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TWO REMARKS ABOUT SPECTRAL ASYMPTOTICS OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. In this paper we show an asymptotic formula for the number of eigenvalues of a pseudodifferential operator. As a corollary we obtain a generalization of the result by Shubin and Tulovskiĭ about the Weyl asymptotic formula. We also consider a version of the Weyl formula for the quasi-classical asymptotics.

In [11] Shubin and Tulovskiĭ discuss the class of pseudodifferential operators

$$Af(x) = \iint e^{2\pi i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi$$

on $\mathbb{R}^N$ whose Weyl symbols are positive and satisfy

(i) $$c|w|^\delta \le a(w) \le C|w|^p,$$

(ii) $$|\partial^\alpha a(w)| \le C_\alpha a(w)^{1-\varrho|\alpha|},$$

(iii) $$|\langle a'(w), w\rangle| \ge Ca(w)^{1-\kappa},$$

where $\delta > 0$, $0 < \kappa < \varrho \le 1$, $p \in \mathbb{N}$ are fixed constants, $\alpha \in \mathbb{N}^{2N}$ and $w \in \mathbb{R}^{2N}$. Such an operator A is selfadjoint on $L^2(\mathbb{R}^N)$, bounded from below, and has a discrete spectrum. Shubin and Tulovskiĭ prove the Weyl asymptotic formula

$$N_A(\lambda) = \iint_{a\le\lambda} dw + O(\lambda^{-\sigma}), \qquad 0 < \sigma < \varrho - \kappa,$$

for the number of eigenvalues of A smaller than or equal to λ.

This has been extended by Hörmander [5] to much more general classes and with better estimates for the error term within the framework of his general Weyl calculus. Then a similar question was considered by Głowacki [4] for the class of positive symbols with the following properties:

(a) $$|\partial^\alpha a(w)| \le C_\alpha a(w)^{1-\varrho}, \qquad |\alpha| > 0,$$

(b) $$|\partial^\alpha a| \le C_\alpha, \qquad |\alpha| \gg 1,$$

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(c) $$\lim_{\|w\|\to\infty} a(w) = \infty.$$

This class is larger than that of Shubin–Tulovskiĭ and is not included in any of Hörmander's classes. However, the Weyl formula still holds. The reader is referred to the papers quoted above for comparison of the estimates of the error term.

The first goal of this note is to show that a special case of Głowacki's theorem still holds even if the condition (b) is dropped. For this we will use a version of Beals' theorem on fractional powers of pseudodifferential operators (see Beals [1]). This reduces the proof to the theorem obtained by Głowacki.

The other part of the paper is devoted to the quasi-classical asymptotics

$$N_{A_{(h)}}(\lambda) = h^{-N}(V(\lambda) + O(h^{1/2})), \qquad h \to 0,$$

where $A_{(h)} = \mathrm{Op}(a^{(h)})$, $a^{(h)}(z) = a(\sqrt{h}z)$, $h > 0$, for positive symbols satisfying

(a′) $$|\partial^\alpha a(w)| \le C_\alpha a(w)^{(1-\varrho|\alpha|)_+}, \qquad \text{with } t_+ = \max(t, 0),\ t \in \mathbb{R},$$

(c′) $$\lim_{\|w\|\to\infty} a(w) = \infty.$$

We use Głowacki's version of the Shubin–Tulovskiĭ method of approximate spectral projectors based on a lemma of Hörmander [5]. This improves results of Roĭtburd [9] (see also [10]).

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1. Pseudodifferential operators. Let V be an N-dimensional vector space and V^* its dual space. Fix a Euclidean norm on V and the dual norm on V^*. The space $W = V \times V^*$ with the product norm will be called the *phase space*. Let $\{e_j\}_{j=1}^N$ be an orthonormal basis in V and $\{e_j\}_{j=N+1}^{2N}$ the dual basis in V^*. For a multindex $\alpha \in \mathbb{N}^{2N}$ and a smooth function f on W we define

$$\partial^\alpha f = \partial_1^{\alpha_1} \ldots \partial_{2N}^{\alpha_{2N}} f, \quad \text{where} \quad \partial_j f(w) = \frac{1}{2\pi i}\frac{d}{dt}\bigg|_{t=0} f(w + te_j).$$

In the phase space W we define the following symplectic form:

$$\sigma(w, v) = y\xi - x\eta,$$

where $w = (x, \xi)$ and $v = (y, \eta)$. Moreover, let $\langle v\rangle = (1 + \|v\|^2)^{1/2}$ for $v \in W$. Then we have *Peetre's inequality*

(1) $$\langle v + w\rangle \le \sqrt{2}\langle v\rangle\langle w\rangle.$$

A strictly positive, continuous function m on W is called a *weight* if

$$m(w+v) \le Cm(w)\langle v\rangle^n$$

for every $w, v \in W$ and some constants $C, n > 0$. In particular, for every weight m we have

$$\frac{1}{C}\langle w\rangle^{-n} \le m(w) \le C\langle w\rangle^n.$$

The set of weights is a group under multiplication. Moreover, if $\alpha \in \mathbb{R}$ and m is a weight, then m^α is also a weight. For a fixed weight m we define $S(m)$ to be the set of all functions $a \in \mathcal{C}^\infty(W)$ such that

$$\max_{|\alpha|\le k} \|m^{-1}\partial^\alpha a\|_\infty < \infty \quad \text{for } k = 0, 1, 2, \ldots$$

Let $\mathcal{S}(V)$ denote the space of Schwartz functions on V. Every function $a \in \mathcal{C}^\infty(W)$ having derivatives with a common polynomial growth (e.g. $a \in S(m)$) defines a continuous endomorphism $A = \mathrm{Op}(a) : \mathcal{S}(V) \to \mathcal{S}(V)$ given by the Weyl formula

$$Af(x) = \iint e^{2\pi i(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi.$$

Such an operator A is called a *pseudodifferential operator* and the function a is the *symbol* of A. We denote by $\mathcal{L}(m)$ the class of operators $\mathrm{Op}(a)$ with $a \in S(m)$.

