From (8) we obtain $Y = Z_\beta \cup \bigcup_{\iota<\beta} Q(Z_\iota)$, where the sets in the sums on the right are disjoint according to (8) and to the hypothesis concerning the sets $Q(Z_\iota)$ for $\iota < \beta$. Thus

$$\overline{\overline{Y}} = \overline{\overline{Z}}_\beta + \sum_{\iota<\beta} \overline{\overline{Q(Z_\iota)}} \leqslant \aleph_\alpha + \aleph_\alpha \cdot \overline{\beta}$$

by virtue of (7) and (9). But $\beta < \omega_{\alpha+1}$ implies $\overline{\beta} \leqslant \aleph_\alpha$. Hence $\overline{\overline{Y}} \leqslant \aleph_\alpha + \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$, contrary to (5).

From (6) we have $Q(Z_\beta) \subset Z_\beta$. That means according to (8) that the sets $Q(Z_\beta)$ and $Q(Z_\iota)$ are disjoint for every $\iota < \beta$.

Therefore the sets Z_β (where $\beta < \omega_{\alpha+1}$) are defined so that $Q(Z_\beta)$ are disjoint and $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}_\beta$ for $\beta < \omega_{\alpha+1}$. Put $z_\beta = q(Z_\beta)$ for $\beta < \omega_{\alpha+1}$.

Now let $\beta < \beta'$. Since $z_\beta \in Q(Z_\beta)$ and $z_{\beta'} \in Q(Z_{\beta'})$ by virtue of (6), we have $z_\beta \neq z_{\beta'}$ and $z_{\beta'}\, non \in Q(Z_\beta)$, that is $q(Z_\beta)\, non R z_{\beta'}$, i. e. $z_\beta\, non R z_{\beta'}$. Hence the set of points z_β (where $\beta < \omega_{\alpha+1}$) has a power $\aleph_{\alpha+1}$, is contained in Y, thus also in X, and $z_\beta non R z_{\beta'}$ for every two of its distinct elements z_β and $z_{\beta'}$. This contradicts $(s_{\alpha+1})$.

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CARTESIAN PRODUCTS AND CONTINUOUS IMAGES

BY

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Studying the question whether the Cartesian product $A \times B$ of continua A and B is a continuous image of A provided that B is a continuous image of A, Siekluckі and Engelking have proved that the answer can be negative already for $A = B$, i. e. for topological squares (1). Their examples are the following:

(i) $A = B = \mathcal{S}$, where $\mathcal{S}$ is the *condensed sinusoid*, i. e. the sum of the curve $\{(x, y): y = \sin 1/x,\ 0 < x \leqslant 1\}$ and the straight segment with end points $(0, -1)$ and $(0, 1)$,

(ii) $A = B = \mathcal{B}$, where $\mathcal{B}$ is the *countable brush*, i. e. the sum of the infinite sequence of straight segments with end points $(0, 1)$ and $(1/i, 0)$ for $i = 1, 2, \ldots$ and the straight segment with end points $(0, 1)$ and $(0, 0)$.

The purpose of this paper is to prove a more general theorem (2), which comprises the cases (i) and (ii) (see especially the corollary).

Let X, Y and Z be arbitrary compact spaces and let p be the projection of Cartesian product $X \times Y$ onto X. We denote by $L(X)$ the set of points of X at which the space X is locally connected and we put $N(X) = X - L(X)$.

(1) *If* $f(X) = Y$ *is a continuous mapping,* $y \in Y$ *and* $f^{-1}(y) \subset \operatorname{Int}(V)$, *then* $y \in \operatorname{Int}(f(V))$.

Proof. Suppose that $\lim y_n = y$ and $y_n \in Y - f(V)$. Then we have $f^{-1}(y_n) \subset X - f^{-1}f(V) \subset X - V$. Applying the compactness of X, let $x_n \in f^{-1}(y_n)$ and $\lim x_n = x'$. Then $f(x_n) = y_n$ and $x_n \in X - V$, that is $x' \in \overline{X - V} = X - \operatorname{Int}(V)$. Thus $x' \in X - f^{-1}(y)$ and hence $f(x') \neq y$, which contradicts the continuity of f.

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(1) See **P 290**, Colloquium Mathematicum 7 (1960), p. 110, and **P 290, R 1**, ibidem, p. 309.

(2) It is a result of a correspondence and discussion at the meeting on 16 December 1959 of the Wrocław Topological Seminar conducted by Professor B. Knaster.

