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APPROXIMATION BY LINEAR COMBINATION OF SZÁSZ–MIRAKIAN OPERATORS

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Introduction. To approximate continuous functions on the interval $[0,\infty)$, O. Szász and G. Mirakian generalized the Bernstein polynomials as follows:

$$S_n(f;x)=\sum_{\nu=0}^{\infty}\phi_{n,\nu}(x)f(\nu/n),$$

where

$$\phi_{n,\nu}(x)=e^{-nx}(nx)^{\nu}/\nu!,\qquad f\in\mathcal{C}[0,\infty).$$

Singh [8] has obtained an estimate for bounded continuous functions in *simultaneous approximation* involving higher derivatives by these operators. Sun [9] has tried to extend this estimate to functions of bounded variation with $O(t^{\alpha t})$ growth of the *derivatives* and has remarked that unfortunately, for continuous derivatives his estimate does not include the case $f'\in\mathrm{Lip}\,1$ on every finite subinterval of $[0,\infty)$. In this case he only obtains

$$S_n^{(r)}(f;x)-f^{(r)}(x)=O(\log n/n),\qquad r=0,1,2,\ldots$$

This degree is worse than the usual degree $1/n$. He put up the question of whether a unified approach can be developed which may improve this estimate for the class $f'\in\mathrm{Lip}\,1$ on every finite subinterval of $[0,\infty)$.

In this paper we present a unified approach which improves the estimate of Sun [9] for continuous functions and moreover, it makes the results of Singh [8] applicable to unbounded functions.

In the sequel $\langle a,b\rangle$ denotes an open interval in $[0,\infty)$ containing the closed interval $[a,b]$ and $\|\cdot\|_{[a,b]}$ means the sup norm on the space $\mathcal{C}[a,b]$.

The mth moment of the Szász–Mirakian operator is defined as

$$V_{n,m}(x)=\sum_{\nu=0}^{\infty}\phi_{n,\nu}(x)\left(\frac{\nu}{n}-x\right)^{m},\qquad m=0,1,2,\ldots$$

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Let $d_0, d_1, \dots, d_k$ be arbitrary but fixed distinct positive integers. Then, following Kasana and Agrawal [5], the *linear combinations* $S_n(f,k,x)$ of $S_{d_jn}(f;x)$, $j=0,1,\dots,k$, are introduced as

$$S_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} S_{d_0n}(f;x) & d_0^{-1} & d_0^{-2} & \dots & d_0^{-k} \\ S_{d_1n}(f;x) & d_1^{-1} & d_1^{-2} & \dots & d_1^{-k} \\ \dots & \dots & \dots & \dots & \dots \\ S_{d_kn}(f;x) & d_k^{-1} & d_k^{-2} & \dots & d_k^{-k} \end{vmatrix},$$

where Δ is the *Vandermonde determinant* obtained by replacing the operator column of the determinant by the entries 1. On simplification this is reduced to

$$S_n(f,k,x) = \sum_{j=0}^{k} C(j,k) S_{d_jn}(f;x),$$

where

$$C(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \qquad k \neq 0, \qquad C(0,0) = 1;$$

and this is the form of linear combinations considered by May [7].

1. To prove the main theorem we need the following auxilliary results.

Lemma 1.1. *For $V_{n,m}(x)$, we have the recurrence relation*

$$nV_{n,m+1}(x) = xV'_{n,m}(x) + mxV_{n,m-1}(x), \qquad m \geq 1.$$

Gröf [2] has proved that:

(a) $V_{n,0} = 1,\ V_{n,1} = 0$;

(b) $V_{n,m}(x)$ is a polynomial in x of degree $[m/2]$ and in n^{-1} of degree $m-1$, $m > 1$;

(c) for all finite x, $V_{n,m}(x) = O(n^{-[(m+1)/2]})$.

Lemma 1.2. *Let $f(t) = O(t^{\alpha t})$ as $t \to \infty$ with $\alpha > 0$, and δ be a positive number. Then*

$$\sum_{|\nu/n - x| > \delta} \phi_{n,\nu}(x) f(\nu/n) = O(e^{-\gamma n}),$$

where γ is a constant depending on f, x and δ.

This lemma is due to Hermann [3]. A better estimate can also be found in [1].

Corollary 1.3. *For $\delta > 0$ and $s = 0,1,\dots,$ we have*

$$\Big\| \sum_{|\nu/n - x| > \delta} \phi_{n,\nu}(x) (\nu/n)^{\alpha\nu/n} \Big\|_{[a,b]} \leq K_s n^{-s},$$

where K_s is a constant depending on s.