We will use the following propositions:

(1.1) PROPOSITION (see [3]). *If $A \in \mathcal{L}(m_1)$ and $B \in \mathcal{L}(m_2)$, then $AB \in \mathcal{L}(m_1m_2)$ and for $k \in \mathbb{N}$ large enough, we have*

$$a \circ b(w) = 4^N \iint \frac{(1+\Delta_u)^k a(w+u)}{\langle 2u\rangle^{2k}} (1+\Delta_v)^k \left[\frac{b(w+v)}{\langle 2v\rangle^{2k}}\right] e^{4\pi i\sigma(v,u)}\, du\, dv,$$

where $a \circ b$ denotes the symbol of AB.

(1.2) PROPOSITION (see [3]). *If $a \in S(m_1)$, $b \in S(m_2)$ and $\partial^\alpha a \in S(m_1')$, $\partial^\alpha b \in S(m_2')$ then $r(a,b) = a \circ b - ab \in S(m_1'm_2')$. Moreover,*

$$\begin{aligned} r(a,b)(w) = {} & \frac{4^N}{2} \sum_{j=1}^N \int_0^1 dt \iint \frac{(1+t^2\Delta_u)^k \partial_j a(w+u)}{\langle 2u\rangle^{2k}} \\ & \times (1+\Delta_v)^k \left[\frac{\partial_{j+N} b(w+v)}{\langle 2v\rangle^{2k}}\right] e^{4\pi i\sigma(v,u)}\, du\, dv \\ & - \frac{4^N}{2} \sum_{j=1}^N \int_0^1 dt \iint \frac{(1+t^2\Delta_u)^k \partial_{j+N} a(w+u)}{\langle 2u\rangle^{2k}} \\ & \times (1+\Delta_v)^k \left[\frac{\partial_j b(w+v)}{\langle 2v\rangle^{2k}}\right] e^{4\pi i\sigma(v,u)}\, du\, dv. \end{aligned}$$

A symbol $a \in S(m)$ is called *elliptic* if for some constants $C, K > 0$,

$$|a(w)| \geq Cm(w) \quad \text{for } |w| \geq K \text{ and}$$
$$\partial^\alpha a \in S(m^{1-\varrho}) \quad \text{for } |\alpha| > 0 \text{ and some } \varrho > 0, \text{ independent of } \alpha.$$

(1.3) PROPOSITION (see [3] and [4]). *Suppose $m \geq c > 0$ is a weight and a is an elliptic symbol in $S(m)$. Let $\mathrm{Op}(a) : \mathcal{S}(V) \to L^2(V)$.*

(i) *If a is real-valued then $\mathrm{Op}(a)$ is essentially selfadjoint.*
(ii) *If $a \geq 0$ then $\mathrm{Op}(a)$ is bounded from below.*
(iii) *If $\lim_{|w|\to\infty} m(w) = \infty$, then the spectrum of $\mathrm{Op}(a)$ is discrete.*
(iv) *If $\lim_{|w|\to\infty} a(w) = 0$, then $\mathrm{Op}(a)$ is compact.*

(1.4) PROPOSITION (see [4]). *If $a \geq 0$ and $\max_{0<|\alpha|\leq 2N+2} |\partial^\alpha a| \leq C$, then*

$$\mathrm{Op}(a) \geq -LC,$$

where L is a constant independent of a.

(1.5) PROPOSITION (see [10]). *If $a \in S(m)$ and $a \geq 0$, then $\mathrm{Op}(H \star a)$ is positive, where $H(w) = e^{(-2\pi\|w\|^2)}$.* (Here $\star$ denotes convolution.)

We will also need the following technical estimates:

(1.6) PROPOSITION (see [4]). *Let a and b be symbols. Then*

$$\|\partial^\alpha r(a,b)\|_1 \leq C \max_{0<|\beta|\leq|\alpha|+2k+1} \|\partial^\beta a\|_1 \max_{0<|\beta|\leq|\alpha|+2k+1} \|\partial^\beta b\|_\infty.$$

Moreover, if $\partial_j a \in S(m_1)$ and $\partial_j b \in S(m_2)$ for $j = 1, \ldots, 2N$, then

$$\|\partial^\alpha r(a,b)\|_\infty \leq C \max_{0<|\beta|\leq|\alpha|+2k+1} \|m_1 \partial^\beta b\|_\infty$$

and

$$\|\partial^\alpha r(a,b)\|_\infty \leq C \max_{0<|\beta|\leq|\alpha|+2k+1} \|m_2 \partial^\beta a\|_\infty.$$

