Let $U = U_0 \cup W$ and $V = V_0 \cup W$ so that $A \cup L(A) \subset U$ and $A \cup M(A) \subset V$ and, moreover, $U \cap V \subset W$. If we put

$$W_0 = U \cap L_0(U) \cap V \cap M_0(V),$$

then W_0 is the desired set. For W_0 is open in virtue of a preceding remark, and it is clear that $A \subseteq U \cap V$. It is readily seen that

$$L(A) \subset B \quad \text{if and only if} \quad A \subset L_0(B).$$

From this we infer that $A \subset W_0$. Now the intersection of R-convex sets is R-convex and it is easily seen that $U \cap L_0(U)$ and $V \cap M_0(V)$ are R-convex. This completes the proof.

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ON A PROBLEM OF V. KLEE CONCERNING THE HILBERT MANIFOLDS

BY

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In his talk at the conference on Functional Analysis in Warsaw, September 1960, V. Klee raised the following problem:

Is it true that every Hilbert manifold (i. e. a connected space locally homeomorphic to the Hilbert space at each of its points) is homeomorphic to the Cartesian product of an n-dimensional manifold (in the classical sense) and of the Hilbert space?

In the present note I give an example answering this question in the negative sense and I consider another analogous problem.

Let H denote the Hilbert space, i. e. the space consisting of all real sequences $\{x_n\}$ with $\sum_{n=1}^{\infty} x_n^2 < +\infty$, metrized by the formula

$$\varrho(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Let Q_n denote the open ball in H with centre $a_n = (3n, 0, 0, \ldots)$ and radius 1. Let B_n denote the boundary of Q_n.

It is clear that every open ball in H is homeomorphic to H; consequently every point of a Hilbert manifold has neighbourhoods with arbitrary small diameters, homeomorphic to H.

Obviously the Cartesian product of H by an n-dimensional manifold (i. e. by a connected space locally homeomorphic with the Euclidean n-space at each of its points) is a Hilbert manifold. In particular the spaces

$$A_n = H \times S^n, \quad n = 1, 2, \ldots,$$

where S^n denotes the Euclidean n-sphere, are Hilbert manifolds. It follows that there exists a homeomorphism h_n mapping H onto an open subset G_n of A_n and one can assume that

$$G_n \subset A_n - (a_1) \times S^n.$$

Setting

$$f_n(x,y) = (a_1, y) \quad \text{for every} \quad (x,y)\in H\times S^n,$$

we get a retraction f_n of space $A_n = H\times S^n$ to the sphere $(a_1)\times S^n$.

Let Y denote the space, which we obtain from the set $H\cup\bigcup_{n=1}^{\infty} A_n$ by matching each point $x\in B_n$ with the point $h_n(x)\in A_n$. This identification may be considered as a continuous map φ of $H\cup\bigcup_{n=1}^{\infty} A_n$ onto Y such that

If $y=\varphi(x)$ with $x\in (H-\bigcup_{n=1}^{\infty} B_n)\cup\bigcup_{n=1}^{\infty}(A_n-h_n(B_n))$, then $\varphi^{-1}(y)=x$.

If $y=\varphi(x)$ with $x\in B_n$, then the set $\varphi^{-1}(y)$ consists of two points x and $h_n(x)$.

If $y=\varphi(x)$ with $x\in h_n(B_n)$, then the set $\varphi^{-1}(y)$ consists of two points x and $h_n^{-1}(x)$.

Let us set

$$Z=\varphi\Big(H\cup\bigcup_{n=1}^{\infty}A_n-\bigcup_{n=1}^{\infty}Q_n-\bigcup_{n=1}^{\infty}h_n(Q_n)\Big).$$

Evidently Z is a connected space, locally homeomorphic to H at every point $z\in Z-\varphi(\bigcup_{n=1}^{\infty}B_n)$. In order to prove that Z is locally homeomorphic to H also at every point $z_0=\varphi(x_0)=\varphi(h_n(x_0))$, where $x_0\in B_n$, it suffices to show that there exists a neighbourhood U of z_0 in Z homeomorphic to an open subset of H.