(2) *If* $f(X) = Y$ *is a continuous mapping,* $y \in Y$ *and* $f^{-1}(y) \subset L(X)$, *then* $y \in L(f(X))$.

Proof. Let U be an open neighbourhood of the point y in Y. Then $f^{-1}(y) \subset f^{-1}(U)$ and $f^{-1}(U)$ is open in X. Since $f^{-1}(y) \subset L(X)$, there exists for each $x \in f^{-1}(y)$ an open neighbourhood U_x of x which is connected and contained in $f^{-1}(U)$. Thus the sum $V = \bigcup_x U_x$, where $x \in f^{-1}(y)$, is an open set and $f^{-1}(y) \subset V$. From (1) we have $y \in \operatorname{Int}(f(V)) = \operatorname{Int}(\bigcup_x f(U_x))$. However, $y \in f(U_x)$ for every $x \in f^{-1}(y)$. Hence the neighbourhood $\bigcup_x f(U_x)$ of the point y is connected and contained in U, because $f(U_x) \subset ff^{-1}(U) = U$ for each $x \in f^{-1}(y)$.

(3) *If* f *is a continuous mapping, then* $N(f(X)) \subset f(N(X))$.

Proof. Since $f(X) - N(f(X)) = L(f(X))$, the theorem is equivalent to the inclusion $f(X) - f(N(X)) \subset L(f(X))$. Putting $y \in f(X) - f(N(X))$ we have $f^{-1}(y) \subset f^{-1}f(X) - f^{-1}f(N(X)) \subset X - N(X) = L(X)$, which, applying (2), implies $y \in L(f(X))$.

(4) $L(X \times Y) = L(X) \times L(Y)$.

Proof. The inclusion $L(X) \times L(Y) \subset L(X \times Y)$ is evident. Let $(x, y) \in L(X \times Y)$. If V is a neighbourhood of x in X, $p^{-1}(V)$ is one of (x, y) in $X \times Y$. Then there exists a neighbourhood U of (x, y) which is connected and contained in $p^{-1}(V)$. It follows that the set $p(U)$ containing the point x is connected, contained in V and open, the projection p being an open mapping. Then $x \in L(X)$. The proof that $y \in L(Y)$ is similar.

(5) $N(X \times Y) = N(X) \times Y \cup X \times N(Y)$.

Proof. We have the identities $X \times Y = L(X) \times L(Y) \cup L(X) \times N(Y) \cup N(X) \times L(Y) \cup N(X) \times N(Y)$ and $N(X) \times Y \cup X \times N(Y) = N(X) \times L(Y) \cup N(X) \times N(Y) \cup L(X) \times N(Y)$ and the sums on the right are those of disjoint sets. Hence $X \times Y = L(X) \times L(Y) \cup N(X) \times Y \cup X \times N(Y)$ and (4) implies $N(X) \times Y \cup X \times N(Y) = X \times Y - L(X) \times L(Y) = X \times Y - L(X \times Y) = N(X \times Y)$.

THEOREM. *If* $N(Y) \neq 0 \neq N(X) \subset Z \subset X$ *and* $N(Z) = 0$, *then* $X \times Y \neq f(X)$ *for every continuous mapping* f.

Proof. Suppose that $X \times Y = f(X)$. It follows from (3) and (5) that $N(X) \times Y \cup X \times N(Y) = N(X \times Y) = N(f(X)) \subset f(N(X)) \subset f(Z)$. Thus $X \times N(Y) \subset f(Z)$. However, the condition $N(Y) \neq 0$ implies $X = p(X \times N(Y)) \subset pf(Z)$. Hence $X = pf(Z)$. We conclude from (3) and from the continuity of the mapping pf that $N(X) = N(pf(Z)) \subset pf(N(Z)) = 0$. This implies that $N(X) = 0$, contrary to our hypothesis.

Applying the theorem for $X = Y$ we obtain

COROLLARY. *If a continuum* X *is not locally connected and the set* $N(X)$ *of points at which* X *is not locally connected is contained in a locally connected continuum lying in* X, *then the Cartesian product* $X \times X$ *is not a continuous image of* X.

Let us note that the hypothesis concerning $N(X)$ in this corollary is essential. For example, a countable Cartesian product $\mathfrak{B}^{\aleph_0}$ of the brush $\mathfrak{B}$ is not locally connected at the point $((0,0),(0,0),\ldots)$ and $\mathfrak{B}^{\aleph_0} \times \mathfrak{B}^{\aleph_0}$ is homeomorphic to $\mathfrak{B}^{\aleph_0}$.

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