(1.7) PROPOSITION (see [4]). *Let $e, f \in S(1)$, $a \in S(m)$ and $\partial_j a \in S(n)$, for $j = 1, \ldots, 2N$. Then for all α and k sufficiently large*

$$\begin{aligned}\|\partial^\alpha r(e,a,f)\|_\infty \leq C\Big(&\max_{0<|\beta|\leq|\alpha|+k} \|n\,\partial^\beta e\|_\infty \|f\|_\infty \\ &+ \max_{0<|\beta|\leq|\alpha|+k} \|\partial^\beta (ea)\|_\infty \max_{0<|\beta|\leq|\alpha|+k} \|\partial^\beta f\|_\infty \\ &+ \max_{0<|\beta|\leq|\alpha|+k} \|\partial^\beta e\|_\infty \max_{0<|\beta|\leq|\alpha|+k} \|n\,\partial^\beta f\|_\infty\Big).\end{aligned}$$

Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$. An operator $T \in L(\mathcal{H})$ is called a *trace operator* if there is an orthonormal basis $\{e_\alpha\}$ such that

$$\sum_\alpha |\langle Te_\alpha, e_\alpha\rangle| < \infty.$$

Then the number

$$\operatorname{Tr} T = \sum_{\alpha} \langle T e_\alpha, e_\alpha \rangle$$

does not depend on the choice of basis and is called the *trace* of T. We let $\|T\|_{\mathrm{Tr}} = \operatorname{Tr}|T|$.

(1.8) PROPOSITION (see [4]). *If $a \in L^1(W)$ and $\mathrm{Op}(a)$ is a trace operator then*

$$\operatorname{Tr}\mathrm{Op}(a) = \iint a(x,\xi)\,dx\,d\xi.$$

(1.9) PROPOSITION (see [4]). *Let $a \in \mathcal{C}^\infty(W)$. If $\partial^\alpha a \in L^1(W)$ for $|\alpha| \le 2N+1$, then $\mathrm{Op}(a)$ extends uniquely to a trace operator A on $L^2(V)$ and*

$$\|A\|_{\mathrm{Tr}} \le C \max_{|\alpha| \le 2N+1} \|\partial^\alpha a\|_{L^1}.$$

Let us also state a lemma which connects the condition (a′) with the definition of weight (the proof of the lemma relies only on Taylor's formula).

(1.10) LEMMA. *If $a \ge C > 0$ and there exists $\varrho > 0$ such that $|\partial^\alpha a| \le C_\alpha a^{(1-|\alpha|\varrho)_+}$ for all α, then $1+a$ is a weight.*

2. Classical asymptotics. Let A be a selfadjoint operator bounded from below with a discrete spectrum. We denote by $N_A(\lambda)$ the number of eigenvalues of A less than or equal to λ (counting with multiplicities). For a positive function $a \in \mathcal{C}^\infty(W)$ we define

$$V_a(\lambda) = \int_{a \le \lambda} dw \quad \text{and} \quad \psi_a(t) = \inf_{a(w)=t} |\langle a'(w), w\rangle|.$$

For $\lambda, m > 0$, we also define

$$\nu(\lambda, m) = \int_{\lambda}^{\lambda+m} \psi_a(t)^{-1}\,dt.$$

(2.1) PROPOSITION. *If for $t \ge \lambda$ we have $\psi_a(t) > 0$ and $\nu(\lambda,m) \le \log 2$, then*

$$V_a(\lambda+m) - V_a(\lambda) \le 2\nu(\lambda,m) V_a(\lambda).$$

Proof. Note that V is increasing, hence differentiable almost everywhere on $\mathbb{R}^+$. By formula (28.41) of Shubin [10] for $t \ge \lambda$ we have

$$V_a'(t)/V_a(t) \le \psi_a(t)^{-1}.$$

Using a simple estimate $e^x - 1 \le xe^x$ valid for $x \in \mathbb{R}$ we obtain

$$\begin{aligned}\frac{V_a(\lambda+m)-V_a(\lambda)}{V_a(\lambda)} &= \exp\Big(\int\limits_{\lambda}^{\lambda+m} \frac{V_a'(t)}{V_a(t)}\,dt\Big) - 1\\ &\le \int\limits_{\lambda}^{\lambda+m} \frac{V_a'(t)}{V_a(t)}\,dt \cdot \exp\Big(\int\limits_{\lambda}^{\lambda+m} \frac{V_a'(t)}{V_a(t)}\,dt\Big)\\ &\le 2\nu(\lambda,m). \ \blacksquare\end{aligned}$$

The following theorem is a particular case of the theorem due to Głowacki (Theorem (2.3) of [4]).

(2.2) THEOREM. *Let m be a weight such that $\lim_{\|w\|\to\infty} m(w) = \infty$ and let b be a positive elliptic symbol in $S(m)$. If*

$$|\partial^\alpha b(w)| \le C_\alpha \quad \textit{for } |\alpha| > 0,$$

and

$$\psi_b(t) \ge Ct^r, \quad \textit{where } 0 < r \le 1,\ t > 0,$$

then for λ large enough

$$\left|\frac{N_{\mathrm{Op}(b)}(\lambda)}{V_b(\lambda)} - 1\right| \le C\lambda^{-r}.$$

We will show that a similar theorem is true with a weaker assumption

$$|\partial^\alpha b(w)| \le C_\alpha b^{1-\varrho}(w) \quad \text{for } |\alpha| > 0,$$

where $0 < \varrho < 1$. For this purpose we will use results due to Beals [1]. In his paper Beals defines pseudodifferential operators by the Kohn–Nirenberg formula, but his method is so general that one can easily apply it to operators defined by the Weyl formula. The next theorem is a special case of Theorem (4.2) of [1].

