COLLOQUIUM MATHEMATICUM

VOL. 80 1999 NO. 1

APPLICATION OF COVERING SETS

BY

KANDASAMY MUTHUVEL (OSHKOSH, WISCONSIN)

This paper contains results concerning covering sets which generalize and unify some known results about the additive subgroups of the reals and the algebraic difference of sets.

Throughout the paper, the set of all real numbers is denoted by $\mathbb{R}$. The algebraic difference of a subset A of R is defined to be $A - A = \{x - y : x, y \in A\}$. Any basis for the vector space of the reals over the rationals is called a *Hamel basis*.

Sierpiński proved that the complement of a Hamel basis is everywhere of the second category. This was generalized in [5]: if H is a Hamel basis and X is a subset of $\mathbb{R}$ of cardinality less than the cardinality of the continuum, then the complement of the algebraic sum $H+X$ is everywhere of the second category.

The above result was improved in [6] by showing that (1) the complement of the algebraic sum $\mathbb{Z}(H) + X$ is everywhere of the second category, where $|X| < |\mathbb{R}|$, and the *Erdős set*, $\mathbb{Z}(H)$, is the set of all finite linear combinations of elements from a Hamel basis H with integer coefficients, and (2) the complement of a finite union of Hamel bases is everywhere of the second category. $\mathbb{Z}(H)+X$ is contained in a proper subgroup of the additive group of the reals because the cardinality of the group generated by X is less than the cardinality of the continuum and the index of the additive subgroup $\mathbb{Z}(H)$ of $\mathbb{R}$ is the cardinality of the continuum.

In this paper, we generalize and unify the above results by showing that the complement of a finite union of proper subgroups of $\mathbb{R}$ is everywhere of the second category (i.e., it is large in the sense of category). We prove a theorem about covering sets which directly implies that the cardinality of the complement of a finite union of proper subgroups of $\mathbb{R}$ is the cardinality of the continuum. We also prove a theorem which generalizes and unifies Theorems 5, 6, and 8 of [3].

NOTATION. We use the standard set theory notation. The set of all real numbers and the set of all natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$ re-

1991 *Mathematics Subject Classification*: Primary 26A03; Secondary 04A15, 54A25.

spectively. If A and B are subsets of an abelian group $(G,+)$, then $A+B$ and $A-B$ stand for $\{x+y : x \in A \text{ and } y \in B\}$ and $\{x-y : x \in A \text{ and } y \in B\}$ respectively. The set $A-A$ is denoted by $D(A)$. The notation $A \setminus B$ stands for the set-theoretic difference of A and B. The set $\mathbb{R} \setminus A$ is denoted by A^{c}. For any set A, $|A|$ denotes the cardinality of A, and for any cardinal number κ, we denote the sets $\{X \subseteq A : |X| < \kappa\}$ and $\{X \subseteq A : |X| = \kappa\}$ by $[A]^{<\kappa}$ and $[A]^{\kappa}$ respectively. If κ is a cardinal number, then κ^+ stands for the cardinal successor of κ. The cardinal numbers are to be identified with initial ordinal numbers. ω is the first infinite ordinal number. We define $\mathbb{Z}(H)$ to be the set of all finite linear combinations of elements from H with integer coefficients.

Definition 1. Let $(G,+)$ be an arbitrary abelian group. For any two subsets $A, X \subseteq G$, we define $\mathrm{Tr}(X,A) = \{t \in G : X + t \subseteq A\}$. A subset A of G is called *$<\kappa$-covering* for G if $\mathrm{Tr}(X,A) \neq \emptyset$ for each $X \in [G]^{<\kappa}$. A set A is called *κ-covering* for G if it is $<\kappa^+$-covering for G.

Lemma 1. *Suppose that A is a subset of an abelian group $(G,+)$. Then A is 2-covering for G if and only if $A - A = G$.*

Proof. Suppose that A is 2-covering for G. Let $g \in G$. Then $\{0,g\}+t \subseteq A$ for some $t \in G$ and hence $g = (g+t)-t \in A - A$. Consequently, $A - A = G$. To prove the converse, suppose that $A - A = G$. If $x, y \in G$, then $x - y = a_1 - a_2$ for some $a_1, a_2 \in A$. Hence $\{x,y\} + a_2 - y \subseteq A$. Thus A is 2-covering for G.

