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*ON THE DEFINITION OF MULTI-VALUED ANALYTIC FUNCTIONS**

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The definition of multi-valued analytic functions is usually based, in known text-books, on the identification of the multi-valued analytic function with its Riemann surface (more exactly: with the set of all "schlicht" elements of the Riemann surface). The multi-valued analytic function is then defined as a set of analytic elements satisfying some additional conditions. This kind of definition is very abstract and, therefore, too difficult to be understood by students. The so-defined multi-valued analytic function is not any function in the ordinary meaning of this word, i. e. it is not an operation which, with some mathematical objects, associates other mathematical objects. Moreover, such a definition conflicts with the usual definition of various special multi-valued analytic functions as, for instance, $\log z$, z^a etc., which, usually, are not defined as sets of analytic elements, but are defined as fonctions which, with some complex numbers, associate some sets of complex numbers.

I think that the definition correct from both the mathematical and the didactic points of view should be as follows. First we define the general notion of multi-valued complex function, i. e. a function which, with some complex numbers, associates some sets of complex numbers (∞ included). Futher on, we formulate some conditions which characterize and distinguish a smaller class of multi-valued functions called analytic. This method is systematically applied in the theory of single-valued functions: first we define the general notion of a single-valued function, later we distinguish some special classes of continuous functions, analytic functions, etc.

I am going to describe more precisely the definition of multi-valued analytic functions by the method mentioned above. The terminology

* This is the English version of a note published formerly in Polish; see R. Sikorski, *O definicji funkcji analitycznej wieloznacznej*, Wiadomości Matematyczne 2 (1956), p. 208-210. .

used is the same as in the book of S. Saks and A. Zygmund, *Analytic functions* (Monografie Matematyczne, Warszawa-Wrocław 1953).

Definition I. Let f and g be meromorphic functions defined in regions G and H respectively. The function g is said to be a *direct continuation* of f provided $G \cap H \neq 0$ and $f(z) = g(z)$ for every $z \in G \cap H$. The function g is said to be a *continuation of f in a region* H_0 (containing the regions G and H) provided there exists a finite sequence of meromorphic functions $f_1, \ldots, f_n$ defined in regions $G_1, \ldots, G_n$ respectively, such that f_{j+1} is a direct continuation of f_j for $j = 1, \ldots, n-1$, $G_j \subset H_0$ for $j = 1, \ldots, n$, and $f_1 = f$, $f_n = g$, $G_1 = G$, $G_n = H$.

Definition II. Suppose that, with every point z in a region G, we have associated a set $F(z)$ of complex numbers, and for at least one $z_0 \in G$ the set $F(z_0)$ is not empty. We say then that we have defined a *multi-valued complex function* F in G.

Note that for some points $z \in G$ the set $F(z)$ may be empty.

Definition III. By a *branch* of a multi-valued complex function F defined in a region G we shall understand any meromorphic function f defined in a region $H \subset G$ and such that $f(z) \in F(z)$ for every $z \in H$.

Definition IV. A multi-valued complex function F defined in a region G is said to be *analytic in* G provided the following conditions are satisfied:

(a) If $z_0 \in G$ and $w_0 \in F(z_0)$, then there exists a branch f of F, defined in a region $H \subset G$ and such that $f(z_0) = w_0$ and $z_0 \in H$.

(b) If meromorphic functions f, g are branches of F, then g is a continuation of f in the region G.

(c) If a meromorphic function f is a branch of F, and a meromorphic function g is a continuation of f in the region G, then g is a branch of F.

If G is the whole Gaussian plane, then F is called simply *analytic*.

Definition V. Let F be a multi-valued analytic function in a region G. The set H of all points $z \in G$ such that $F(z)$ is not empty is called the *natural subregion* of F. If G is the whole Gaussian plane, H is called the *natural region* of F.

The notion of the analytic continuation along a curve is superflous in the elementary lecture on multi-valued analytic functions. It suffices to use only the notion of continuation given in Definition I. For instance, the notion of functions arbitrarily continuable in a region can be defined as follows:

Definition VI. A multi-valued analytic function F in a region G is said to be *arbitrarily continuable in* G provided the following conditions are satisfied:

(a) G is the natural subregion of F.

(b) If f is a branch of F, defined in a region H ($\bar{H} \subseteq G$) bounded by a regular Jordan curve C, then every point in C is a point of continuability of f.

The monodromy theorem can be proved very easy if the above definition is assumed. We omit the proof of this theorem and of other theorems on multi-valued analytic functions because the proofs are the same or simpler than in the case of the classical definition.

The above definition of multi-valued analytic function is not, of course, any definition of its Riemann surface (or the subset of "schlicht" elements). The notion of multi-valued analytic function and of its Riemann surface (or the subset of smooth elements) are different, although there is a natural one-to-one correspondence between them. The first notion has a very simple analytic character, the second one is a rather complicated topological notion.

In an elementary lecture on analytic functions, when student's knowledge of topology is rather small, it is difficut to develop the theory of Riemann surfaces. It suffices to mention that every multi-valued analytic function determines uniquely, in a natural way, a surface (called its *Riemann surface*) and a single-valued function on it.

Observe that the above method of definition of multi-valued analytic functions can be applied also in other cases, e. g. for multi-valued harmonic functions.

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