LEMMA 1.4. *If $C(j,k)$, $j = 0,1,\dots,k$, are defined as in the previous section then*

$$\sum_{j=0}^{k} C(j,k) d_j^{-m} = \begin{cases} 1, & m = 0, \\ 0, & m = 1,\dots,k. \end{cases}$$

May [7] has proved this lemma using Lagrange polynomials. A simpler exposition can be seen as Lemma 2 of Kasana [4].

LEMMA 1.5. *There exist polynomials $T_{p,q,r}(x)$ independent of n and ν such that*

$$x^r \frac{d^r}{dx^r} \phi_{n,\nu}(x) = \sum_{\substack{2p+q \le r \\ p,q \ge 0}} n^p (\nu - nx)^q T_{p,q,r}(x) \phi_{n,\nu}(x).$$

This can be proved by induction; for a detailed proof we refer the reader to Kasana *et al.* [6].

2. We state and prove our main result as follows.

THEOREM. *Let f be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^{\alpha t})$ as $t \to \infty$, for some $\alpha > 0$. If $f^{(r+1)} \in \mathcal{C}\langle a,b \rangle$, then, for n sufficiently large,*

$$\|S_n^{(r)}(f,k,\cdot) - f^{(r)}\|_{[a,b]} \le C_1 n^{-1/2} \omega(f^{(r+1)}; n^{-1/2}) + C_2 n^{-(k+1)},$$

where $C_1 = C_1(k,r)$, $C_2 = C_2(k,r,f)$ and $\omega(f^{(r+1)};\delta)$ is the modulus of continuity of $f^{(r+1)}$ on $\langle a,b \rangle$ defined as

$$\omega(f^{(r+1)};\delta) = \sup_{x \in \langle a,b \rangle} \sup_{|h| \le \delta} |\varDelta_h f^{(r+1)}(x)|.$$

Proof. Write

$$f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(r+1)}(\xi) - f^{(r+1)}(x)}{(r+1)!}(t-x)^{r+1}\chi(t) + \varepsilon(t,x)(1-\chi(t)),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $\langle a,b \rangle$. As

$$S_n^{(r)}(f,k,x) = \sum_{j=0}^{k} C(j,k) S_{d_j n}^{(r)}(f;x)$$

we have

$$\begin{aligned} S^{(r)}_{d_jn}(f;x) &= \sum_{\nu=0}^{\infty} \phi^{(r)}_{d_jn,\nu}(x) f\left(\frac{\nu}{d_jn}\right) \\ &= \sum_{\nu=0}^{\infty} \phi^{(r)}_{d_jn,\nu}(x) \Bigg[\sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}\left(\frac{\nu}{d_jn}-x\right)^i \\ &\quad + \frac{f^{(r+1)}(\xi)-f^{(r+1)}(x)}{(r+1)!}\left(\frac{\nu}{d_jn}-x\right)^{r+1}\chi\left(\frac{\nu}{d_jn}\right) \\ &\quad + \varepsilon\left(\frac{\nu}{d_jn},x\right)\left(1-\chi\left(\frac{\nu}{d_jn}\right)\right).\Bigg] \end{aligned}$$

Thus,

$$\begin{aligned} S^{(r)}_{n}(f,k,x) &= \sum_{j=0}^{k}\sum_{\nu=0}^{\infty} C(j,k)\phi^{(r)}_{d_jn,\nu}(x) \Bigg[\sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}\left(\frac{\nu}{d_jn}-x\right)^i \\ &\quad + \frac{f^{(r+1)}(\xi)-f^{(r+1)}(x)}{(r+1)!}\left(\frac{\nu}{d_jn}-x\right)^{r+1}\chi\left(\frac{\nu}{d_jn}\right) \\ &\quad + \varepsilon\left(\frac{\nu}{d_jn},x\right)\left(1-\chi\left(\frac{\nu}{d_jn}\right)\right)\Bigg], \\ &= I_{n,1}+I_{n,2}+I_{n,3} \qquad \text{(say)}. \end{aligned}$$

Now,

$$\begin{aligned} I_{n,1} &= \sum_{i=0}^{r+1}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k} C(j,k)\sum_{\nu=0}^{\infty}\phi^{(r)}_{d_jn,\nu}(x)\left(\frac{\nu}{d_jn}-x\right)^i \\ &= \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{r+1}\frac{f^{(i)}(x)}{i!}\sum_{l=0}^{i}\binom{i}{l}(-x)^{i-l}\sum_{\nu=0}^{\infty}\phi^{(r)}_{d_jn,\nu}(x)\left(\frac{\nu}{d_jn}\right)^l \\ &= \sum_{j=0}^{k} C(j,k)\sum_{i=0}^{r+1}\frac{f^{(i)}(x)}{i!}\sum_{l=0}^{i}\binom{i}{l}(-x)^{i-l}\sum_{\nu=0}^{\infty} S^{(r)}_{d_jn}(t^l;x). \end{aligned}$$