(2.3) THEOREM. *Let $m \ge c > 0$ be a weight, $A > 0$ have an elliptic symbol and $A \in \mathcal{L}(m)$. Then $A^s \in \mathcal{L}(m^s)$ for $s \in \mathbb{R}$.*

(2.4) COROLLARY. *Let $m \ge c > 0$ be a weight and a be a positive elliptic symbol in $S(m)$ such that $\mathrm{Op}(a) > 0$ and $\partial^\alpha a \in S(m^{1-\varrho})$ for $|\alpha| > 0$. Let $a^{\circ s}$ denote the symbol of $\mathrm{Op}(a)^s$. Then*

$$a^{\circ s} = a^s + r_s,$$

where $r_s \in S(m^{s-\varrho})$.

Proof. Let us consider the case when $-1 < s < 0$. Then for $A = \mathrm{Op}(a)$,

$$A^s = -\frac{\sin \pi s}{\pi} \int\limits_0^\infty t^s (t+A)^{-1}\,dt$$

(see [7]). Hence

$$a^{\circ s} = -\frac{\sin \pi s}{\pi} \int\limits_0^\infty t^s a_t \, dt,$$

where a_t denotes the symbol of the operator $(t+A)^{-1}$. In this way we obtain

$$r_s = -\frac{\sin \pi s}{\pi} \int\limits_0^\infty t^s (a_t - (a+t)^{-1}) \, dt.$$

We claim that

$$a_t - (a+t)^{-1} \in S((m+t)^{-1-\varrho})$$

uniformly with respect to t. In fact we have

$$a_t - (a+t)^{-1} = (a+t)^{-1}(a_t(a+t) - 1) = -r(a_t, a)(a+t)^{-1},$$

but $\partial^\beta a \in S(m^{1-\varrho})$ for $|\beta| > 0$ and $a_t \in S((m+t)^{-1})$ uniformly with respect to t, so also $r(a_t, a) \in S((m+t)^{-\varrho})$ uniformly with respect to t (Proposition (1.2)). Hence our claim is true.

In this way we obtain

$$|\partial^\beta r_s| \le -\frac{\sin \pi s}{\pi} \int\limits_0^\infty t^s C_s (m+t)^{-1-\varrho} \, dt \le C_s m^{s-\varrho},$$

so that $r_s \in S(m^{s-\varrho})$. To prove the corollary for $s \in \mathbb{R}$ we only have to remark that $r_s \in S(m^{s-\varrho})$ implies $r_{2s} \in S(m^{2s-\varrho})$ and $r_{-s} \in S(m^{-s-\varrho})$. ∎

(2.5) Corollary. *Let $m \ge c > 0$ be a weight, $B > 0$ have an elliptic symbol and $B \in \mathcal{L}(m)$. Then for all $s \in \mathbb{R}$ and $R \in \mathcal{L}(m^s)$ there exists a constant L such that*

$$\|Rf\| \le L \|B^s f\| \quad \textit{for } f \in \mathcal{S}(V).$$

P r o o f. By Theorem (2.3) we have $B^{-s} \in \mathcal{L}(m^{-s})$, so the Calderón–Vaillancourt theorem [2] shows that the operator RB^{-s} is bounded. Hence $\|RB^{-s} f\| \le L\|f\|$, which implies that $\|Rf\| \le L\|B^s f\|$. ∎

(2.6) Corollary. *Let $m \ge c > 0$ be a weight, $B > 0$ have an elliptic symbol and $B \in \mathcal{L}(m)$. Then for all $s \in \mathbb{R}$ and $R \in \mathcal{L}(m^s)$ there exists a constant L such that*

$$|\langle Rf, f\rangle| \le L \langle B^s f, f\rangle \quad \textit{for } f \in \mathcal{S}(V).$$

P r o o f. We have $R = B^{s/2} B^{-s/2} R$, so

$$\langle Rf, f\rangle = \langle B^{s/2} B^{-s/2} Rf, f\rangle = \langle B^{-s/2} Rf, B^{s/2} f\rangle \le \|B^{-s/2} Rf\| \|B^{s/2} f\|.$$

Note that by Theorem (2.3) the operator $B^{-s/2}$ is in $\mathcal{L}(m^{-s/2})$, therefore $B^{-s/2} R \in \mathcal{L}(m^{s/2})$; so by Corollary (2.5) we obtain $\|B^{-s/2} Rf\| \le$

$L\|B^{s/2}f\|$. In this way

$$\langle Rf, f\rangle \le L\|B^{s/2}f\|^2 = L\langle B^s f, f\rangle. \ \blacksquare$$

(2.7) LEMMA. *Let m be a weight such that $\lim_{\|w\|\to\infty} m(w) = \infty$ and let b be a positive elliptic symbol in $S(m)$. If*

$$|\partial^\alpha b(w)| \le C_\alpha \quad \textit{for } |\alpha| > 0$$

and

$$\psi_b(t) \ge Ct^r, \quad \textit{where } 0 < r \le 1,$$

then for $s \ge 1$ and λ large enough

$$\left|\frac{N_{\mathrm{Op}(b^s)}(\lambda)}{V_{b^s}(\lambda)} - 1\right| \le C\lambda^{-r/s}.$$