Consider the set

$$P_n = \mathop{E}_{x\in H}[1\leqslant \varrho(x,a_n)<2]$$

and the inversion i_n defined in $H-(a_n)$ by the formula

$$i_n(x) = a_n + \frac{x-a_n}{\varrho(x,a_n)^2}.$$

Setting

$$\psi_n(z) = \begin{cases} i_n[\varphi^{-1}(z)\cap H] & \text{for every point} \quad z\in\varphi(P_n),\\ h_n^{-1}[\varphi^{-1}(z)\cap A_n] & \text{for every point} \quad z\in\varphi(h_n(P_n)),\end{cases}$$

we easily see that ψ_n is a homeomorphism which maps the open neighbourhood $U=\varphi(P_n\cup h_n(P_n))$ of the point z_0 in space Z onto the set $P_n\cup i_n(P_n)$, open in H. Thus the proof that Z is a Hilbert manifold is concluded.

Now let us observe that the homeomorphism h_n maps the closed ball $\bar{Q}_n = Q_n\cup B_n$ onto a closed subset of the space A_n. Manifestly the set $\bar{Q}_n$, as a convex subset of H, is an absolute retract (in the generalized sense, see [1], p. 358) and consequently there exists a retraction r_n of A_n to the set $h_n(\bar{Q}_n)$.

Now let us fix an index n_0 and let us set

$$\vartheta_{n_0}(x) = a_{n_0} + \frac{x-a_{n_0}}{\varrho(x,a_{n_0})} \quad \text{for every point} \quad x\in H-Q_{n_0}$$

and

$$g_{n_0}(z) = \begin{cases} z & \text{if} \quad z\in\varphi(A_{n_0}-h_{n_0}(Q_{n_0})),\\ \varphi\vartheta_{n_0}(x) & \text{if} \quad z=\varphi(x) \text{ with } x\in H-\bigcup_{n=1}^{\infty}Q_n,\\ \varphi\vartheta_{n_0}h_n^{-1}r_n(x) & \text{if} \quad z=\varphi(x) \text{ with } x\in A_n-h_n(Q_n), \text{ where } n\neq n_0.\end{cases}$$

One sees easily that g_{n_0} is a retraction of the space Z to the set $\varphi(A_{n_0}-h_{n_0}(Q_{n_0}))\supset\varphi(A_{n_0}-G_{n_0})\supset\varphi((a_1)\times S^{n_0})$. It follows that $\varphi_0 f_{n_0}\varphi_0^{-1}g_{n_0}$, where $\varphi_0=\varphi|A_{n_0}-h_{n_0}(Q_{n_0})$, is a retraction of the space Z to the topological sphere $\varphi((a_1)\times S^{n_0})$. Consequently, for every natural n_0, the n_0-th Betti number $p_{n_0}(Z)$ is $\geqslant 1$ and we conclude that Z is not homeomorphic to the Cartesian product of H by any n-dimensional manifold.

Now let us call an *ω-manifold* every connected space which is locally homeomorphic to the Hilbert cube, i. e. to the subset Q^{ω} of Hilbert space H, consisting of all points $(x_1, x_2, \ldots, x_n, \ldots)$ satisfying the inequality

$$0\leqslant x_n\leqslant\frac{1}{n} \quad \text{for every} \quad n=1,2,\ldots$$

By a theorem of Keller ([3], p. 757), the Hilbert cube Q^{ω} is topologically homogeneous, i. e., for every two points $x, y\in Q^{\omega}$, there exists a homeomorphism h of Q^{ω} onto itself such that $h(x)=y$. If we observe that, for the point $(0,0,\ldots,0,\ldots)$ of Q^{ω} there exists neighbourhoods (in Q^{ω}) with arbitrarily small diameters, homeomorphic to Q^{ω}, we conclude that every point of an ω-manifold has arbitrarily small neighbourhoods homeomorphic to Q^{ω}, because, for positive $\varepsilon\leqslant 1$ sufficiently small, the map f_{ε} defined by the formula

$$f_{\varepsilon}(x_1,x_2,\ldots,x_n,\ldots) = (\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n, \ldots)$$

is a homeomorphism mapping Q^{ω} onto a neighbourhood of the point $(0,0,\ldots,0,\ldots)$ in Q^{ω} with arbitrarily small diameter.