But $S_{d_jn}(t^l;x)$ is a polynomial in x of degree exactly l and the coefficient of x^l is 1. So, for $0 \le l < r$, $S_{d_jn}(t^l;x) = 0$ and, for $l = r$, we have $S_{d_jn}(t^l;x) = r!$. Further,

$$\begin{aligned} I_{n,1} &= \sum_{j=0}^{k} C(j,k)\Bigg[f^{(r)}(x) + \frac{f^{(r+1)}(x)}{(r+1)!}\Bigg\{\binom{r+1}{r}(-x)S^{(r)}_{d_jn}(t^r;x) \\ &\quad + \binom{r+1}{r+1}(-x)^0 S^{(r)}_{d_jn}(t^{r+1};x)\Bigg\}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= f^{(r)}(x) + \sum_{j=0}^{k} C(j,k) \\
&\quad \times \left[\frac{f^{(r+1)}(x)}{r+1)!} \{(-x)(r+1)! + S_{d_j n}^{(r)}(t^{(r+1)};x)\} \right] \\
&= f^{(r)}(x) + f^{(r+1)}(x) \sum_{j=0}^{k} C(j,k) \Big[-x + \frac{1}{(r+1)!} S_{d_j n}^{(r)}(t^{(r+1)};x) \Big] \\
&= f^{(r)}(x) + f^{(r+1)}(x) \sum_{j=0}^{k} C(j,k) \\
&\quad \times \left[-x + \frac{1}{(r+1)!} \left\{ (r+1)!x + \frac{r(r+1)}{2d_j n} r! \right\} \right] \\
&= f^{(r)}(x) + f^{(r+1)}(x) \sum_{j=0}^{k} C(j,k) \left[-x + \left\{ x + \frac{r}{2d_j n} \right\} \right] \\
&= f^{(r)}(x) + f^{(r+1)}(x) \frac{r}{2n} \sum_{j=0}^{k} \frac{C(j,k)}{d_j n} = f^{(r)}(x),
\end{aligned}
$$

since $\sum C(j,k)/(d_j n) = 0$, by Lemma 1.4. Thus, if $S_n^{(r)}(f,k,x) = I_{n,1} + I_{n,2} + I_{n,3}$, then $S_n^{(r)}(f,k,x) - f^{(r)}(x) = I_{n,2} + I_{n,3}$.

To estimate $I_{n,2}$ it is sufficient to consider it without the linear combination. Let

$$
I_{n,2} \equiv \sum_{\nu=0}^{\infty} \phi_{n,\nu}^{(r)}(x) \frac{f^{(r+1)}(\xi) - f^{(r+1)}(x)}{(r+1)!} \Big(\frac{\nu}{n} - x \Big)^{r+1} \chi\Big(\frac{\nu}{n}\Big).
$$

Then, using Lemmas 1.5 and 1.2, we get for $t \in \langle a,b \rangle$ and $\delta > 0$,

$$
\begin{aligned}
I_{n,2} &\le \sum_{\nu=0}^{\infty} \sum_{\substack{2p+q \le r \\ p,q \ge 0}} n^p |\nu - nx|^q \frac{|T_{p,q,r}(x)|}{x^r} \phi_{n,\nu}(x) \\
&\quad \times \frac{|f^{(r+1)}(\xi) - f^{(r+1)}(x)|}{(r+1)!} \Big| \frac{\nu}{n} - x \Big|^{r+1} \chi\Big(\frac{\nu}{n}\Big) \\
&\le \sum_{\substack{2p+q \le r \\ p,q \ge 0}} n^p \sum_{\nu=0}^{\infty} \phi_{n,\nu}(x) |\nu - nx|^q \frac{|T_{p,q,r}(x)|}{(r+1)! x^r} \\
&\quad \times \left\{ 1 + \frac{|\nu/n - x|}{\delta} \right\} \omega(f^{(r+1)};\delta) \Big| \frac{\nu}{n} - x \Big|^{r+1}
\end{aligned}
$$

$$\leq M_1(r)\omega(f^{(r+1)};\delta)\sum_{\substack{2p+q\leq r\\ p,q\geq 0}} n^{p+q}\sum_{\nu=0}^{\infty}\phi_{n,\nu}(x)$$
$$\times\left(\left|\frac{\nu}{n}-x\right|^{q+r+1}+\frac{|\nu/n-x|^{q+r+2}}{\delta}\right),$$

where

$$M_1(r)=\sup_{a\leq x\leq b}\sup_{\substack{2p+q\leq r\\ p;q\geq 0}}\frac{|T_{p,q,r}(x)|}{(r+1)!x^r}.$$

