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STABILITY OF THE FIXED-POINT PROPERTY

BY

V. KLEE* (SEATTLE AND COPENHAGEN)

As is well-known, the fixed-point property is possessed by every compact absolute retract A; if the mapping φ of A into itself is continuous, then some point of A is invariant under φ. We show here that for such an A there is in the following sense a sort of stability about the fixed-point property; if the mapping φ of A into itself is nearly continuous, then some point of A is nearly invariant under φ. An example is given of a plane continuum in which the fixed-point property persists but fails to satisfy the stability condition.

Consider a topological space X and a metric space (M, ϱ). For $\varepsilon > 0$, a mapping φ of X into M will be called *ε-continuous* provided each point x of X admits a neighborhood U_x such that the ϱ-diameter of the set φU_x is at most ε. For $\delta \geqslant 0$, a *δ-invariant* point for a mapping ξ of M into M is a point $p \in M$ such that $\varrho(\xi p, p) \leqslant \delta$; ξ will be called a *δ-mapping* provided each point of M is δ-invariant for ξ.

1. PROPOSITION. *Suppose X and Y are topological spaces, M a metric space, f a continuous mapping of X into Y, φ an ε-continuous mapping of Y into M, and ξ a δ-mapping of M into M. Then $\xi\varphi f$ is an $(\varepsilon+2\delta)$-continuous mapping of X into M.*

Proof. Consider an arbitrary point $x \in X$. Since φ is ε-continuous, there is a neighborhood V of fx such that $\operatorname{diam}\varphi V \leqslant \varepsilon$. And since f is continuous, there is a neighborhood U_x of x such that $fU_x \subset V$. Then $\operatorname{diam}\varphi f U_x \leqslant \varepsilon$. Since ξ is a δ-mapping, for arbitrary $u, u' \in U_x$ we have

$$\begin{aligned}\varrho(\xi\varphi fu, \xi\varphi fu') &\leqslant \varrho(\xi\varphi fu, \varphi fu) + \varrho(\varphi fu, \varphi fu') + \varrho(\varphi fu', \xi\varphi fu') \\ &\leqslant \delta + \varepsilon + \delta.\end{aligned}$$

Consequently $\operatorname{diam}\xi\varphi f U_x \leqslant \varepsilon + 2\delta$ and the proof is complete.

2. PROPOSITION. *Suppose P is a compact convex polyhedron in a finite-dimensional normed linear space, and φ is an ε-continuous mapping of P*

* Research fellow of the Alfred P. Sloan Foundation.

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into P. Then there exists a continuous mapping g of P into P such that $\|gp-\varphi p\| \leqslant \varepsilon$ for all $p \in P$. Consequently some point of P is ε-invariant under φ.

Proof. By compactness of P and ε-continuity of φ, P must admit a finite open covering $\mathcal{U}$ such that $\operatorname{diam}\varphi U \leqslant \varepsilon$ for each $U \in \mathcal{U}$. Let $\mathcal{S}$ be a simplicial subdivision of P such that every member of $\mathcal{S}$ lies in some member of $\mathcal{U}$. Let $gv = \varphi v$ for each vertex v of $\mathcal{S}$, and then extend g affinely over all the simplices in $\mathcal{S}$. Then g is a continuous map of P into P, so by Brouwer's fixed-point theorem there exists $p_0 \in P$ for which $gp_0 = p_0$. Now consider an arbitrary point $p \in P$, and let V be the set of all vertices of the carrier of p in $\mathcal{S}$. Then $\operatorname{diam}(\varphi p \cup \varphi V) \leqslant \varepsilon$. Thus each point of φV lies within ε of φp, and since spheres are convex the same is true of the entire set $\operatorname{conv}\varphi V$. But of course $gp \in \operatorname{conv}\varphi V$ and consequently $\|gp-\varphi p\| \leqslant \varepsilon$. In particular, $\varepsilon \geqslant \|gp_0-\varphi p_0\| = \|p_0-\varphi p_0\|$.

3. Theorem. *Suppose C is a compact convex subset of a normed linear space, φ is an ε-continuous mapping of C into C, and $\varepsilon' > \varepsilon$. Then some point of C is ε'-invariant under φ.*

Proof. Choose $\delta \in]0, (\varepsilon'-\varepsilon)/3[$. Since C is compact and hence totally bounded, there exists a finite set $F \subset C$ such that for each point $x \in C$ there exists $\xi x \in F$ for which $\|x-\xi x\| < \delta$. Let P denote the convex hull of F. Since ξ is a δ-mapping of C into P it follows from Proposition 1 that $\xi\varphi$ is an $(\varepsilon+2\delta)$-continuous mapping of C into P. By Proposition 2 there exists $p_0 \in P$ such that $\|\xi\varphi p_0 - p_0\| \leqslant \varepsilon+2\delta$. But then

$$\|\varphi p_0-p_0\| \leqslant \|\varphi p_0-\xi\varphi p_0\|+\|\xi\varphi p_0-p_0\| \leqslant \delta+(\varepsilon+2\delta) < \varepsilon',$$

and the proof is complete.

