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QUANTUM LOGICS WITH CLASSICALLY DETERMINED STATES

BY

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1. Preliminaries. The standard notions on quantum logics we deal with in this paper are taken from [4] and [11]. Let us briefly recall the basic ones. By a *quantum logic* we mean a triple $(L, \leq, ')$, where L is a set partially ordered by $\leq$ and $'$ is an orthocomplementation relation on L, such that the following properties are satisfied:

(i) a least and a greatest element, 0 and 1, exist in L,
(ii) if $a \leq b$, then $b' \leq a'$, and $a = a''$ $(a, b \in L)$,
(iii) $a \vee a' = 1,\ a \wedge a' = 0$ $(a \in L)$,
(iv) if $a \leq b$, then $b = a \vee (b \wedge a')$ (the orthomodular law).

By a *state* on L we mean a probability measure on L. Thus, a mapping $s : L \to [0, 1]$ is a state on L if

(i) $s(1) = 1$,
(ii) $s(a \vee b) = s(a) + s(b)$ provided $a \leq b'$ $(a, b \in L)$.

Let us denote by $\mathscr{S}(L)$ the set of all states on L. By a standard argument, $\mathscr{S}(L)$ is a convex subset of the topological linear space $\mathbb{R}^L$, and moreover, $\mathscr{S}(L)$ is compact with respect to the topology of $\mathbb{R}^L$ (in other words, $\mathscr{S}(L)$ is compact in the pointwise topology). Notice that $\mathscr{S}(L)$ may be quite poor or even empty (see [3]).

Let us now introduce the main notion of this paper. We say that L is *state-classically-determined* (abbr.: L is SCD) if there is a Boolean subalgebra B of L such that, for any state $s \in \mathscr{S}(L)$, if $a \in L$ then there is $b \in B$ with $b \leq a$ so that $s(a) = s(b)$. We are interested in the ques-

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tion of when the SCD logics must be classical. Obviously, the character of the problem changes when we pass from the finitely additive setup to the σ-additive setup or completely additive setup of the problem. It should be noted that the SCD logics may also be relevant to certain considerations of noncommutative measure theory (see [5]).

Prior to presenting our results, let us recall the notion of concrete logic (see [10]). This notion will play a prominent rôle in the sequel. A quantum logic is called *concrete* (alias *set-representable*) if L is isomorphic (in the natural sense based on the quantum logic morphism) to a collection of subsets of a set. Thus, equivalently, the triple $L = (\Delta, \leq, ')$, where Δ is a collection of subsets of a set S with $\leq$ meaning the set inclusion and $'$ meaning the set complement, is a concrete logic if the following three conditions are satisfied:

(i) $\emptyset \in \Delta$,
(ii) if $A \in \Delta$, then $S - A \in \Delta$,
(iii) if $A, B \in \Delta$ and $A \cap B = \emptyset$, then $A \cup B \in \Delta$.

We sometimes write $L = (S, \Delta)$ (instead of $L = \Delta$) when we need to refer to the underlying set S. Observe that a concrete logic is a Boolean algebra if and only if it is closed under the formation of intersections (i.e., if $L = \Delta$ then L is Boolean if and only if $A \cap B \in \Delta$ for any pair $A, B \in \Delta$).

2. Results. Let us first consider the "finitely additive" SCD logics. Thus, the definitions of the previous section apply without changes. Let L be a logic. If the state space $\mathscr{S}(L)$ is empty (see [3]), then L is SCD by definition. Continuing in this vein, it can be seen that there are logics with exactly one state which are SCD. Indeed, if L is stateless, then the logic $L \times \{0, 1\}$ is such an example (see [9]). Obviously, we would rather be interested in logics which have a fairly large set of states. Let us say that L *has enough states* if the following condition is satisfied: If $a \not\leq b$ in L, then there is a state $s \in \mathscr{S}(L)$ such that $s(a) = 1$ and $s(b) < 1$. Naturally, if L is a Boolean algebra, then L has enough states. The result we want to prove first says that the converse is true for the SCD logics. (Surprisingly, a one-line proof does not seem to be available—we were expecting to use the compactness of the state space plus fairly nontrivial intrinsic properties of logics.)