Lemma 2. (a) *Suppose that A is $<\kappa$-covering for an abelian group $(G,+)$, κ is an infinite cardinal number, and $X \in [G]^{<\kappa}$. Then* $\mathrm{Tr}(X,A) - \mathrm{Tr}(X,A) = G$.

(b) *Suppose that A is 2κ-covering for an abelian group $(G,+)$, κ is a finite cardinal number, and $X \in [G]^{\kappa}$. Then* $\mathrm{Tr}(X,A) - \mathrm{Tr}(X,A) = G$.

Proof. (a) Let $g \in G$. Since A is $<\kappa$-covering for G, $(X \cup (X+g)) + t \subseteq A$ for some $t \in G$. Consequently, $t \in \mathrm{Tr}(X,A)$ and $g + t \in \mathrm{Tr}(X,A)$. Hence $g = (g+t) - t \in \mathrm{Tr}(X,A) - \mathrm{Tr}(X,A)$. Thus $\mathrm{Tr}(X,A) - \mathrm{Tr}(X,A) = G$.

The proof of (b) is similar.

Theorem 1. *Suppose that $(G,+)$ is an abelian group, κ is an infinite cardinal number, A is $<\kappa$-covering for G, H is a subgroup of G such that $|G/H| \geq \kappa$, and $X \in [G]^{<\kappa}$. Then $A \setminus (H + X)$ is $<\kappa$-covering for G.*

Proof. Let $Y \in [G]^{<\kappa}$. By Lemma 2(a), $\mathrm{Tr}(Y,A) - \mathrm{Tr}(Y,A) = G$. To show that $Y + t \subseteq A \setminus (H + X)$ for some $t \in \mathrm{Tr}(Y,A)$, suppose that $(Y+t) \cap (H+X) \neq \emptyset$ for every $t \in \mathrm{Tr}(Y,A)$. Then $\mathrm{Tr}(Y,A) \subseteq H + X - Y$ and hence $G = \mathrm{Tr}(Y,A) - \mathrm{Tr}(Y,A) \subseteq (H + X - Y) - (H + X - Y) =$

$H + X - X + Y - Y$. Consequently, $|G/H| \leq |X - X + Y - Y| < \kappa$. This contradicts the hypothesis that $|G/H| \geq \kappa$ and thus the proof is complete.

Any uncountable abelian group, in particular $\mathbb{R}$, can be written as a countable union of its proper subgroups [7]. However, it follows from the following corollary that $\mathbb{R}$ is not a finite union of proper subgroups of $\mathbb{R}$.

Corollary 1. *The complement of a finite union of proper subgroups of the reals is* $<\omega$*-covering for* $\mathbb{R}$ *and hence the cardinality of such a complement is the cardinality of the continuum.*

The above corollary follows from the previous theorem together with the fact that $|\mathbb{R}/H| \geq \omega$ for any proper subgroup H of $\mathbb{R}$, and the cardinality of any κ-covering subset of $\mathbb{R}$, where $\kappa \geq 2$, is the cardinality of the continuum.

Remark 1. Corollary 1 is true if $\mathbb{R}$ is replaced by any abelian group G with the property that every proper subgroup of G is of infinite index, and the cardinality of the continuum is replaced by the cardinality of G. For example, since the set of all nonzero complex numbers, $\mathbb{C}^*$, is an abelian group under the ordinary multiplication and $\mathbb{C}^*$ contains no proper subgroup of finite index, the cardinality of the complement of a finite union of proper subgroups of $\mathbb{C}^*$ is the cardinality of the continuum.

If H is a Hamel basis, then $|\mathbb{R}/\mathbb{Z}(H)| = |\mathbb{R}|$. Hence we have the following.

Corollary 2. *The complement of a finite union of Hamel bases is* $<|\mathbb{R}|$*-covering for* $\mathbb{R}$ *and the cardinality of such a complement is the cardinality of the continuum.*

Corollary 2 is "best possible" because under the assumption of the continuum hypothesis, the set of all nonzero real numbers can be written as a countable union of linearly independent sets (in particular Hamel bases) [4].