Further, using the Schwarz inequality and Lemma 1.1, we observe that

$$|I_{n,2}|\leq M_1(r)\omega(f^{(r+1)};\delta)\sum_{\substack{2p+q\leq r\\ p,q\geq 0}} n^{p+q}\left\{O(n^{-(q+r+1)/2})+\frac{1}{\delta}O(n^{-(q+r+2)/2})\right\}.$$

Choosing $\delta=n^{-1/2}$, we get

$$\|I_{n,2}\|_{[a,b]}\leq C_1(k,r)n^{-1/2}\omega(f^{(r+1)};n^{-1/2}).$$

For $t\in[0,\infty)\setminus\langle a,b\rangle$, we can choose $\delta>0$ such that $|t-x|>\delta$ for all $x\in[a,b]$ and we also have $\varepsilon(t,x)=O(f(t))$. By the Schwarz inequality, Lemma 1.5, Lemma 1.1 and Corollary 1.3, $I_{n,3}$ is estimated as

$$|I_{n,3}(x)|=\sum_{j=0}^{k}\sum_{|\nu/(d_jn)-x|>\delta}\sum_{\substack{2p+q\leq r\\ p,q\geq 0}}|C(j,k)|(d_jn)^p|\nu-d_jnx|^q$$
$$\times\phi_{d_jn,\nu}(x)\frac{|T_{p,q,r}(x)|}{x^r}O\left(f\left(\frac{\nu}{d_jn}\right)\right)$$
$$\leq M_2(r,f)\sum_{j=0}^{k}|C(j,k)|\sum_{\substack{2p+q\leq r\\ p,q\geq 0}}(d_jn)^{p+q}$$
$$\times\sum_{|\nu/(d_jn)-x|>\delta}\phi_{d_jn,\nu}(x)\left|\frac{\nu}{d_jn}-x\right|^q\left(\frac{\nu}{d_jn}\right)^{\alpha\nu/(d_jn)}$$
$$\leq M_2(r,f)\sum_{j=0}^{k}\sum_{\substack{2p+q\leq r\\ p,q\geq 0}}|C(j,k)|(d_jn)^{p+q}$$
$$\times\left(\sum_{\nu=0}^{\infty}\phi_{d_jn,\nu}(x)\left(\frac{\nu}{d_jn}-x\right)^{2q}\right.$$
$$\left.\times\sum_{|\nu/(d_jn)-x|>\delta}\phi_{d_jn,\nu}(x)\left(\frac{\nu}{d_jn}\right)^{2\alpha\nu/(d_jn)}\right)^{1/2},$$

or

$$\begin{aligned}\|I_{n,3}\|_{[a,b]} &\le M_2(r,f)\sum_{j=0}^{k}\sum_{\substack{2p+q\le r\\ p,q\ge 0}}|C(j,k)|\\ &\qquad\times (d_jn)^{p+q}O((d_jn)^{-q/2})O((d_jn)^{-s/2})\\ &= M_3(r,f)\sum_{j=0}^{k}|C(j,k)|(d_jn)^{-(s-r)/2}\\ &= C_2(k,r,f)n^{-(k+1)} \qquad \text{if } s\ge 2k+r+2.\end{aligned}$$

Combining the estimates of $I_{n,1}$, $I_{n,2}$ and $I_{n,3}$, we obtain the required result.

COROLLARY. *If, in addition to the hypothesis of the above theorem,* $f^{(r+1)} \in \mathrm{Lip}_M\,\beta$ *for some* $M > 0$ *and* $0 < \beta \le 1$ *on the interval* $\langle a, b\rangle$, *then*

$$\|S_n^{(r)}(f,k,\cdot) - f^{(r)}\|_{[a,b]} \le C_3 n^{-(\beta+1)/2} + C_2 n^{-(k+1)},$$

where $C_3 = MC_1$.

Further, for $k = 0$ and $\beta = 1$, this is reduced to the desired estimate

$$S_n^{(r)}(f;x) - f^{(r)}(x) = O(1/n)$$

on every finite subinterval of $[0,\infty)$.

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