THEOREM 1. *Let L be a quantum logic and let L have enough states. If L is SCD*, *then L is Boolean.*

Proof. Let us first make two observations. The first relates the class of logics we are interested in to the Jauch–Piron property for states (see [10], [2], [11], etc., for discussions of this property relevant to the foundation of quantum mechanics).

Observation 1. *Under the assumptions of Theorem* 1, *there is a Boolean subalgebra B of L so that the following strong form of Jauch–Piron property is satisfied*: *If $s(a) = s(b) = 1$ $(a, b \in L,\ s \in \mathscr{S}(L))$, then there is an element $c \in B$ with $c \leq a,\ c \leq b$ such that $s(c) = 1$.*

Indeed, there are elements $e, f \in B$ such that $e \leq a, f \leq b$ and $s(e) = s(f) = 1$. Since both e, f belong to B, we easily see that $s(e \wedge f) = 1$. Thus, setting $c = e \wedge f$, we obtain $s(c) = 1$.

Observation 2. *Under the assumptions of Theorem* 1, *L is a concrete logic.*

Indeed, all we have to show is that the following implication holds true: If $a \wedge b = 0$ in L $(a, b \in L)$, then $a \leq b'$. It is known (see [8]) that then L must have enough two-valued states, and this easily implies (see e.g. [14]) that L is concrete.

Suppose that $a \not\leq b'$. Then there is a state $s \in \mathscr{S}(L)$ such that $s(a) = 1$ and $s(b') < 1$. Equivalently, there is a state $s \in \mathscr{S}(L)$ such that $s(a) = 1$ and $s(b) > 0$. Since L is SCD, there is a Boolean subalgebra B of L and elements $c, d \in B$ so that $c \leq a,\ d \leq b$ and $s(a) = s(c)$ and $s(d) = s(b)$. If we set $e = c \wedge d$, then $s(e) = s(b) > 0$. It follows that $e \neq 0$. Since $e \leq a, e \leq b$, we see that $a \wedge b$ either does not exist or is distinct from 0. Thus, $a \wedge b \neq 0$ and this is what we were to show.

Let us take up the proof of Theorem 1. We may assume that L is concrete (Observation 2). Thus, $P = \varDelta$, where $\varDelta \subset \exp S$. Write $B = \widetilde{\varDelta}$ for the Boolean subalgebra of $\varDelta$ guaranteed by the assumptions of Theorem 1 $(\widetilde{\varDelta} \subset \exp S)$. Take two elements of $\varDelta$, some sets A and B, and consider $A \cap B$. We want to show that $A \cap B \in \varDelta$. Put $\mathscr{S}_{A,B} = \{s \in \mathscr{S}(\varDelta) \mid s(A) = s(B) = 1\}$. Then $\mathscr{S}_{A,B}$ is a closed subset of $\mathscr{S}(\varDelta)$ and therefore $\mathscr{S}_{A,B}$ is compact. Further, put $\mathscr{C} = \{C \in \widetilde{\varDelta} \mid C \subset A \cap B\}$ and let, for any $C \in \mathscr{C}$, $\mathscr{S}_C = \{s \in \mathscr{S}_{A,B} \mid s(C) > 0\}$. Then the set $\{\mathscr{S}_C \mid C \in \mathscr{C}\}$ forms an open covering of $\mathscr{S}_{A,B}$ (one makes use of Observation 1). Since $\mathscr{S}_{A,B}$ is compact, we see that there is a finite subcollection of $\mathscr{C}$, some sets $C_1, \ldots, C_n$, such that $\mathscr{S}_{A,B} = \bigcup_{i \leq n} \mathscr{C}_{C_i}$. It remains to be shown that $\bigcup_{i \leq n} C_i = A \cap B$. This will suffice, for $\bigcup_{i \leq n} C_i$ belongs to $\widetilde{\varDelta}$ and therefore to $\varDelta$. Suppose on the contrary that $\bigcup_{i \leq n} C_i \neq A \cap B$. Then there is a point $p \in S$ such that $p \in (A \cap B) - \bigcup_{i \leq n} C_i$. Consider now the (two-valued) state $s_p \in \mathscr{S}(\varDelta)$ concentrated in p. Thus, $s_p(D) = 1$ if and only if $p \in D$. It follows that $s_p \in \mathscr{S}_{A,B}$ and therefore $s_p \notin \bigcup_{i \leq n} \mathscr{C}_{C_i}$. This is a contradiction. The proof of Theorem 1 is complete.

