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ON A THEOREM IN THE THEORY OF RELATIONS AND A SOLUTION OF A PROBLEM OF KNASTER

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In 1933 Turán raised the following problem. Let an arbitrary finite set $f(x)$ correspond to every real number x. Two distinct numbers x and y are said to be *independent* if $x \notin f(y)$ and $y \notin f(x)$. A subset S' of the set S of real numbers is said to be *independent* if any two of its elements are independent. Turán then asked: does there always exist an infinite independent set? G. Grünwald has proved that the answer is affirmative and Lázár has proved that there exists an independent set of power $\mathfrak{c}$.

Ruziewicz then asked the following question: Suppose that $\overline{\overline{S}} = \mathfrak{m}$ ($\overline{\overline{S}}$ denotes the cardinal number of the set S) and that to every $x \in S$ there corresponds a subset $f(x)$ of S satisfying $\overline{\overline{f(x)}} < \mathfrak{n} < \mathfrak{m}$ where $\mathfrak{n} < \mathfrak{m}$ is a cardinal number which does not depend on x. Does there always exist an independent subset S' of S of power $\mathfrak{m}$? Sierpiński, Ruziewicz, Lázár and Sophie Piccard have proved (see [3] and [4]) this without using any hypothesis if $\mathfrak{m}$ is regular or if $\mathfrak{m}$ is the sum of countably many cardinals less than $\mathfrak{m}$.

Assuming the generalized continuum hypothesis $2^{\aleph_k} = \aleph_{k+1}$ Erdös (see [1]) has proved that the answer to the question of Ruziewicz is always affirmative. It is not known if this can be proved without using any hypothesis.

It is clear that if we only assume $\overline{\overline{f(x)}} < \mathfrak{m}$ (instead of $\overline{\overline{f(x)}} < \mathfrak{n} < \mathfrak{m}$), no two elements have to be independent. To see this let $\{X_\alpha\}$, $1 \leqslant \alpha < \Omega_m$, be a well-ordering of the set S. Put $f(X_\alpha) = \{X_\beta\}$, $1 \leqslant \beta < \alpha$. Clearly $\overline{\overline{f(X_\alpha)}} < \mathfrak{m}$ for every α and no two elements are independent.

We are going to prove the following

THEOREM. *Let $S = \{X_\alpha\}$, $1 \leqslant \alpha < \Omega_m$. Assume that there exists a fixed ordinal $\beta < \Omega_m$ so that for every α $(1 \leqslant \alpha < \Omega_m)$ the ordinal type of the (well-ordered) set $f(X_\alpha)$ is less than β. Then there exists an independent set of power $\mathfrak{m}$.*

First of all we can assume that the cardinal $\mathfrak{m}$ has an immediate predecessor (i. e. is not a limit cardinal). For if $\mathfrak{m}$ were a limit cardinal, then there would clearly exists an $\mathfrak{n}$ satisfying $\bar{\beta} < \mathfrak{n} < \mathfrak{m}$ ($\bar{\beta}$ is the cardinal number whose power equals the power of a well-ordered set of ordinal type β) and our theorem follows from the positive answer given to the problem of Ruziewicz.