Proof. First let us study the functions V_{b^s} and $N_{\mathrm{Op}(b^s)}$ (our assumptions imply that $\mathrm{Op}(b^s)$ is bounded from below and has a discrete spectrum (Proposition (1.3))). Denoting the symbol b^s by a we obtain

$$V_a(\lambda) = V_b(\lambda^{1/s}).$$

Note that by Corollary (2.4) we have $b^s = b^{\circ s} + r$, where $r \in S(m^{s-1})$. Let A denote $\mathrm{Op}(a)$ and B denote $\mathrm{Op}(b)$. We know that $b > 0$ so there is a constant M such that $B + M > 0$ (Proposition (1.3)). We also have $r \in S(m^{s-1}) \subset S((m+M)^{s-1})$, so by Corollary (2.6),

$$-L(B+M)^{s-1} \le \mathrm{Op}(r) \le L(B+M)^{s-1}.$$

Hence

$$B^s - L(B+M)^{s-1} \le A \le B^s + L(B+M)^{s-1}$$

and therefore, using the "mini-max principle" [8], we obtain

$$b_n^s - L(b_n+M)^{s-1} \le a_n \le b_n^s + L(b_n+M)^{s-1}$$

where a_n and b_n denote the nth eigenvalues of the operators A and B, respectively. In this way increasing the constants L and M if necessary, we obtain

$$b_n^s - Lb_n^{s-1} - M \le a_n \le b_n^s + Lb_n^{s-1} + M,$$

so

$$\{n\in\mathbb{N} : b_n^s + Lb_n^{s-1} + M \le \lambda\} \subset \{n \in \mathbb{N} : a_n \le \lambda\} \subset \{n\in\mathbb{N} : b_n^s - Lb_n^{s-1} - M \le \lambda\}.$$

But it is easy to see that for λ large enough $b_n^s - Lb_n^{s-1} - M \le \lambda$ implies $b_n \le \lambda^{1/s}+L$. In fact we can consider the function $f(x) = x^s - Lx^{s-1} - M - \lambda$ defined for $x>0$. It has only one zero with positive derivative and for large λ,

$$f(\lambda^{1/s}+L) = \lambda^{1/s}((\lambda^{1/s}+L)^{s-1} - \lambda^{(s-1)/s}) - M \ge 0.$$

Similarly $b_n \leq \lambda^{1/s} - L$ implies $b_n^s + Lb_n^{s-1} + M \leq \lambda$. Therefore

$$\{n \in \mathbb{N} : b_n \leq \lambda^{1/s} - L\} \subset \{n \in \mathbb{N} : a_n \leq \lambda\} \subset \{n \in \mathbb{N} : b_n \leq \lambda^{1/s} + L\}$$

so

$$N_B(\lambda^{1/s} - L) \leq N_A(\lambda) \leq N_B(\lambda^{1/s} + L).$$

Denote $\lambda^{1/s} - L$ by μ_1 and $\lambda^{1/s} + L$ by μ_2. Then

$$\frac{N_B(\mu_1)}{V_b(\mu_1 + L)} \leq \frac{N_A(\lambda)}{V_a(\lambda)} \leq \frac{N_B(\mu_2)}{V_b(\mu_2 - L)}.$$

So using Proposition (2.1) we obtain

$$\frac{N_B(\mu_1)}{V_b(\mu_1)(2\nu(\mu_1, L) + 1)} \leq \frac{N_A(\lambda)}{V_a(\lambda)} \leq \frac{N_B(\mu_2)}{V_b(\mu_2)}(2\nu(\lambda^{1/s}, L) + 1).$$

Note that

$$\left|\frac{N_B(\mu_1)}{V_b(\mu_1)(2\nu(\mu_1, L) + 1)} - 1\right| \leq \left|\frac{N_B(\mu_1)}{V_b(\mu_1)} - 1\right| + 2\nu(\mu_1, L)$$

and

$$\left|\frac{N_B(\mu_2)}{V_b(\mu_2)}(2\nu(\lambda^{1/s}, L) + 1) - 1\right| \leq \left|\frac{N_B(\mu_2)}{V_b(\mu_2)} - 1\right| + 3\nu(\lambda^{1/s}, L).$$

Obviously the symbol b satisfies the assumptions of Theorem (2.2), therefore

$$\left|\frac{N_B(\mu_i)}{V_b(\mu_i)} - 1\right| \leq C\mu_i^{-r} \quad \text{for } i = 1, 2.$$

Now we only need to note that μ_1^{-r}, μ_2^{-r}, $\nu(\mu_1, L)$ and $\nu(\lambda^{1/s}, L)$ are estimated by $M\lambda^{-r/s}$, where M is a constant. Finally, we conclude that there is a constant C such that for λ large enough

$$\left|\frac{N_A(\lambda)}{V_a(\lambda)} - 1\right| \leq C\lambda^{-r/s}. \ \blacksquare$$

(2.8) THEOREM. *Let m be a weight such that $\lim_{\|w\| \to \infty} m(w) = \infty$ and let a be a positive elliptic symbol in $S(m)$. If*

$$|\partial^\alpha a(w)| \leq C_\alpha a^{1-\varrho}(w) \quad \textit{for } |\alpha| > 0,$$

and

$$\psi_a(t) \geq Ct^r,$$

where $0 < \varrho \leq 1$, $0 < r \leq 1$ and $r + \varrho > 1$, then for λ large enough

$$\left|\frac{N_{\mathrm{Op}(a)}(\lambda)}{V_a(\lambda)} - 1\right| \leq C\lambda^{-(r+\varrho-1)}.$$

Proof. Note that the symbol a^ϱ satisfies the assumptions of Lemma (2.7). In fact, denoting a^ϱ by b, we obtain

$$|\partial^\alpha b(w)| \leq C_\alpha \quad \text{and} \quad \psi_b(t) \geq Ct^{r/\varrho + 1 - 1/\varrho}.$$

