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A NOTE ON CONVEXITY

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The theorem of this note extends a result of Nachbin [1] and Ward [3].

We suppose that X is a Hausdorff space and that R is a binary relation on X; that means that R is a subset of $X\times X$ (xRy if and only if $(x,y)\in R$).

We say that R is *struct* [3] on X if it is a closed non-void transitive subset of $X\times X$, i. e. the relation R is transitive.

A closed subset A of X is called *R-convex* if $a, a'\in A$, $x\in X$ and aRx, xRa' implies $x\in A$.

THEOREM. *If A is a compact R-convex subset of the compact Hausdorff space X, where R is a struct on X, and if W is an open set containing A, then there exists an open R-convex set W_0 with $A\subset W_0\subset W$.*

Proof. Let

$$L(A) = p\big((X\times A)\cap R\big) \quad \text{and} \quad M(A) = q\big((A\times X)\cap P\big),$$

where p and q are the projections of $X\times X$ on the first and second coordinates. It is well known that the projection of the Cartesian product of a compact space and any space on the non-compact factor is a closed map. Hence $L(A)$ and $M(A)$ are closed.

We write also

$$L_0(U) = X\setminus M(X\setminus U) \quad \text{and} \quad M_0(V) = X\setminus L(X\setminus V)$$

and it follows from the above that if U and V are open, then $L_0(U)$ and $M_0(V)$ are open [3].

Let us put

$$C(A) = L(A)\cap M(A).$$

It is obvious that A is R-convex if and only if $C(A)\subset A$.

The sets $L(A)\setminus W$ and $M(A)\setminus W$ are disjoint and closed. Hence there exist disjoint open sets U_0 and V_0 with $L(A)\setminus W\subset U_0$ and $M(A)\setminus W\subset V_0$.

Let $U = U_0 \cup W$ and $V = V_0 \cup W$ so that $A \cup L(A) \subset U$ and $A \cup M(A) \subset V$ and, moreover, $U \cap V \subset W$. If we put

$$W_0 = U \cap L_0(U) \cap V \cap M_0(V),$$

then W_0 is the desired set. For W_0 is open in virtue of a preceding remark, and it is clear that $A \subseteq U \cap V$. It is readily seen that

$$L(A) \subset B \quad \text{if and only if} \quad A \subset L_0(B).$$

From this we infer that $A \subset W_0$. Now the intersection of R-convex sets is R-convex and it is easily seen that $U \cap L_0(U)$ and $V \cap M_0(V)$ are R-convex. This completes the proof.

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ON A PROBLEM OF V. KLEE CONCERNING THE HILBERT MANIFOLDS

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In his talk at the conference on Functional Analysis in Warsaw, September 1960, V. Klee raised the following problem:

Is it true that every Hilbert manifold (i. e. a connected space locally homeomorphic to the Hilbert space at each of its points) is homeomorphic to the Cartesian product of an n-dimensional manifold (in the classical sense) and of the Hilbert space?

In the present note I give an example answering this question in the negative sense and I consider another analogous problem.

Let H denote the Hilbert space, i. e. the space consisting of all real sequences $\{x_n\}$ with $\sum_{n=1}^{\infty} x_n^2 < +\infty$, metrized by the formula

$$\varrho(\{x_n\}, \{y_n\}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Let Q_n denote the open ball in H with centre $a_n = (3n, 0, 0, \ldots)$ and radius 1. Let B_n denote the boundary of Q_n.

It is clear that every open ball in H is homeomorphic to H; consequently every point of a Hilbert manifold has neighbourhoods with arbitrary small diameters, homeomorphic to H.

Obviously the Cartesian product of H by an n-dimensional manifold (i. e. by a connected space locally homeomorphic with the Euclidean n-space at each of its points) is a Hilbert manifold. In particular the spaces

$$A_n = H \times S^n, \quad n = 1, 2, \ldots,$$

where S^n denotes the Euclidean n-sphere, are Hilbert manifolds. It follows that there exists a homeomorphism h_n mapping H onto an open subset G_n of A_n and one can assume that

$$G_n \subset A_n - (a_1) \times S^n.$$