Assume next that $\mathfrak{m}$ has an immediate predecessor, i. e. that $\mathfrak{m} = \aleph_{k+1}$ (in this case our proof will not use the continuum hypothesis). Let S_1 be a maximal independent subset of $S = \{X_a\}$, $1 \leqslant a < \Omega_{k+1}$. That is S_1 is independent and if $Z \epsilon S$ is not in S_1, then the set $Z \cup S_1$ is not independent. If S_1 has power $\aleph_{k+1}$ our theorem is proved. Thus we can assume that S_1 and every other independent set has power less than $\aleph_{k+1}$ and we shall arrive at a contradiction. Consider the set $S_1 \cup f(S_1)$ ($f(S_1) = \bigcup_{x \epsilon S_1} f(x)$). $S_1 \cup f(S_1)$ has a power less than $\aleph_{k+1}$ (since $\overline{\overline{f(S)}} \leqslant \aleph_k$ and since $\aleph_{k+1}$ is regular it is not cofinal with Ω_{k+1}) and therefore there exists a least ordinal a_1 which is larger than the index δ of any element X of $S_1 \cup f(S_1)$. Since S_1 is a maximal independent set, we immediately infer that if $\gamma \geqslant a_1$ then $f(X_\gamma) \cap S_1$ cannot be empty (since $X_\gamma \cup S_1$ is not independent and by construction $X_\gamma \notin f(S_1)$). Now let S_2 be a maximal independent set in $\{X_\gamma\}$, $a_1 \leqslant \gamma < \Omega_{k+1}$; by our assumption S_2 has a power less than $\aleph_{k+1}$ and we can define a_2 as the least ordinal which is larger than the index of any element of X_δ of $S_2 \cup f(S_2)$. Let $\eta < \beta$ be any ordinal. Suppose that for every $\xi < \eta$ we have already defined an increasing sequence a_ξ and maximal independent sets S_ξ where the index of each element of S_ξ is greater than $a_{\xi'}$ for every $\xi' < \xi$ and where a_ξ is the least ordinal greater than the index of any element of $S_\xi \cup f(S_\xi)$. We proceed by transfinite induction. Let S_η be a maximal independent set amongst the elements $\{X_\tau\}$ where τ runs through the ordinals $< \Omega_{k+1}$ which are greater than a_ξ for every $\xi < \eta$. By our assumption S_η has power $< \aleph_{k+1}$. Define a_η as the least ordinal greater than the index of any element of $S_\eta \cup f(S_\eta)$. Thus the sets S_η and the ordinal a_η are defined for every $\eta < \beta$. Since $\bar{\beta} \leqslant \aleph_k$, there exists a least ordinal δ such that $a_\eta < \delta$ for each $\eta < \beta$. $X_\delta \cup S_\eta$ is not independent (by the maximality of S_η) and since by construction $X_\delta \notin f(S_\eta)$, $f(X_\delta) \cap S_\eta$ is not empty for every $\eta < \beta$. But since the index of every element of S_η is greater than $a_{\eta'}$ for every $\eta' < \eta$ and is less than a_η, $f(X_\delta)$ clearly contains a well-ordered subset of ordinal type β. This contradiction proves our theorem.

Knaster [2] poses the following question: as is well known, Sierpiński [5] has proved that $\mathfrak{c} = \aleph_1$ is equivalent to the possibility of decomposing the plane into two sets A and B so that every horizontal line $x = t$ intersects A in a denumerable set and every vertical line $y = t$ intersects B in a denumerable set. Now let t_ξ, $1 < \xi < \Omega_1$, be a well-ordering of the real numbers. Is it possible to decompose the plane into two sets A and B so that there should exist an ordinal $\beta < \Omega_1$ such that every horizontal line $x = t$ intersects A in a set of ordinal type $< \beta$ and every vertical line $y = t$ intersects B in a set of ordinal type $< \beta$ (i. e. the ordinal type of the sequence t of the points (t, t_ξ) in A is less than β for every t)?

Knaster remarks that Sierpiński's original decomposition does not have this property and conjectures (see [2]) that such a decomposition is impossible. We are going to prove this and in fact will show that if A is such that every horizontal line $x = t$ intersects it in a set of ordinal type $< \beta$ then B (the complement of A) contains a square of power $\aleph_1$, i. e. there exists a subset S_1 of the reals of power $\aleph_1$ so that for every $x \epsilon S_1$, $y \epsilon S_1$, $x \neq y$, the point (x, y) belongs to B. (Clearly, the condition $x \neq y$ cannot be omitted since all the points (x, x) could be in A).

Let t be any real number. Define $f(t)$ as the set of all t_ξ where (t, t_ξ) belongs to A. By assumption $f(t)$ has an ordinal type less than β. Thus by our theorem there exists an independent set S_1 of power $\aleph_1$. By definition of $x \epsilon S_1$, $y \epsilon S_1$, $x \neq y$, thus the point (x, y) belongs to B. Thus our assertion is proved.

Remark. It is easy to prove by the method of Sierpiński [5] that if A is such that every horizontal line $x = t$ intersects A in a set which is not everywhere dense, then there is a vertical line $y = t$ which intersects B in a set of power $\mathfrak{c}$.

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