So from Lemma (2.7) it follows that

$$\left|\frac{N_{\mathrm{Op}(a)}(\lambda)}{V_a(\lambda)} - 1\right| \le C\lambda^{-\sigma},$$

where $\sigma = (r/\varrho + 1 - 1/\varrho)\varrho = r + \varrho - 1$. ■

(2.9) Example. Let us assume that a symbol a satisfies conditions analogous to those of Tulovskiĭ and Shubin:

(i′) $$0 \le a(w) \in S(m), \quad a \text{ elliptic and } \lim_{\|w\|\to\infty} m(w) = \infty,$$

(ii′) $$|\partial^\alpha a(w)| \le C_\alpha a(w)^{1-\varrho} \quad \text{for } |\alpha| > 0,$$

(iii) $$|\langle a'(w), w\rangle| \ge Ca(w)^{1-\kappa}.$$

Then (iii) implies that

$$\psi_a(t) \ge Ct^{1-\kappa},$$

so by Theorem (2.8),

$$N_{\mathrm{Op}(a)}(\lambda) = V_a(\lambda) + O(\lambda^{-(\varrho-\kappa)}),$$

which is stronger than the result of Tulovskiĭ and Shubin. Moreover, by Lemma (1.10) the conditions (i) and (ii) imply that $a+1$ is a weight and also that a is an elliptic symbol in $S(a+1)$. This means that the condition (i′) is satisfied with $m = a+1$. An easy observation that (ii) implies (ii′) proves that the conditions (i′), (ii′), (iii) are more general than (i)–(iii).

3. Quasiclassical asymptotics. As before let A be a selfadjoint operator defined on a dense subspace of a Hilbert space, with spectrum $\mathrm{Sp}\, A$ discrete and bounded from below. Let $N_A(\lambda)$ be the spectral function of A. The following result is due to L. Hörmander [5].

(3.1) Lemma. *Let E be a selfadjoint trace operator such that AE is bounded. If $(I-E)(A-\lambda)(I-E) \ge -l$, then*

$$N_A(\lambda - 4l) \le \mathrm{Tr}\, E + 2\|E - E^2\|_{\mathrm{Tr}}.$$

If $E(\lambda - A)E \ge -k$, then

$$N_A(\lambda + 4k) \ge \mathrm{Tr}\, E - 2\|E - E^2\|_{\mathrm{Tr}}.$$

For $\delta > 0$ let $\chi(t, \lambda, \delta)$ be a smooth function with the following properties:

$$\chi(t,\lambda,\delta) = \begin{cases} 1 & \text{if } t \le \lambda, \\ 0 & \text{if } t \ge \lambda + 2\delta, \end{cases}$$

$$|(\partial/\partial t)^k \chi(t,\lambda,\delta)| \le C_k \delta^{-k}.$$

For any $h > 0$ and any λ we can define a smooth function

$$e_h(z) = \chi(a(z), \lambda, h^{1/2}),$$

where $a \geq 0$ is the symbol of some pseudodifferential operator A with the property that $\lim_{|w|\to\infty} a(w) = \infty$. Let $a^{(h)}(w) = a(h^{1/2}w)$. Define E_h to be the operator with symbol $e_h^{(h)}$. The function $e_h^{(h)}$ satisfies

$$e_h^{(h)}(z) = \begin{cases} 1 & \text{if } a(h^{1/2}z) \leq \lambda, \\ 0 & \text{if } a(h^{1/2}z) \geq \lambda + 2h^{1/2}, \end{cases}$$

and

$$\partial^\gamma e_h^{(h)}(z) = \sum_{\substack{k=0 \\ m_1+\ldots+m_k=|\gamma|}}^{|\gamma|} C_{m,k}\, h^{|\gamma|/2} \frac{\partial^k \chi}{\partial z^k}(a(h^{1/2}z), \lambda, h^{1/2}) \partial_1^{m_1} a(h^{1/2}z) \ldots \partial_k^{m_k} a(h^{1/2}z).$$

If there is a $\varrho > 0$ such that for every γ we have C_γ satisfying

$$|\partial^\gamma a(z)| \leq C_\gamma a^{(1-|\gamma|\varrho)_+}(z),$$

then we can summarize all the above results to get

$$|\partial^\gamma e_h^{(h)}(z)| \leq C_{\gamma,\lambda}. \tag{2}$$

With the above assumptions we have the following lemma:

(3.2) Lemma. *E_h is a trace operator and*

$$\|E_h^2 - E_h\|_{\mathrm{Tr}} \leq Ch^{-N+1/2}.$$

Proof. The symbol of the operator $E_h - E_h^2$ can be written as $e_h^{(h)}(z) - e_h^{(h)} \circ e_h^{(h)}(z) = e_h^{(h)}(z) - e_h^{(h)} e_h^{(h)}(z) - r(e_h^{(h)}, e_h^{(h)})(z)$. By definition, Proposition (1.6) and (2), for h small enough, we get

$$\|\partial^\gamma (e_h^{(h)} - (e_h^{(h)})^2)\|_1 \leq Ch^{-N}(V(\lambda + 2h^{1/2}) - V(\lambda)),$$
$$\|\partial^\gamma (r(e_h^{(h)}, e_h^{(h)}))\|_1 \leq Ch^{-N}(V(\lambda + 2h^{1/2}) - V(\lambda)).$$

By Proposition (1.9) and Lebesgue's differentiation theorem ([8], Theorem 9.2, p. 226), for almost every λ we have

$$\|E_h - E_h^2\|_{\mathrm{Tr}} \leq C \max_{|\alpha|\leq 2N+2} \|\partial^\alpha (e_h^{(h)}(z) - e_h^{(h)} \circ e_h^{(h)}(z))\|_1$$
$$\leq Ch^{-N}(V(\lambda + 2h^{1/2}) - V(\lambda)) \leq Ch^{-N}h^{1/2}. \ \blacksquare$$

The main purpose of this section is to prove the following theorem.

