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ON TWO EXTENSIONS OF THE HARDY-LANDAU THEOREM

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1. L. Jeśmanowicz [4] recently gave a simple proof of the Hardy-Landau theorem ([3], Theorem 64) for the convergence of series summable by Cesàro means, supposing the order of the means to be *positive real*, and using properties of the Hölder and Kronecker operators. I give here direct proofs of two extensions of the Hardy-Landau theorem, viz. Theorem A and Theorem B, equally simple in principle, using certain difference formulae of Bosanquet ([2], § 3.1). My proof of Theorem A follows a method employed by Rajagopal for a more general purpose ([6], § 3) and my proof of Theorem B follows a method as given by Bosanquet (e. g. [2], Theorem 6) in illustration of how his difference formulae may be used.

Throughout this note $\{s_n\}$ stands for a real sequence and $\{S_n^a\}$, $\{C_n^a\}$, where $a > 0$, are the sequences of Cesàro sums and Cesàro means respectively, of order a, of the sequence $\{s_n\}$. Thus

$$C_n^a = S_n^a / E_n^a,$$

where S_n^a is the coefficient of x^n in $(1-x)^{-a}\sum s_n x^n$, and E_n^a is the coefficient of x^n in $(1-x)^{-a-1}$, and (C, a)-summability of $\{s_n\}$ to l (finite) is defined by $C_n^a \to l$ $(n \to \infty)$. Let us note that

$$E_n^a \sim \frac{n^a}{\Gamma(a+1)} \quad (n \to \infty).$$

Following Bosanquet, we may define below differences of positive integral order p of the sequence $\{s_n\}$. For positive integers h, k,

$$\Delta_h^1 s_n = s_{n+h} - s_n, \quad \Delta_h^p s_n = \Delta_h^1 \Delta_h^{p-1} s_n \quad \text{for} \quad p = 1, 2, 3, \ldots \; (\Delta_h^0 s_n = s_n),$$

$$\Delta_{-k}^1 s_n = s_n - s_{n-k}, \quad \Delta_{-k}^p s_n = \Delta_{-k}^1 \Delta_{-k}^{p-1} s_n \quad \text{for} \quad p = 1, 2, 3, \ldots$$

$$(\Delta_{-k}^0 s_n = s_n;\; s_j = 0 \text{ for } j \leqslant 0);$$

so that

$$\Delta_h^p s_n = \sum_{r=0}^{p} (-1)^r \binom{p}{r} s_{n+(p-r)h},$$

$$\Delta_{-k}^p s_n = \sum_{r=0}^{p} (-1)^r \binom{p}{r} s_{n-rk}, \quad \text{if} \quad n > pk.$$

In the above notation, Bosanquet's difference formulae already referred to may be stated thus.

LEMMA. *For positive integers h, k, p, we have*

$$(1) \qquad \Delta_h^p S_n^p = \sum_{\nu_1=1}^{h} \sum_{\nu_2=1}^{h} \dots \sum_{\nu_p=1}^{h} s_{n+\nu_1+\nu_2+\dots+\nu_p},$$

$$(2) \qquad \Delta_{-k}^p S_n^p = \sum_{\nu_1=1}^{k} \sum_{\nu_2=1}^{k} \dots \sum_{\nu_p=1}^{k} s_{n+p-\nu_1-\nu_2-\dots-\nu_p}, \quad \textit{if} \quad n > kp.$$

The proof of the lemma, by induction on p and using the fact $S_n^{p+1} = S_0^p + S_1^p + \dots + S_n^p$, is quite simple.

2. A neat proof of the Hardy-Landau theorem, in the following more general form, may be based on the above lemma:

THEOREM A. *If $\{s_n\}$ is a sequence summable (C, p) to l for a positive integer p and if the sequence is slowly increasing in the Schmidt sense* (1) *which is (in the now familiar form)*

$$(3) \qquad \limsup_{n\to\infty} \max_{n<n'\leqslant \lambda n} (s_{n'} - s_n) = \omega(\lambda) \downarrow 0 \quad \textit{as} \quad \lambda \downarrow 1,$$

then the sequence is convergent to l.

Proof. From (1) we have at once

$$(4) \qquad -s_n = I + J,$$

where

$$I = -\frac{\Delta_h^p S_n^p}{h^p} \quad \text{and} \quad J = \frac{1}{h^p} \sum_{\nu_1=1}^{h} \sum_{\nu_2=1}^{h} \dots \sum_{\nu_p=1}^{h} (s_{n+\nu_1+\nu_2+\dots+\nu_p} - s_n).$$

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(1) It may be mentioned here that the essential idea of slow oscillation of a sequence was introduced by K. Ananda-Rau independently of others, in connection with $(C, 1)$-summability, in a paper presented to the London Mathematical Society as early as 1919 but published much later in 1924 [1].