Note that, in view of the above result, there are concrete Jauch–Piron logics which are not SCD. Indeed, there exist non-Boolean concrete Jauch–Piron logics (see [7]).

Let us now pass to the σ-additive setup of our question. For the σ-additive (σ-complete) setup we translate the notions we deal with into the language of σ-complete logics. Thus,

(i) a *σ-complete logic* is a logic L which is closed under the formation of suprema of countable families of mutually orthogonal elements in L,

(ii) a *σ-additive state* on L is defined in the standard way,

(iii) the symbol $\mathscr{S}_\sigma(L)$ denotes the set of all σ-additive states on L.

A *state-classically-determined* logic (an SCD_σ logic) will now mean a σ-complete logic L which has a Boolean σ-algebra B as its sublogic such that, for any $s \in \mathscr{S}_\sigma(L)$ and any $a \in L$, there is $b \in B$ with $b \leq a$ so that $s(a) = s(b)$. Apart from the examples of SCD_σ logics with extremally poor state spaces, every Boolean σ-algebra is an SCD_σ logic. We are going to show that there are many others, in particular, that there are many non-Boolean concrete ones.

Let us first recall a few notions and make some conventions. A mapping $f : L_1 \to L_2$ between two (σ-complete) logics is called a *morphism* if

(i) $f(1) = 1$,

(ii) $f(a') = f(a)'$ $(a \in L_1)$, and

(iii) $f(\bigvee_{i\in\mathbb{N}} a_i) = \bigvee_{i\in\mathbb{N}} f(a_i)$ provided $a_i \leq a_j'$, $i \neq j$.

A morphism $f : L_1 \to L_2$ is called an *embedding* if f is injective and the inequality $f(a) \leq f(b)'$ implies $a \leq b'$ $(a, b \in L)$. If $f : L_1 \to L_2$ is an embedding then L_1 is called a *sublogic* of L_2. (Obviously, if L_1 is a sublogic of L_2 and L_1 is not Boolean, then L_2 is not Boolean either.)

Our main result in this section says that every concrete (σ-complete) logic L_1 which is not "very big" is a sublogic of a concrete SCD σ-complete logic L_2. In proving the result, we use some technique of [2] and [6]. We make the following convention. Let $\aleph_0$ be the countable cardinal and let $\aleph_{n+1}$ be the successor cardinal of $\aleph_n$. Put $\alpha = \sup\{\aleph_n \mid n \in \mathbb{N}\}$. Let S be a set. We say that S is *not very large* if the cardinality of S, written $\operatorname{card} S$, is less than α. (It seems plausible that the logics encountered in applications to quantum theories are not very large. The important technical aspect of the notion is that if S is not very large and if $\exp S$ means the σ-algebra of all subsets of S, then every σ-additive probability measure on $\exp S$ must live on a countable subset of S; see e.g. [13].)

