, D) *construction* ([8]).

As stated before, if we leave out the assumption "S is finitely generated over R" in Theorem 1.2, the following example shows, among other facts, that $R \subset S$ may satisfy RAI without satisfying the altitude inequality.

Example 4.1. This example provides an extension $R \subset S$ of domains such that S is not a finitely generated domain over R, and

(a) $R \subset S$ satisfies RAF without satisfying the altitude inequality,

(b) $R \subset S$ is not a residually algebraic extension (hence it does not satisfy RVAF).

Let S be a valuation domain with maximal ideal M and residue field $S/M = K$. Let $R = \varphi^{-1}(k)$, where $\varphi : S \to S/M$ is the natural epimorphism and k is a subfield of K. Assume that $k \subset K$ is a transcendental extension.

$R \subset S$ satisfies RAF, since $\mathrm{ht}_S M = \mathrm{ht}_R M + \mathrm{tr.deg}[S : R]$. However, $\mathrm{ht}_{\mathrm{v}R} M = \dim_{\mathrm{v}} R = \dim S + \mathrm{tr.deg}[K : k] > \dim S = \mathrm{ht}_{\mathrm{v}S} M$. Thus, $R \subset S$ does not satisfy RVAF. On the other hand, we have $\mathrm{ht}_S M + \mathrm{tr.deg}[S/M : R/M] > \mathrm{ht}_R M + \mathrm{tr.deg}[S : R]$. Hence, $R \subset S$ does not satisfy the altitude inequality.

It is obvious that S is not a finitely generated domain over R.

The next example supplies an algebraic extension $R \subset S$ of domains where S is finitely generated over R, $R \subset S$ satisfies the altitude formula and it is not incomparable (hence not residually algebraic).

Example 4.2. Let R be a domain such that $R[X]$ is catenarian and R' is not a Prüfer domain (for instance $R = \mathbb{Z}[X]$). Therefore by [2, Théorème 2.6], there exists $u \in \mathrm{qf}(R)$ such that the extension $R \subset R[u] = S$ is not incomparable. We consider the epimorphism of R-algebras $\varphi : R[X] \to R[u]$, $X \mapsto u$. We have $S = R[u] \simeq R[X]/P$ where $P = \mathrm{Ker}\,\varphi$. The extension $R \subset S$ satisfies the altitude formula. This comes from the following lemma.

Lemma [12, Théorème 2.2]. *Let R be a domain, n a nonzero positive integer and P a prime ideal of $R[n]$ such that $P \cap R = (0)$. Then the extension $R \subset R[n]/P$ satisfies the altitude formula if and only if for any prime ideal Q of $R[n]$ containing P, we have* $\mathrm{ht}(Q/P) = \mathrm{ht}\, Q - \mathrm{ht}\, P$.

The following example exhibits a marked difference between the classical altitude formula (or inequality) and RAF (resp. RAI).

Example 4.3. (a) *There is an algebraic extension $R \subset S$ of domains satisfying RAI without satisfying RAF or RVAF.*

We consider two incomparable valuation domains V and W with the same quotient field such that $\dim W > \dim V$. We denote by M and N the maximal ideals of V and W respectively. Let $S = V \cap W$. Then S is a Prüfer domain with $M' = M \cap S$ and $N' = N \cap S$ as maximal ideals. Also we can assume that $S/M \simeq S/N \simeq K$, where K is a subfield of $\mathrm{qf}(V)$. We set $I = M' \cap N'$ and $R = \varphi^{-1}(K)$ where $\varphi : S \to S/I$ is the natural epimorphism. Then $S = R'$ ([8]). We have $\dim V = \mathrm{ht}\, M' = \mathrm{ht}_{\mathrm{v}} M' < \mathrm{ht}\, I + \mathrm{tr.deg}[S : R] = \mathrm{ht}_{\mathrm{v}} I + \mathrm{tr.deg}[S : R] = \dim W$. Hence, the extension

$R \subset S$ does not satisfy RAF and RVAF. However, $R \subset S$ satisfies RAI because it is an integral extension.

(b) *There exists a ring extension $R \subset S$ satisfying the valuative altitude formula without satisfying RAI.*

Let R be a non-locally Jaffard domain. Then there exists a positive integer n such that the extension $R \subset S = R[n]$ does not satisfy the altitude inequality. Hence, by Theorem 1.2, $R \subset S$ does not satisfy RAI, although it satisfies the valuative altitude formula [12, Théorème 2.1].

(c) *There exists an integral extension satisfying the altitude formula without satisfying the valuative altitude formula. Hence, it satisfies RAF without satisfying RVAF.*

(d) *There exists an integral extension satisfying the valuative altitude formula without satisfying the altitude formula. Therefore, it satisfies RAF without satisfying RVAF.*

(e) *If R is a locally Jaffard domain, then the extension $R \subset R[n]$ satisfies the altitude formula without satisfying RAF or RVAF.*

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