Writing $S_n^p = S^p(n)$ for convenience, we obtain, from the definition of $\Delta_h^p S_n^p$,

$$I = -\sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{S^p(n+\overline{p-r}h)}{(n+\overline{p-r}h)^p} \left(\frac{n}{h} + p - r\right)^p.$$

Given $\lambda > 1$, we can choose h (corresponding to n) so that $h \leqslant (\lambda-1)n/p < h+1$ and hence $h \to \infty$ with n while $n/h \to p/\lambda - 1$. Remembering that summability (C, p) of $\{s_n\}$ to l means

$$\frac{S^p(n+\overline{p-r}h)}{(n+\overline{p-r}h)^p} \to \frac{l}{\Gamma(p+1)} \quad \text{as} \quad n, h \to \infty,$$

we then get, as $n \to \infty$,

$$I \to -\sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{l}{\Gamma(p+1)} \left(\frac{p}{\lambda-1} + p - r\right)^p$$

$$= -\frac{l}{\Gamma(p+1)} \sum_{r=0}^{p} (-1)^r \binom{p}{r} (\varkappa + p - r)^p$$

$$(5) \qquad = -\frac{l}{\Gamma(p+1)} \Delta_1^p \varkappa^p = -l \quad (\varkappa = p/\lambda - 1),$$

where we define $\Delta_1^p \varkappa^p$ exactly like $\Delta_1^p s(n)$ and use the fact that $\Delta_1^p \varkappa^p = \Gamma(p+1)$. Again

$$J \leqslant \max_{n<n'\leqslant n+ph} (s_{n'} - s_n), \quad \text{where} \quad ph \leqslant (\lambda-1)n,$$

so that, by (3),

$$(6) \qquad \limsup_{n\to\infty} J \leqslant \omega(\lambda).$$

Using (5), (6) in (4) and letting $\lambda \downarrow 1$, we find that

$$(7) \qquad \limsup_{n\to\infty} (-s_n) \leqslant -l, \quad \text{or} \quad \liminf_{n\to\infty} s_n \geqslant l.$$

Next, starting from (2), we obtain the relation

$$(4') \qquad s_n = I' + J',$$

where

$$I' = \frac{\Delta_{-k}^p S_n^p}{k^p} \quad \text{and} \quad J' = \frac{1}{k^p} \sum_{\nu_1=1}^{k} \sum_{\nu_2=1}^{k} \dots \sum_{\nu_p=1}^{k} (s_n - s_{n+p-\nu_1-\nu_2-\dots-\nu_p}).$$

Here $I' \to l$ as $n \to \infty$ exactly as $I \to -l$ in (4) and

$$J' \leqslant \max_{n-pk<n'\leqslant n} (s_n - s_{n'}), \quad \text{where} \quad pk \leqslant (1-\theta)n,\ 0<\theta<1,$$

so that, by (3),

$$\limsup_{n\to\infty} J' \leqslant \omega\left(\frac{1}{\theta}\right). \tag{6'}$$

We now deduce from (4′), by substitution for $\lim I'$ and $\limsup J'$ as $n\to\infty$, and by letting $\theta \uparrow 1$, the relation

$$\limsup_{n\to\infty} s_n \leqslant l. \tag{7'}$$

(7) and (7′) together yield the desired conclusion $\lim s_n = l$.

3. The next theorem, which can also be proved by means of Bosanquet's difference formulae, includes the Hardy-Landau theorem as a special case.

THEOREM B. *Let $W(x)$, $V(x)$ be two positive-valued functions of $x>0$ such that there are constants $0<\eta<1$, $H>0$ which satisfy the condition*

$$\left.\begin{matrix} W(x')/W(x) \\ V(x')/V(x) \end{matrix}\right\} < H \quad \text{for} \quad |x'-x| \leqslant \eta x, \quad x > x_0. \tag{8}$$

Also, for the sequence $\{s_n\}$ and a positive integer p, let

$$S_n^{p+1} = o\{W(n)\} \quad \text{as} \quad n\to\infty, \tag{9}$$

$$s_n = O_L\{V(n)\} \quad \text{as} \quad n\to\infty, \tag{10}$$

$$\{W(n)/V(n)\}^{1/p+1} = O(n) \quad \text{as} \quad n\to\infty. \tag{11}$$

Then

$$S_n^1 = o[\{W(n)\}^{1/p+1}\{V(n)\}^{p/p+1}] \quad \text{as} \quad n\to\infty. \tag{12}$$