Now a metric space M will be said to have the *proximate fixed-point property* provided for each $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ such that every τ_ε-continuous mapping of M into M admits an ε-invariant point.

4. Proposition. *If a metric space M has the proximate fixed-point property, then so has every compact retract of M.*

Proof. Let r be a continuous retraction of M onto a compact set $Y \subset M$, and consider an arbitrary $\varepsilon > 0$. We wish to produce $\eta > 0$ such that every η-continuous mapping of Y into Y admits an ε-invariant point. By using the compactness of Y it is easy to see that for a sufficiently small $\delta \in]0, \varepsilon/2[$, $\varrho(x, rx) < \varepsilon/2$ whenever $x \in M$ and $\varrho(x, Y) \leqslant \delta$. Since M has the proximate fixed-point property, there exists $\eta > 0$ such that every η-continuous mapping of M into M admits a δ-invariant point. Now consider an arbitrary η-continuous mapping φ of Y into Y. Then φr is η-continuous by Proposition 1 and hence φr admits a δ-invariant point $x \in M$. Since $\varrho(\varphi rx, x) \leqslant \delta$ and $\varphi rx \in Y$, we have $\varrho(x, Y) \leqslant \delta$ and hence $\varrho(x, rx) < \varepsilon/2$. But also $\delta < \varepsilon/2$, and thus

$$\varrho(\varphi rx, rx) \leqslant \varrho(\varphi rx, x)+\varrho(x, rx) < \varepsilon,$$

and the proof of Proposition 4 is complete for rx is ε-invariant under φ.

5. Proposition. *If a compact metric space X has the proximate fixed-point property, then so has every metric homeomorph of X.*

Proof. Let h be a homeomorphism of X onto a metric space Y, and consider an arbitrary $\varepsilon > 0$. There exists $\delta_1 > 0$ such that $\varrho(x_1, x_2) \leqslant \delta_1$ implies $\varrho(hx_1, hx_2) \leqslant \varepsilon$. Since X has the proximate fixed-point property, there exists $\delta_2 > 0$ such that every δ_2-continuous mapping of X into X admits a δ_1-invariant point. And there exists $\delta_3 > 0$ such that $\varrho(y_1, y_2) \leqslant \delta_3$ implies $\varrho(h^{-1}y_1, h^{-1}y_2) \leqslant \delta_2$. Now consider an arbitrary δ_3-continuous mapping φ of Y into Y. Each point of Y admits a neighborhood V for which $\operatorname{diam}\varphi V < \delta_3$, and hence each point x of X admits a neighborhood U_x for which $\operatorname{diam}\varphi h U_x < \delta_3$. But then $\operatorname{diam} h^{-1}\varphi h U_x \leqslant \delta_2$ and $h^{-1}\varphi h$ is δ_2-continuous, so there exists $x_0 \in X$ for which $\varrho(h^{-1}\varphi hx, x) \leqslant \delta_1$. Thus $\varrho(\varphi hx_0, hx_0) \leqslant \varepsilon$ and hx_0 is an ε-invariant point for φ.

From a well-known embedding theorem in conjunction with the results 3-5 we have

6. Theorem. *Every compact metric absolute retract has the proximate fixed-point property.*

It is evident that a compact metric space which has the proximate fixed-point property must also have the fixed-point property. We wish now to describe a plane continuum K which has the fixed-point property but lacks the proximate fixed-point property. For each $t \in T = [0, 6+4/\pi]$, let the point ζt in the Cartesian plane be defined as follows:

$$\text{for } t \in \begin{cases} [0, 3], \\ [3, 3+2/\pi], \\ [3+2/\pi, 6+2/\pi], \\ [6+2/\pi, 6+4/\pi[, \end{cases} \quad \zeta t = \begin{cases} (0, 1-t), \\ (t-3, -2), \\ (2/\pi, t-5-2/\pi), \\ (6+4/\pi-t, \sin 1/(6+4/\pi-t)). \end{cases}$$

Let $K = \{\zeta t \colon t \in T\}$. Clearly K is a continuum. Now consider a continuous mapping f of K into K, and for each $t \in T$ let $\eta t = \zeta^{-1} f \zeta t \in T$. If always $\eta t \geqslant t$ then clearly $\zeta 0 \in \operatorname{cl} fK$, whence $\zeta 0 = f\zeta s$ for some $s \in T$ and $\eta s = 0 \leqslant s$. Thus in any case there exists $s \in T$ such that $\eta s \leqslant s$. And η is continuous on $[0, s]$ so there is a smallest number $t_0 \in [0, s]$ for which $\eta t_0 \leqslant t_0$. Then $\eta t_0 = t_0$, whence $f\zeta t_0 = \zeta t_0$ and K must have the fixed-point property.