An ω-manifold is said to be *closed* if it is compact. Evidently the Cartesian product of Q^ω by an n-dimensional closed manifold is a closed ω-manifold. Moreover the Cartesian product of Q^ω by a Euclidean ball, and more generally, by a compact n-dimensional manifold with a boundary, is a closed ω-manifold.

Let us observe that a closed ω-manifold is locally an absolute retract and consequently (by a theorem of Yajima [4]; see also [2]) it is a compact ANR-set. It follows that every closed ω-manifold is acyclic in almost all dimensions and the Betti numbers of it are finite. However there exist closed ω-manifolds which are not homeomorphic to the Cartesian product of Q^ω by any n-dimensional closed manifold. In fact, let P denote the plane set which we obtain by removing from a disk K of the interiors of two small disks K_1, K_2 lying in the interior of K. The Cartesian product $M = P \times Q^\omega$ is a closed ω-manifold and the set P is a deformation retract of M. Consequently, $H_1(M, \mathfrak{A}) \simeq \mathfrak{A}^2$ and $H_n(M, \mathfrak{A})$ is trivial for every $n \neq 1$.

However those conditions are neither satisfied by any n-dimensional manifold M_n, hence nor by any space homeomorphic with the Cartesian product $M_n \times Q^\omega$.

P 335. *Is it true that the Cartesian product of a connected and not empty polytope* (*or more generally, of a compact, not empty ANR-set*) *by Q^ω is always a closed ω-manifold?*

P 336. *Is every closed ω-manifold homeomorphic to the Cartesian product of a connected polytope by Q^ω?*

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SUR UN PROBLÈME DE K. URBANIK CONCERNANT LES ENSEMBLES LINÉAIRES

PAR

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$\mathcal{R}$ étant l'ensemble des nombres réels, soit A^a, où $A \subset \mathcal{R}$ et $a \in \mathcal{R}$, l'image de translation de l'ensemble A à la distance a, c'est-à-dire l'ensemble des nombres de la forme $a + \alpha$ où $\alpha \in A$.

Le problème suivant a été posé par K. Urbanik:

P 337. Est-ce qu'aucun ensemble compact de nombres réels n'est non-dense dans toute somme finie de ses images de translation?

Ce problème peut être formulé en ces termes: a-t-on

$$A - \overline{(A^{a_1} \cup \ldots \cup A^{a_n}) - A} \neq 0$$

pour tout $A \subset \mathcal{R}$ compact et tout système $a_1, \ldots, a_n$ d'éléments de $\mathcal{R}$?

Le but de cette communication est d'établir le théorème qui suit et qui constitue une solution (affirmative) du problème pour $n = 2$:

THÉORÈME. *Si $a \in \mathcal{R}$, $\beta \in \mathcal{R}$ et l'ensemble $A \subset \mathcal{R}$ est compact, on a*

$$A - \overline{(A^a \cup A^\beta) - A} \neq 0. \tag{1}$$

La démonstration fera l'usage essentiel de deux lemmes et d'un théorème dû à Ramsey.

LEMME 1. *Si un ensemble $A \subset \mathcal{R}$ n'est pas un ensemble-frontière dans $\mathcal{R}$, on a* (1) *pour $a \in \mathcal{R}$ et $\beta \in \mathcal{R}$ quelconques.*

Démonstration. On a $\mathcal{R} - \overline{\mathcal{R} - A} \neq 0$ par hypothèse, $\mathcal{R} - \overline{\mathcal{R} - A} \subset A$ toujours et $A^a \cup A^\beta \subset \mathcal{R}$ par définition. Par conséquent, $0 \neq \mathcal{R} - \overline{\mathcal{R} - A} = A \cap (\mathcal{R} - \overline{\mathcal{R} - A}) = A - \overline{\mathcal{R} - A} \subset A - \overline{(A^a \cup A^\beta) - A}$, donc (1).

LEMME 2. *Soient $M \in \mathcal{R}$, $p \in \mathcal{R}$ et $\{p_i\}$ une suite telle que l'on a pour tout* $i = 1, 2, \ldots$

$$p_i \in \mathcal{R}, \quad |p_i| < M, \tag{2}$$

$$p_i = p - (k_i a + l_i \beta), \tag{3}$$