THEOREM 2. *Suppose that $L_1 = (S, \Delta_1)$ is a concrete σ-complete logic and suppose that S is not very large. Then L_1 can be embedded into a concrete σ-complete SCD_σ logic $L_2 = (W, \Delta_2)$ so that W is not very large. Hence, in particular, there exist concrete non-Boolean σ-complete SCD_σ logics.*

P r o o f. Since S is not very large, the set S is either finite or $\operatorname{card} S = \aleph_n$ for some $n \in \mathbb{N}$. Choose a set T as follows: If $\operatorname{card} S$ is finite, we take T so that $\operatorname{card} T = \aleph_1$, and if $\operatorname{card} S = \aleph_n$, we take T so that $\operatorname{card} T = \aleph_{n+1}$. Put $W = T \times S$. Then W is not very large. Define a collection $\varDelta_2 \subset \exp W$ by the following requirement: $A \in \varDelta_2$ if and only if

$$\operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap A = \{t\} \times U \text{ for some } U \in \varDelta_1\}) \leq \chi_0.$$

Let us verify that $\varDelta_2$ is a σ-complete logic. Obviously, $\emptyset \in \varDelta_2$. Further, since $\varDelta_1$ is closed under the formation of complements, we see that

$$\begin{aligned}&\operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap A = \{t\} \times U \text{ for some } U \in \varDelta_1\})\\ &\quad = \operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap (W - A) = \{t\} \times U \text{ for some } U \in \varDelta_1\}).\end{aligned}$$

It follows that if $A \in \varDelta_2$ then $W - A \in \varDelta_2$. Now we assume that $\{A_n \mid n \in \mathbb{N}\}$ is a collection of mutually disjoint elements of $\varDelta_2$. Put $A = \bigcup_{n\in\mathbb{N}} A_n$. Then

$$\begin{aligned}&\operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap A = \{t\} \times U \text{ for some } U \in \varDelta_1\}\\ &\leq \sum_{n\in\mathbb{N}} \operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap A_n = \{t\} \times U \text{ for some } U \in \varDelta_1\}.\end{aligned}$$

Thus, if every A_n belongs to $\varDelta_2$, then so does $\bigcup_{n\in\mathbb{N}} A_n$. We have checked that $\varDelta_2$ is a σ-complete logic.

We now assert that $\varDelta_1$ is a sublogic of $\varDelta_2$. Indeed, if we define the mapping $e : \varDelta_1 \to \varDelta_2$ by setting $e(C) = T \times C$ $(C \in \varDelta_1)$, we immediately see that e is an embedding. Thus, $\varDelta_1$ is a sublogic of $\varDelta_2$.

Finally, we want to show that $\varDelta_2$ is SCD_σ. Consider first the Boolean σ-algebra $\widetilde{B} = (W, \widetilde{\Sigma})$, where $\widetilde{\Sigma} = \{V \times S \mid V \subset T\}$. Since, for any $A \in \widetilde{\Sigma}$, we have

$$T - \{t \in T \mid (\{t\} \times S) \cap A = \{t\} \times U \text{ for some } U \in \varDelta_1\} = \emptyset,$$

we infer that $\widetilde{\Sigma}$ is a sublogic of $\varDelta_2$. Moreover, $\widetilde{\Sigma}$ is isomorphic (as a Boolean σ-algebra) to the σ-algebra $\exp T$. Further, consider the Boolean σ-algebra $\widetilde{\widetilde{B}} = (W, \widetilde{\widetilde{\Sigma}})$, where $\widetilde{\widetilde{\Sigma}}$ is generated by all sets of the form $\{t\} \times X$ $(t \in T, X \subset S)$. Obviously, for any $A \in \widetilde{\widetilde{\Sigma}}$ we have

$$\operatorname{card}(T - \{t \in T \mid (\{t\} \times S) \cap A = \{t\} \times U\} \leq \chi_0.$$

Thus, $\widetilde{\widetilde{\Sigma}}$ is a sublogic of $\varDelta_2$. Finally, let $B = (W, \Sigma)$ be the Boolean σ-algebra generated by $\widetilde{\Sigma} \cup \widetilde{\widetilde{\Sigma}}$. We shall show that for any state $s \in \mathscr{S}(\varDelta_2)$ and any $A \in \varDelta_2$ there is a set $D \in \Sigma$ such that $D \subset A$ and $s(D) = s(A)$.

