A NOTE ON CONVEXITY

BY

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The theorem of this note extends a result of Nachbin [1] and Ward [3].

We suppose that $X$ is a Hausdorff space and that $R$ is a binary relation on $X$; that means that $R$ is a subset of $X \times X$ (a $R$ y if and only if $(x, y) \in R$).

We say that $R$ is strict [3] on $X$ if it is a closed non-void transitive subset of $X \times X$, i.e., the relation $R$ is transitive.

A closed subset $A$ of $X$ is called $R$-convex if $a, a' \in A$, $a \in X$ and $aR a'$ implies $a' \in A$.

Theorem. If $A$ is a compact $R$-convex subset of the compact Hausdorff space $X$, where $R$ is a strict on $X$, and if $W$ is an open set containing $A$, then there exists an open $R$-convex set $W_0$ with $A \subseteq W_0 \subseteq W$.

Proof. Let

$$L(A) = \{ (X \times A) \cap R \} \quad \text{and} \quad M(A) = \{ (A \times X) \cap P \},$$

where $p$ and $q$ are the projections of $X \times X$ on the first and second coordinates. It is well known that the projection of the Cartesian product of a compact space and any space on the non-compact factor is a closed map. Hence $L(A)$ and $M(A)$ are closed.

We write also

$$L_0(U) = X \setminus M(X \setminus U) \quad \text{and} \quad M_0(V) = X \setminus L(X \setminus V)$$

and it follows from the above that if $U$ and $V$ are open, then $L_0(U)$ and $M_0(V)$ are open [3].

Let us put

$$\mathcal{C}(A) = L(A) \setminus M(A).$$

It is obvious that $A$ is $R$-convex if and only if $\mathcal{C}(A) \subseteq A$.

The sets $L(A) \setminus W$ and $M(A) \setminus W$ are disjoint and closed. Hence there exist disjoint open sets $U_0$ and $V_0$ with $L(A) \setminus W \subseteq U_0$ and $M(A) \setminus W \subseteq V_0$. 
Let $U = U_0 \cup W$ and $V = V_0 \cup W$ so that $A \cup L(A) \subset U$ and $A \cup M(A) \subset V$ and, moreover, $U \cap V \subset W$. If we put

$$W_x = U \cap L_x(U) \cap V \cap M_x(V),$$

then $W_x$ is the desired set. For $W_x$ is open in virtue of a preceding remark, and it is clear that $A \subset U \cap V$. It is readily seen that

$$L(A) \subset B \quad \text{if and only if} \quad A \subset L_x(B).$$

From this we infer that $A \subset W_x$. Now the intersection of $R$-convex sets is $R$-convex and it is easily seen that $U \cap L_x(U)$ and $V \cap M_x(V)$ are $R$-convex. This completes the proof.

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ON A PROBLEM OF V. KLEE
CONCERNING THE HILBERT MANIFOLDS

BY

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In his talk at the conference on Functional Analysis in Warsaw, September 1960, V. Klee raised the following problem:

Is it true that every Hilbert manifold (i.e., a connected space locally homeomorphic to the Hilbert space at each of its points) is homeomorphic to the Cartesian product of an $n$-dimensional manifold (in the classical sense) and of the Hilbert space?

In the present note I give an example answering this question in the negative sense and I consider another analogous problem.

Let $H$ denote the Hilbert space, i.e., the space consisting of all real sequences $(a_n)$ with $\sum_{n=1}^{\infty} a_n^2 < +\infty$, metrized by the formula

$$d((x_n), (y_n)) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$ 

Let $Q_n$ denote the open ball in $H$ with centre $a_n = (3n, 0, 0, \ldots)$ and radius 1. Let $B_n$ denote the boundary of $Q_n$.

It is clear that every open ball in $H$ is homeomorphic to $H$; consequently every point of a Hilbert manifold has neighbourhoods with arbitrary small diameters, homeomorphic to $H$.

Obviously the Cartesian product of $H$ by an $n$-dimensional manifold (i.e., by a connected space locally homeomorphic with the Euclidean $n$-space at each of its points) is a Hilbert manifold. In particular the spaces

$$A_n = H \times S^n, \quad n = 1, 2, \ldots,$$

where $S^n$ denotes the Euclidean $n$-sphere, are Hilbert manifolds. It follows that there exists a homeomorphism $h_n$ mapping $H$ onto an open subset $G_n$ of $A_n$ and one can assume that

$$G_n \subset A_n -(a_n) \times S^n.$$