There exist disjoint subsets A and B of $\mathbb{R}$ such that $\mathbb{R} = A \cup B$ and $D(A)$ and $D(B)$ contain no nonempty open interval. However, the following corollary implies that if $\mathbb{R} = A \cup B$, then either the group generated by A or the one generated by B is $\mathbb{R}$.

Corollary 3. *If a set* A *is* $<\omega$*-covering for* $\mathbb{R}$ *and* A *is a finite union of subsets* A_i *of the reals then, for some* i, *the group generated by* $A_i, \mathbb{Z}(A_i)$, *is* $\mathbb{R}$.

Proof. Suppose that $A = \bigcup_{i=1}^{n} A_i$ for some $n \in \mathbb{N}$. If $\mathbb{Z}(A_i) \neq \mathbb{R}$ for every i, then by applying Theorem 1 a finite number of times, we find that $A \setminus \bigcup_{i=1}^{n} A_i = \emptyset$ is $<\omega$-covering for $\mathbb{R}$, which is impossible. Hence the result.

The following theorem implies that the complement of a union of fewer than continuum many translates of the Erdős set $\mathbb{Z}(H)$, where H is a Hamel basis, is everywhere of the second category [6].

THEOREM 2. *The complement of a finite union of proper subgroups of the reals is everywhere of the second category.*

LEMMA 3. *Suppose that F is a first category subset of the real line $\mathbb{R}$ and I is a nonempty open interval. Then the set $\mathbb{N}\cdot(I\setminus F)=\{nx : n\in\mathbb{N}$ and $x\in I\setminus F\}$ is $<\omega$-covering for $\mathbb{R}$.*

Proof. Let $X\in[\mathbb{R}]^{<\omega}$. Since nI is an interval whose length tends to infinity as $n\to\infty$, there exists a nonempty open interval J such that $J+X\subseteq mI$ for some $m\in\mathbb{N}$. The set X being finite and F of the first category implies that $(\mathbb{N}\cdot F)-X$ is of the first category. Since J is a nonempty open interval, J is of the second category and $J\not\subseteq(\mathbb{N}\cdot F)-X$. This implies that $(j+X)\cap(\mathbb{N}\cdot F)=\emptyset$ for some $j\in J$. Consequently, for some $j\in J$, $j+X\subseteq(\mathbb{N}\cdot I)\setminus(\mathbb{N}\cdot F)\subseteq\mathbb{N}\cdot(I\setminus F)$. This shows that $\mathbb{N}\cdot(I\setminus F)$ is $<\omega$-covering for $\mathbb{R}$.

Proof of Theorem 2. Suppose that the conclusion of the theorem is false. Then there exists a nonempty open interval I such that $(\bigcup_{i=1}^n G_i)^{\mathrm{c}}\cap I$ is of the first category, where G_i's are proper subgroups of $\mathbb{R}$ and $n\in\mathbb{N}$. Denote $(\bigcup_{i=1}^n G_i)^{\mathrm{c}}\cap I$ by F. Then $I\setminus F\subseteq\bigcup_{i=1}^n G_i$. Since G_i's are subgroups of $\mathbb{R}$, $\mathbb{N}\cdot(I\setminus F)\subseteq\bigcup_{i=1}^n G_i$. Denote $\mathbb{N}\cdot(I\setminus F)$ by A. By Lemma 3, A is $<\omega$-covering for $\mathbb{R}$. Now by Theorem 1, $A\setminus G_1$ is $<\omega$-covering for $\mathbb{R}$ and again by Theorem 1, $(A\setminus G_1)\setminus G_2=A\setminus(G_1\cup G_2)$ is $<\omega$-covering for $\mathbb{R}$. Continuing in this way, we conclude that $A\setminus\bigcup_{i=1}^n G_i$ is $<\omega$-covering for $\mathbb{R}$. This is impossible because $A\setminus\bigcup_{i=1}^n G_i=\emptyset$. Thus the proof is complete.

It is interesting to compare the following corollary with Theorem 1.