We first claim that there is a countable subset P of W so that $s(P) = 1$. Indeed, since $\widetilde{\Sigma}$ is a sublogic of $\varDelta_2$, we may view s as a state on $\widetilde{\Sigma}$. But $\widetilde{\Sigma} = \exp T$ and T is not very large. By [13], s must be concentrated on a countable set. Thus, there is a countable set K with $K \subset T$ such that $s(K \times S) = 1$. But Σ contains all subsets of $K \times S$ and therefore s can be also viewed as a state on $\exp(K \times S)$. It follows that s must be concentrated on a countable set and therefore there is a countable subset P of $K \times S$ such that $s(P) = 1$.

The rest is easy. Given $A \in \Delta_2$, we easily see that if we put $D = P \cap A$, we have $D \subset A$ and $s(D) = s(A)$. The proof is complete.

It may be worth noting that we have proved slightly more in Theorem 2: the desired Boolean sublogic B was in fact a sublogic of the centre of the logic constructed (i.e., B consisted of absolutely compatible elements of Δ_2; see e.g. [10]). Thus, for the σ-complete case the notion is meaningful within "nonclassical logics" even in its stronger central form.

Let us briefly consider the stability properties of the class of σ-complete SCD_σ logics. It turns out that this class is fairly large, including both concrete and nonconcrete logics (as is known, Boolean σ-algebras are often nonconcrete; see e.g. [12]).

PROPOSITION 3. (i) *Suppose that* L_2 *is an epimorphic image of* L_1 *and* L_1 *is* SCD_σ. *Then so is* L_2.

(ii) *Let* $\{L_\alpha \mid \alpha \in I\}$ *be a collection of logics. Let* L_α *be* SCD_σ *for any* $\alpha \in I$ *and let* I *be not very big. Then the direct product* $\prod_{\alpha \in I} L_\alpha$ *is an* SCD_σ *logic. Hence, in particular, a countable product of* SCD_σ *logics is an* SCD_σ *logic.*

Proof. (i) Suppose that $f : L_1 \to L_2$ is a logic epimorphism. Let s_2 be a state on L_2. Then the composite mapping $s_1 = s_2 \cdot f$ is a state on L_1 and therefore there is a Boolean σ-algebra B_1 of L so that, for any $a \in L_1, s_1(a) = s_1(b)$ for some $b \in B$ with $b \le a$. Since a logic morphism maps Boolean sublogics into Boolean sublogics (see e.g. [10]), the set $B_2 = f(B_1)$ is a Boolean sublogic of L_2. Suppose that $c_2 \in L_2$. Choose an arbitrary $c_1 \in L$ such that $f(c_1) = c_2$. Then there is an element $d_1 \in L$ with $d_1 \in B,\ d_1 \le c_1$ so that $s_1(d_1) = s_1(c_1)$. Put $d_2 = f(d_1)$. Then $d_2 \le c_2$ and $s_2(d_2) = s(f(d_1)) = s_1(d_1) = s_1(c_1) = s_2(f(c_1)) = s_2(c_2)$.

(ii) Let $L = \prod_{\alpha \in I} L_\alpha$ and let B_α $(\alpha \in I)$ be the Boolean sublogic of L_α which makes L_α an SCD_σ logic. Let $B = \prod_{\alpha \in I} B_\alpha$ and let $s \in \mathscr{S}(L)$. Since I is not very large, we can show easily that there is a countable subset J of I and states $s_j \in \mathscr{S}(L)$ such that, for a partition of unity $\sum_{j \in J} \alpha_j = 1$, we can write $s = \sum_{j \in J} \alpha_j \cdot \widetilde{s}_j$, where $\widetilde{s}_j \in \mathscr{S}(L)$ are states defined so that $\widetilde{s}_j(a) = \widetilde{s}_j(a_1, a_2, \ldots, a_j, \ldots) = s_j(a_j)$. With the help of the Boolean σ-algebra B, it is easy to show by "coordinate-wise reasoning" that for any $a \in L$ there is $b \in B$ with $b \le a$ such that $s(a) = s(b)$. This concludes the proof.