(3.3) Theorem. *Let m be a weight such that $\lim_{|z|\to\infty} m(z) = \infty$. Let $a \geq C > 0$ be an elliptic symbol in the class $S(m)$ such that*

$$|\partial^\gamma a(z)| \leq C_\gamma a^{(1-|\gamma|\varrho)_+}(z) \quad \textit{for some } \varrho > 0. \tag{3}$$

Let $A_{(h)}$ be the operator with symbol $a^{(h)}$. Then for almost all λ,

$$N_{A_{(h)}}(\lambda) = h^{-N}(V(\lambda) + O(h^{1/2})).$$

Moreover, *to obtain the above asymptotics it is enough to assume only that* $a \ge C > 0$ *is a smooth function satisfying the inequality* (3) *and such that* $\lim_{|z|\to\infty} a(z) = \infty$.

(3.4) LEMMA. *For h small enough we have the estimate*

$$\|E_h(\lambda - A_{(h)})(I - E_h)\| \le Ch^{1/2}.$$

Proof. The symbol of the operator $E_h(\lambda - A_{(h)})(I - E_h)$ is

$$e_h^{(h)} \circ (\lambda - a^{(h)}) \circ (1 - e_h^{(h)})(z) = a_{\lambda,h}(z) + r_{\lambda,h}(z),$$

where

$$a_{\lambda,h}(z) = e_h^{(h)}(\lambda - a^{(h)})(1 - e_h^{(h)})(z),$$
$$r_{\lambda,h}(z) = r(e_h^{(h)}, \lambda - a^{(h)}, 1 - e_h^{(h)})(z).$$

By (2) and Proposition (1.7),

$$\max_{|\alpha|\le 2N+2} \|\partial^\alpha a_{\lambda,h}\|_\infty \le Ch^{1/2}, \qquad \max_{|\alpha|\le 2n+2} \|\partial^\alpha r_{\lambda,h}\|_\infty \le Ch^{1/2}.$$

The proof can now be completed by using the Calderón–Vaillancourt theorem (for references see [2]). ■

(3.5) PROPOSITION. *For h small enough*

$$E_h(\lambda - A_{(h)})E_h \ge -Ch^{1/2}.$$

Proof. We can write

$$E_h(\lambda - A_{(h)})E_h = E_h(\lambda - A_{(h)}) - E_h(\lambda - A_{(h)})(I - E_h).$$

The symbol of the first operator on the right hand side is $a_{\lambda,h}(z) - r_{\lambda,h}(z)$, where now $a_{\lambda,h}(z) = e_h^{(h)}(\lambda - a^{(h)})(z)$ and $r_{\lambda,h}(z) = r(e_h^{(h)}, a_h^{(h)})(z)$. Moreover, $a_{\lambda,h}(z) \ge -2h^{1/2}$. By (2), for $|\alpha| > 0$ we have $|\partial^\alpha a_{\lambda,h}| \le Ch^{1/2}$. Therefore Proposition (1.4) gives us

$$\mathrm{Op}(a_{\lambda,h}) \ge -Lh^{1/2}.$$

Now by Proposition (1.6) we get $\|\partial^\alpha r_{\lambda,h}\|_\infty \le Ch^{1/2}$, so by the Calderón–Vaillancourt theorem

$$\|\mathrm{Op}(r_{\lambda,h})\| \le Ch^{1/2}.$$

Finally, we can prove the proposition by combining all the above with Lemma (3.4). ■

(3.6) PROPOSITION. *For* $h > 0$ *small enough*

$$(I - E_h)(A_{(h)} - \lambda)(I - E_h) \ge -Ch^{1/2}.$$

Proof. We have a similar decomposition to the one before:

$$(I - E_h)(A_{(h)} - \lambda)(I - E_h) = (A_{(h)} - \lambda)(I - E_h) + E_h(\lambda - A_{(h)})(I - E_h).$$

The symbol of the operator $(A(h)-\lambda)(I-E_h)$ is $a_{\lambda,h}(z)-r_{\lambda,h}(z)$, where now $a_{\lambda,h}(z)=(a^{(h)}-\lambda)(1-e_h^{(h)})(z)\ge 0$ and $r_{\lambda,h}(z)=r(a^{(h)},e_h^{(h)})(z)$. Notice, therefore, that we only need to estimate $\operatorname{Op}(a_{\lambda,h})$. It is easy to see that

$$|\partial^\alpha a_{\lambda,h}|\le C_{\alpha,h}h^{1/2}a_{\lambda,h}$$

for $|\alpha|>0$. Moreover, for every $b\in S(m)$ and $k\in\mathbb{N}$,

$$\begin{aligned}&(\delta-H)^{\star k}\star b(w)\\&\quad=\int\limits_{[0,1]^k}\int\limits_{W^k} b^{(k)}\Big(w-\sum_{j=1}^k t_jv_j\Big)(v_1,\dots,v_k)H(v_1)\dots H(v_k)\,dv\,dt,\end{aligned}$$

therefore

$$|\partial^\alpha(\delta-H)^{\star k}\star a_{\lambda,h}|\le C_{\lambda,h,k}h^{1/2}a_{\lambda,h}.$$

Notice also that for $k+|\alpha|>[\varrho^{-1}]$, where $[\ ]$ denotes the greatest integer function, we get

$$|\partial^\alpha(\delta-H)^{\star k}\star a_{\lambda,h}|\le C_{\lambda,\alpha,k}h^{1/2}.$$

Since we can decompose $a_{\lambda,h}$ as

$$a_{\lambda,h}=H\star\sum_{k=0}^n(\delta-H)^{\star k}\star a_{\lambda,h}+(\delta-H)^{\star(n+1)}\star a_{\lambda,h},$$

where $n=[\varrho^{-1}]$, the above calculations and Propositions (1.4) and (1.5) give us $\operatorname{Op}(a_{\lambda,h})\ge -Ch^{1/2}$. ■

(3.7) FACT. $\operatorname{Tr}E_h=h^{-N}V(\lambda)(1+O(h^{1/2}))$.