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Now for $n = 1, 2, \ldots$, let $\varepsilon_n = 2/(4n+1)\pi$ and let γ_n be a homeomorphism of the interval $[0, 3+2/\pi]$ onto the interval $[3+2/\pi, 6+4/\pi-\varepsilon_n]$ which maps the endpoints in the indicated order. Let $\varphi_n(\zeta t) = \zeta\gamma_n t$ for $t \in [0, 3+2/\pi]$, $\varphi_n(\zeta t) = \zeta\gamma_n^{-1}t$ for $t \in [3+2/\pi, 6+4/\pi-\varepsilon_n]$, and complete the definition of φ_n by setting $\varphi_n\zeta t = \varphi_n(0, \sin 1/(6+4/\pi-t))$ for $t \in [6+4/\pi-\varepsilon_n, 6+4/\pi]$. It is easily verified that φ_n is continuous at each point of $K\backslash\{\zeta(3+2/\pi)\}$, and that for each $\varepsilon' > \varepsilon_n$, the point $\zeta(3+2/\pi)$ admits a neighborhood U in K such that $\operatorname{diam}\varphi_n U < \varepsilon'$. Thus φ_n is $2\varepsilon_n$-continuous. But it can be verified further that $\varrho(\zeta t, \varphi_n\zeta t) \geqslant 2/\pi - \varepsilon_n$ for all $t \in T$, and consequently the plane continuum K does not have the proximate fixed-point property.

Thus far we have confined our attention to metric spaces. But this was only for the sake of simplicity, and generalizations to uniform spaces are almost immediate. Proposition 2 is easily extended to cover "nearly upper semicontinuous" mappings which associate with each point of P a closed convex subset of P. The resulting generalization of Kakutani's fixed-point theorem [3] can be applied after the manner of Theorem 3 above to a compact convex set in an arbitrary locally convex Hausdorff linear space. This leads to an extension of the fixed-point theorem of Fan [1] and Glicksberg [2]. From a rather special case of that extension, the following result can be deduced:

7. THEOREM. *Suppose X is a compact Hausdorff space which is an absolute retract for such spaces. Then for each open covering $\mathcal{U}$ of X there exists a finite open covering $\mathcal{V}$ of X which has the following property:*

if φ is any mapping of X into X such that each point of X admits a neighborhood N_x for which φN_x lies in some member of $\mathcal{V}$, then there exists a point $x_0 \in X$ such that x_0 and φx_0 lie together in some member of $\mathcal{U}$.

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SUR LES FONCTIONS QUASICONTINUES AU SENS DE S. KEMPISTY

PAR

S. MARCUS (BUCAREST)

1. Propriétés de structure. A et A' étant deux espaces topologiques, désignons par f une fonction définie dans A et ayant ses valeurs dans A'.

En généralisant la notion de quasicontinuité, introduite par Kempisty [4], convenons de dire que f est *quasicontinue* au point $x \in A$ lorsque, pour tout voisinage U de x et pour tout voisinage V de $f(x)$, il existe un ensemble ouvert $G \subset U$ tel que $f(G) \subset V$.

B et B' étant deux espaces métriques avec les distances ϱ et ϱ' respectivement, désignons par φ une fonction définie dans B et ayant ses valeurs dans B'. Elle sera dite *avoisinée* („neighborly" selon Morse et Bledsoe [1]) au point $x \in B$ lorsqu'il existe pour tout $\varepsilon > 0$ une sphère $S \subset B$ telle que $\varrho(x, y) + \varrho'(\varphi(x), \varphi(y)) < \varepsilon$ quel que soit $y \in S$.

Elle sera dite *avoisinée au sens large* („neighborly *" selon Bledsoe [1]) au point $x \in B$ lorsqu'il existe pour tout $\varepsilon > 0$ une sphère $S \subset B$ telle que $\varrho(x, y) + \varrho'(\varphi(y), \varphi(z)) < \varepsilon$ quels que soient $y \in S$ et $z \in S$.

Enfin, appelons la fonction φ *apparentée* („cliquiss" selon Thielman [8] et [9]) au point $x \in B$ lorsque pour tout $\varepsilon > 0$ et pour toute sphère ouverte S contenant x, il existe une sphère $S_1 \subset S$ telle que $\varrho'(\varphi(y), \varphi(z)) < \varepsilon$ quels que soient $y \in S_1$ et $z \in S_1$.

Toutes les quatre notions sont des généralisations de celle de continuité. Les relations suivantes entre elles (pour les fonctions φ définies dans un espace B métrique) sont faciles à démontrer:

(i) *La quasicontinuité d'une fonction φ en un point $x \in B$ équivaut à l'avoisinement de cette fonction en même point.*

(ii) *L'apparentage d'une fonction φ en un point $x \in B$ équivaut à l'avoisinement au sens large de cette fonction en même point.*

(iii) *La quasicontinuité d'une fonction φ en un point $x \in B$ entraîne l'apparentage de cette fonction en x.*