COROLLARY 4. *If H is a proper subgroup of $\mathbb{R}$ and X is an infinite subset of $\mathbb{R}$ with $|X|<|\mathbb{R}/H|$, then $\mathbb{R}\setminus(H+X)$ is everywhere of the second category.*

Proof. Since the group generated by X and the set X itself have the same cardinality, $|\mathbb{Z}(X)|<|\mathbb{R}/H|$. Consequently, $H+\mathbb{Z}(X)$ is a proper subgroup of $\mathbb{R}$ and hence the result.

REMARK 2. Corollary 4 generalizes Theorem 3 of [6]: "The complement of a union of fewer than continuum many translates of the Erdős set $\mathbb{Z}(H)$, where H is a Hamel basis, is everywhere of the second category." Theorem 2 generalizes Theorem 5 of [6]: "The complement of a finite union of Hamel bases is everywhere of the second category."

The following theorem shows that if an abelian group $(G,+)$ is the set-theoretic union of finitely many cosets, $G=\bigcup_{i=1}^n(g_i+S_i)$, where $g_i\in G$

and S_i's are proper subgroups of G, then the index of S_i is finite for at least two values of i. This is more general than a result of [2] (or see Lemma 1 of [7]). It is obvious that if G is a union of two cosets, $G = g_1 + S_1 \cup g_2 + S_2$, then the index of S_i in G is 2 for $i = 1, 2$.

THEOREM 3. *Let A be a subset of an abelian group $(G, +)$. If A is 2^κ-covering for G, where $1 \leq \kappa < \omega$, and H is a subgroup of G with $|G/H| > 2^{2\kappa-2} - 2^{\kappa-1} + 1$, then for every $g \in G$, $A \setminus (H + g)$ is $2^{\kappa-1}$-covering for G.*

Proof. Let X be a subset of G and $|X| = 2^{\kappa-1}$. For $g \in G$, $|X \cup (X + g)| \leq 2(2^{\kappa-1}) = 2^\kappa$. If $X + t \not\subseteq A \setminus (H + g)$ for every $t \in \mathrm{Tr}(X, A)$, then $\mathrm{Tr}(X, A) \subseteq H + g - X$ and, by Lemma 2(b), $G = \mathrm{Tr}(X, A) - \mathrm{Tr}(X, A) \subseteq H - X + X$. This implies that $|G/H| \leq |-X + X| \leq |X|^2 - |X| + 1 = 2^{2k-2} - 2^{k-1} + 1$. Thus the proof is complete.

COROLLARY 5. *Suppose $S_1, \ldots, S_n$ are proper subgroups of an abelian group $(G, +)$ and G is the set-theoretic union of finitely many cosets, $G = \bigcup_{i=1}^n (g_i + S_i)$, where $g_i \in G$. Then the index of S_i in G is at most $2^{2n-2} - 2^{n-1} + 1$ for at least two values of i.*

Proof. If the index of S_i in G is greater than $2^{2n-2} - 2^{n-1} + 1$ for each i, then by the previous theorem, $G \setminus (S_1 + g_1)$ is 2^{n-1}-covering for G and continuing we conclude that $\emptyset = G \setminus \bigcup_{i=1}^n (g_i + S_i)$ is $2^{n-n} = 1$-covering for G, a contradiction. Hence the index of S_i is at most $2^{2n-2} - 2^{n-1} + 1$ for at least one i. If it is the case for exactly one i, say $i = 1$, then $G \setminus \bigcup_{i=2}^n (g_i + S_i) \subseteq g_1 + S_1$ is $2^{n-(n-1)} = 2$-covering for G. Consequently, by Lemma 1, $G = (g_1 + S_1) - (g_1 + S_1) = S_1$. This contradicts the fact that S_1 is a proper subgroup of G.

If P is a nonempty perfect subset of $\mathbb{R}$, then there is a subset M of $\mathbb{R}$ with Lebesgue measure 0 such that $P + M = \mathbb{R}$ (see [1]). It is not known whether for any measure zero set A, there exists an $\aleph_0$-covering set Y for $\mathbb{R}$ such that $A + Y$ is of measure zero. However, we prove the following.