Proof. By Proposition (1.9), E_h is a trace operator. Moreover, by Proposition (1.8),

$$\operatorname{Tr}E_h=\int e_h^{(h)}(z)\,dz.$$

So according to the definition of e_h,

$$h^{-N}V(\lambda)\le\operatorname{Tr}E_h\le Ch^{-N+1/2}+h^{-N}V(\lambda),$$

which is equivalent to

$$\frac{\operatorname{Tr}E_h}{h^{-N}V(\lambda)}-1=O(h^{1/2}).\ ■$$

(3.8) REMARK. *The operator $A_{(h)}E_h$ is bounded.*

Proof. The symbol of this operator is $a^{(h)}(z)e_h^{(h)}(z)+r(a^{(h)},e_h^{(h)})(z)$. The derivatives of the first summand are bounded because it is a smooth, compactly supported function, and of the other one by Proposition (1.6). Thus our claim follows from the Calderón–Vaillancourt theorem (see Proposition (1.15) of [4]). ■

Proof of Theorem (3.3). By Proposition (1.3) the operator $A_{(h)}$ is essentially selfadjoint and bounded from below. Therefore we can apply Lemma (3.1) to its closure, which we will also denote by $A_{(h)}$. By Lemma (3.1) and Propositions (3.5) and (3.6) we have

$$N_{A_{(h)}}(\lambda - Ch^{1/2}) \le \operatorname{Tr} E_h + 2\|E_h^2 - E_h\|_{\operatorname{Tr}}$$

and

$$N_{A_{(h)}}(\lambda + Ch^{1/2}) \ge \operatorname{Tr} E_h - 2\|E_h^2 - E_h\|_{\operatorname{Tr}},$$

for any $h > 0$ and λ, parameters of the symbol $e_h^{(h)}$. So

$$N_{A_{(h)}}(\lambda) \le \operatorname{Tr} E_{h,\lambda_h^+} + 2\|E_{h,\lambda_h^+}^2 - E_{h,\lambda_h^+}\|_{\operatorname{Tr}}$$

and

$$N_{A_{(h)}}(\lambda) \ge \operatorname{Tr} E_{h,\lambda_h^-} - 2\|E_{h,\lambda_h^-}^2 - E_{h,\lambda_h^-}\|_{\operatorname{Tr}},$$

where E_{h,λ_h^+} is the operator with symbol $e_{h,\lambda+Ch^{1/2}}^{(h)}$ and E_{h,λ_h^-} has symbol $e_{h,\lambda-Ch^{1/2}}^{(h)}$. By the proofs of Lemma (3.2) and Fact (3.7) it is easy to see that

$$\|E_{h,\lambda_h}^2 - E_{h,\lambda_h}\|_{\operatorname{Tr}} \le Ch^{-N+1/2}$$

and

$$|\operatorname{Tr} E_{h,\lambda_h} - h^{-N} V(\lambda)| \le Ch^{-N+1/2}.$$

Therefore the first part of our theorem is established. To prove the second part it is enough to notice by Lemma (1.10) that $a+1$ is a weight and a is an elliptic symbol in the class $S(a+1)$. ∎

(3.9) EXAMPLE. Let $p(w_1, \ldots, w_{2N}) \ge c > 0$ be a hypoelliptic polynomial (i.e. $|\partial^\alpha p(\xi)/p(\xi)| \le C|\xi|^{-c|\alpha|}$, $|\alpha| > 0$, see [6], Theorem 11.1.3). In particular $\lim_{|w|\to\infty} p(w) = \infty$. By the hypoellipticity of p, condition (3) is satisfied. Therefore Theorem (3.3) gives us the spectral asymptotics for $\operatorname{Op}(p^{(h)})$.

(3.10) EXAMPLE. Let $l(t) = \ln(1+t)$ and $l^n(t) = l \circ \ldots \circ l(t)$ for $n \in \mathbb{N}$. Let $c_n(w_1, \ldots, w_{2N}) = l^n(\langle w_1, \ldots, w_{2N}\rangle)$. Since $|\partial^\gamma(\langle x\rangle^b)| \le C_\gamma \langle x\rangle^{b-|\gamma|}$, we have, for $|\alpha| > 0$,

$$|\partial^\alpha c_n(w)| \le C_\alpha \langle w\rangle^{-|\alpha|} \le C_\alpha.$$

Since $\lim_{|w|\to\infty} c_n(w) = \infty$, we can apply Theorem (3.3) to $\operatorname{Op}(c_n^{(h)})$.

(3.11) EXAMPLE. Finally, we mention that in our theory we can also consider Schrödinger operators with potential of logarithmic growth. Thus, as before, we can obtain our spectral asymptotics for the operator with symbol $\langle x\rangle^{-1}\xi^2 + \log(\langle x\rangle)$.

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