Let us finally consider the complete state-classically-determined logics (abbr.: $\mathrm{SCD}_{\mathrm{comp}}$). A logic L is said to be *complete* if every collection in L of orthogonal elements possesses a supremum in L. Upon the obvious translation of all notions into the complete setup, we can formulate the following result (by $\mathscr{S}_{\mathrm{comp}}(L)$ we denote the set of all completely additive measures on L).

THEOREM 4. *Let L be a complete quantum logic. Let L have enough completely additive states and let L be SCD_{comp}. Then L is a complete Boolean algebra.* (*Hence*, *in particular*, *if a logic of projections in a von Neumann algebra $\mathscr{A}$ is SCD_{comp}*, *then $\mathscr{A}$ is commutative.*)

Proof. Using the reasoning employed in Observation 2, we immediately see that L has to be concrete (with respect to finite operations!). Thus, $L = (\Omega, \Delta)$, where Δ is a concrete logic. Let $A, B \in \Delta$. We have to show that $A \cap B \in \Delta$. This is sufficient since a complete Boolean logic must obviously be a complete Boolean algebra.

Consider the set $A \cap B$. Suppose that $A \cap B \neq \emptyset$. Thus, $A \not\subset \Omega - B$. Put $\mathscr{S} = \{s \in \mathscr{S}_{\mathrm{comp}}(\Delta) \mid s(A) = 1 \text{ and } s(B) > 0\}$. Since L has enough completely additive states and since L is $\mathrm{SCD}_{\mathrm{comp}}$, we easily see that $\mathscr{S} \neq \emptyset$. Write $\mathscr{S} = \{s_\alpha \mid \alpha < \beta \text{ for a cardinal number } \beta\}$. Thus, $\mathscr{S}$ becomes well-ordered. Consider the state $s_1 \in \mathscr{S}$. Since L is $\mathrm{SCD}_{\mathrm{comp}}$, there is a set $C_1 \in \Delta$ such that $C_1 \subset A \cap B$ and $s_1(C_1) > 0$. It follows that $C_1 \neq \emptyset$. If $C_1 = A \cap B$, we are done. If $C_1 \neq A \cap B$, then there is a least index $\alpha_1 < \beta$ so that $s_{\alpha_1} \in \mathscr{S}$ and $s_{\alpha_1}(A - C_1) = 1$, $s_{\alpha_1}(B - C_1) > 0$. Since L is $\mathrm{SCD}_{\mathrm{comp}}$, there is a set C_{α_1} so that $s_{\alpha_1}(C_{\alpha_1}) > 0$ and $C_{\alpha_1} \subset (A - C_1) \cap (B - C_1)$. It follows that $C_{\alpha_1} \neq \emptyset$. Moreover, $C_1 \cap C_{\alpha_1} = \emptyset$ and $C_1 \cup C_{\alpha_1} \subset A \cap B$. In the next step we obtain $C_{\alpha_{\alpha_1}}$, etc. If we continue this way, taking the unions of the previously constructed sets C_γ for the limit cardinals, we obtain a disjoint covering $\{C_\delta \mid \delta \in I\}$ of $A \cap B$ consisting of sets belonging to Δ. Since Δ is complete, the supremum $\bigvee_{\delta \in I} C_\delta$ exists in Δ. But $\bigvee_{\delta \in I} C_\delta$ must be a subset of both A and B. Consequently, $\bigvee_{\delta \in I} C_\delta = A \cap B$ and therefore $A \cap B \in \Delta$. This completes the proof.

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