Proof. In (1) we can replace s_n by S_n^1 and hence S_n^p by S_n^{p+1} and obtain

$$-S_n^1 = I_1 + J_1, \tag{13}$$

where

$$I_1 = -\frac{\Delta_h^p S_n^{p+1}}{h^p} \quad \text{and} \quad J_1 = \frac{1}{h^p}\sum_{\nu_1=1}^{h}\sum_{\nu_2=1}^{h}\dots\sum_{\nu_p=1}^{h}(S^1_{n+\nu_1+\nu_2+\dots+\nu_p} - S_n^1).$$

We first suppose that h is subject to the preliminary conditions $h \leqslant \eta n/p$, $h\to\infty$ with n. Then, given any small $\varepsilon>0$, we see that, in virtue of (9) and (8), $S_n^{p+1} = S^{p+1}(n)$ satisfied the condition

$$|S^{p+1}(n+\overline{p-r}h) < \left(\frac{\varepsilon}{2}\right)^{p+1} W(n+\overline{p-r}h)$$
$$< \left(\frac{\varepsilon}{2}\right)^{p+1} HW(n) \quad \text{for} \quad n>n_0,\ r=0,1,2,\dots,p,$$

since $n \leqslant n+(p-r)h \leqslant n(1+\eta)$. Hence, in (13),

$$h^p|I_1| = \left|\sum_{r=0}^{p}(-1)^r\binom{p}{r}S^{p+1}(n+\overline{p-r}h)\right| \leqslant \varepsilon^{p+1}HW(n) \quad \text{for} \quad n>n_0. \tag{14}$$

On the other hand, in virtue of (10) and (8),

$$J_1 = \frac{1}{h^p}\sum_{\nu_1=1}^{h}\sum_{\nu_2=1}^{h}\dots\sum_{\nu_p=1}^{h}(s_{n+1}+s_{n+2}+\dots+s_{n+\nu_1+\nu_2+\dots+\nu_p}) \tag{15}$$
$$\geqslant -\frac{K}{h^p}\sum_{\nu_1=1}^{h}\sum_{\nu_2=1}^{h}\dots\sum_{\nu_p=1}^{h}\{V(n+1)+V(n+2)+\dots+V(n+\nu_1+\dots+\nu_p)\}$$
$$\geqslant -KhpHV(n) \quad \text{for} \quad n>n_1,$$

since $n < n+\nu_1+\nu_2+\dots+\nu_p \leqslant n+ph \leqslant n(1+\eta)$.

From (13), (14), and (15) we get

$$-S_n^1 \geqslant -\frac{\varepsilon^{p+1}}{h^p}HW(n) - hKpHV(n) \quad \text{for} \quad n>\max(n_1,n_0). \tag{16}$$

Here the most advantageous choice of h (for a given n) is that which makes the right-hand member maximum, i. e., the choice is $h = \varepsilon\{W(n)/KV(n)\}^{1/p+1}$ which is in conformity with our preliminary conditions on h in consequence of (11). (16) gives us, with this choice of h,

$$-S_n^1 \geqslant -\varepsilon C(p,H,K)\{W(n)\}^{1/p+1}\{V(n)\}^{p/p+1} \quad \text{for} \quad n>\max(n_1,n_0),$$

i. e. $S_n^1 = o_R[\{W(n)\}^{1/p+1}\{V(n)\}^{p/p+1}]$ as $n\to\infty$.

We can establish the above relation with o_R changed to o_L by starting from (2) instead of from (1) as in the above work, and repeating our arguments in all essential respects. Thus (12) is proved as required.

The significance of Theorem B lies in the fact that it includes the following two well-known results.

If $W(x) = x^{p+1}$, $V(x) = K$ in Theorem B, we have the following result:

COROLLARY B_1. *If a sequence is summable* $(C, p+1)$ *to zero (or to any* l*) for a positive integer* p *and bounded below, then the sequence is* $(C, 1)$*-summable to zero (or to* l*).*

If, in Theorem B, we replace s_n by $S_n^{-1} = s_n - s_{n-1}$ and hence S_n^{p+1} by S_n^p, we have the following generalisation of a theorem of Mordell [5] for the case $(C, 1)$:

COROLLARY B_2. *Theorem* B *can be restated with the hypotheses* (9), (10) *changed to*

$$S_n^p = o\{W(n)\}, \qquad s_n - s_{n-1} = O_L\{V(n)\}$$

respectively, and the conclusion (12) *changed so that the place of* S_n^1 *is taken by* s_n.

The case $W(n) = n^p$, $V(n) = n^{-1}$ of Corollary B_2 is the Hardy-Landau theorem proved by Jeśmanowicz [4].

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