THEOREM 4. *If H is a Hamel basis, then there is a $<|\mathbb{R}|$-covering set A for $\mathbb{R}$ such that $H + A = \mathbb{R}$ and $H \cap A = \emptyset$. On the other hand, under the assumption of the continuum hypothesis, if A is κ-covering for $\mathbb{R}$, where $\omega \leq \kappa$, then there exists a Hamel basis H such that $H + A = \mathbb{R}$.*

Proof. Let H be a Hamel basis. Then it follows from Corollary 2 that H^{c} is $<|\mathbb{R}|$-covering for $\mathbb{R}$. To prove that $H + H^{\mathrm{c}} = \mathbb{R}$, let $r \in \mathbb{R}$. If $r \notin H + H$, then $r - h \notin H$ for every $h \in H$ and hence $r = (r - h) + h \in H^{\mathrm{c}} + H$. If $r \in H + H$, then $r = h_i + h_j$ for some $h_i, h_j \in H$ and hence $r = (h_i + h_j - h_k) + h_k \in H^{\mathrm{c}} + H$, where $h_k \in H$ and $h_i \neq h_k \neq h_j$. This completes the proof of the first part of the theorem.

To prove the second part, we need the following lemma.

LEMMA 4. *If A is κ-covering for $\mathbb{R}$, $\omega \leq \kappa$, and $X \in [\mathbb{R}]^{\leq\kappa}$, then $\bigcap_{x\in X}(A-x)$ is κ-covering for $\mathbb{R}$.*

For, if $Y \in [\mathbb{R}]^{\kappa}$ then, since $|Y+X| = \kappa$, there exists $t \in \mathbb{R}$ such that $Y + X + t \subseteq A$ and hence $Y + t \subseteq \bigcap_{x\in X}(A-x)$. Thus $\bigcap_{x\in X}(A-x)$ is κ-covering for $\mathbb{R}$.

Under the assumption of the continuum hypothesis, by a theorem of Erdős–Kakutani [4], the set of all nonzero real numbers is the union of countably many linearly independent sets. So let $\{0\}^{\mathrm{c}} = \bigcup_{1\leq i<\omega} H_i$, where H_i's are Hamel bases. Note that if H is a Hamel basis then so is $-H$. The proof is complete if $-H_i + A = \mathbb{R}$ or $H_i + A = \mathbb{R}$ for some i. Suppose $H_i + A \neq \mathbb{R}$ and $-H_i + A \neq \mathbb{R}$ for every i. Let $r_i \in (H_i + A)^{\mathrm{c}}$. Then $-H_i \cap (A - r_i) = \emptyset$ for every i and hence

$$\bigcup_{1\leq i<\omega}(-H_i) \cap \bigcap_{1\leq i<\omega}(A - r_i) = \emptyset.$$

But $\bigcup_{1\leq i<\omega}(-H_i) = \{0\}^{\mathrm{c}}$ and, by Lemma 4, the cardinality of $\bigcap_{1\leq i<\omega}(A-r_i)$ is the cardinality of the continuum. This implies that

$$\bigcup_{1\leq i<\omega}(-H_i) \cap \bigcap_{1\leq i<\omega}(A - r_i) \neq \emptyset,$$

a contradiction. Thus the proof is complete.

It follows from Theorem 4 of [3] that there exists a set C, in fact a Bernstein set, such that $(D(C) \cup D(C^{\mathrm{c}}))^{\mathrm{c}}$ is everywhere of the second category. Theorems 5, 6 and 8 of [3] were proved to answer some questions relating to Bernstein sets.

The following theorem generalizes and unifies Theorems 5, 6 and 8 of [3].

THEOREM 5. *If A, B, and C are subsets of the reals such that $A \subseteq C \cup D(C), B \subseteq C^{\mathrm{c}} \cup D(C^{\mathrm{c}})$, $A \cap D(C^{\mathrm{c}}) = \emptyset$ and $B \cap D(C) = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.*

LEMMA 5. *If A is not 2-covering for $\mathbb{R}$, then $D(A) \subseteq D(A^{\mathrm{c}})$.*

Proof. By assumption, there exist a, b in $\mathbb{R}$ such that $\{a, b\} + r \not\subseteq A$ for every $r \in \mathbb{R}$. Hence $a + r \in A^{\mathrm{c}}$ or $b + r \in A^{\mathrm{c}}$. This implies that $\mathbb{R} \subseteq (A^{\mathrm{c}} - a) \cup (A^{\mathrm{c}} - b) \subseteq \mathbb{R}$ and hence $\mathbb{R} = \mathbb{R} + a = A^{\mathrm{c}} \cup (A^{\mathrm{c}} - b + a)$. Thus $A \subseteq A^{\mathrm{c}} - b + a$ and $D(A) \subseteq D(A^{\mathrm{c}} - b + a) = D(A^{\mathrm{c}})$.

Proof of Theorem 5. If $D(C) = \mathbb{R}$ or $D(C^{\mathrm{c}}) = \mathbb{R}$, then the result is trivial. Suppose that $D(C) \neq \mathbb{R} \neq D(C^{\mathrm{c}})$. By Lemma 1, C and C^{c} are not 2-covering for $\mathbb{R}$ and, by Lemma 5, $D(C) \subseteq D(C^{\mathrm{c}})$ and $D(C^{\mathrm{c}}) \subseteq D(C)$. Hence $D(C) = D(C^{\mathrm{c}})$. Now it follows from the hypotheses that $A \subseteq C$ and

$B \subseteq C^c$. If $0 \in C$, then $C \subseteq D(C)$ and, since $A \subseteq C$ and $D(C) = D(C^c)$, we have $A = A \cap D(C) = A \cap D(C^c) = \emptyset$. Similarly if $0 \in C^c$, then $B = \emptyset$. Thus the proof is complete.

REMARK 3. Theorem 5 of [3]: "If $A \subseteq C, B \subseteq C^c$ and $A \cap D(C^c) = \emptyset = B \cap D(C)$, then either A or B are empty" and Theorem 6 of [3]: "If $A \subseteq D(C), B \subseteq D(C^c)$ and $A \cap D(C^c) = \emptyset = B \cap D(C)$, then either A or B are empty" follow directly from the previous theorem.

Theorem 8 of [3]: "If $D(C^c) \subseteq D(C) \subseteq \mathbb{R}$, then at least one of $\subseteq$ is an equality" can be verified as follows: if $D(C) \neq \mathbb{R}$, then, by Lemma 1, C is not 2-covering for $\mathbb{R}$ and, by Lemma 5, $D(C) \subseteq D(C^c)$ and hence $D(C) = D(C^c)$.

Corollary 9 of [3]: "If C and C^c form any decomposition of $\mathbb{R}$, where $|C| < |\mathbb{R}|$, then $D(C^c) = \mathbb{R}$" follows directly from the fact that C^c is 2-covering for $\mathbb{R}$, in fact C^c is $<|\mathbb{R}|$-covering for $\mathbb{R}$ [see Theorem 1] and hence $D(C^c) = \mathbb{R}$ [see Lemma 1].

Corollary 10 of [3]: "If B is any rationally independent set, then $D(B^c) = \mathbb{R}$" follows again from the fact that $(\mathbb{Z}(B))^c$ is 2-covering for $\mathbb{R}$ [see Theorem 1] ($\mathbb{Z}(B)$ is the set of all finite linear combinations of elements of B with integer coefficients) and, by Lemma 1, $D((\mathbb{Z}(B))^c) = \mathbb{R}$. Note that $(\mathbb{Z}(B))^c \subseteq B^c$. Consequently, $D((\mathbb{Z}(B))^c) \subseteq D(B^c) = \mathbb{R}$.

There exists no analytic Hamel basis (see [8]).

PROBLEM. Does there exist a Hamel basis whose complement is analytic?

Acknowledgments. The author is grateful to Professor Jacek Cichoń for useful comments on the original version of the paper.

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Department of Mathematics
University of Wisconsin–Oshkosh
Oshkosh, Wisconsin 54901-8601
U.S.A.
E-mail: muthuvel@vaxa.cis.uwosh.edu

Received 1 July 1996;
revised 10 September 1998