ON THE IMPOSSIBILITY OF EMBEDDING OF THE SPACE $L$ IN CERTAIN BANACH SPACES

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In this note we shall prove the following

**Theorem.** The space $L$ of all absolutely summable real-valued functions defined on the interval $[0, 1]$ is not isomorphic to a subspace of a separable $B$-space $X^*$ conjugate to the $B$-space $X$, as well as to a subspace of a $B$-space with an unconditional basis.

The impossibility of the embedding of the space $L$ in a conjugate separable $B$-space was first proved by Gelfand (see [4], p. 265). Recently a similar proof was given by Dieudonné [2]. The arguments of Gelfand and Dieudonné are based on the representation of linear operators in $L$ by kernels. Our proof is quite different and is based on the fact that for every perfect set $T C[0, 1]$ of a positive Lebesgue measure there is a bounded measurable function which is equivalent to no function belonging to the first Baire class on $T$. The alternative proofs that the space $L$ has no unconditional basis are given in [5] and [6].

**Remark.** All results in this paper remain valid if we replace the space $L$ by the space $L(Q, \mu)$, where $\mu$ is a non-purely atomic measure defined on the $\sigma$-field of all Borel sets in a compact metric space $Q$, and $L(Q, \mu)$ denotes the space of all absolutely summable real-valued functions $f$ defined on $Q$, under the norm $\|f\| = \int|f(q)|\mu(dq)$.

By $L^0$ we shall denote the space of all real-valued essentially bounded functions $\varphi$ defined on the interval $[0, 1]$ with the norm $\|\varphi\| = \text{ess sup}|\varphi|$. In the sequel by measure of a set $T C[0, 1]$ we shall mean the Lebesgue measure of this set and we shall denote it by $m(T)$.

**Lemma 1.** Let $E$ be a separable subspace of $L^0$. Then there is a perfect set $T$ with positive measure such that every function $\varphi$ in $E$ is equivalent to a function $\varphi$, the restriction of which is continuous on $T$. 
Proof. Let \((a_n)\) be a dense denumerable set in \(E\) and \(0 < \epsilon < 1\). According to the Lebesgue theorem we define a decreasing sequence of perfect sets \((T_n)\) such that \(\text{mes} T_n < \text{mes} T_n', \epsilon 2^{-n-1}\) and the function \(\alpha_n\) is equivalent to a function restriction of which is continuous on \(T_n\) for \(n = 1, 2, \ldots\). Let us put \(T = \bigcap_{n=1}^\infty T_n\). We omit the easy proof that \(T\) is a required set.

**Lemma 2.** Let \(T\) be a perfect set in \([0, 1]\) with positive measure. Then there exists a bounded measurable real function \(\varphi_0\) such that \(\varphi_0(t) = 0\) for \(t \not\in T\) and \(\varphi_0\) is not equivalent to any function belonging to the first Baire class on \(T\).

**Proof.** Let us put, for a measurable function \(\varphi\),
\[
\Omega_\varphi(\varphi, t) = \sup_{n} \lim_{n} |\varphi(t_n') - \varphi(t_n')|,
\]
where the supremum is extended on all sequences \((t_n')\) and \((t_n')\) in \(T\) with \(\lim_{n} t_n' = \lim_{n} t_n' = t\),
\[
\text{ess } \Omega_\varphi(\varphi, t) = \inf \Omega_\varphi(\varphi, t),
\]
where the infimum is extended on all functions \(\varphi\) equivalent to \(\varphi\).

It is well known that if the function \(\varphi\) belongs to the first Baire class on a perfect set \(T\), then for every \(\epsilon > 0\) the set \(\{t \in T : \Omega_\varphi(\varphi, t) \geq 1\}\) is closed and nowhere dense in \(T\). Hence, to prove our Lemma it is sufficient to show that there exists a bounded measurable function \(\varphi_0\) vanishing outside the set \(T\) and such that \(\text{ess } \Omega_\varphi(\varphi_0, t) \geq 1\) for every \(t\) in \(T\).

Let \((t_n)\) be a dense denumerable set in \(T\). We shall consider the following sets of functions:
\[
F = \{ \varphi \in L^\infty : \|\varphi\| \leq 1 \text{ and } \varphi(t) = 0 \text{ for } t \notin T\},
\]
\[
F_n = \{ \varphi \in F : \text{ess } \Omega_\varphi(\varphi_0, t) \geq 1 \} \quad (n = 1, 2, \ldots).
\]

The set \(F\) under the metric
\[
\rho(\varphi', \varphi'') = \frac{1}{\epsilon} \int |\varphi'(t) - \varphi''(t)| dt
\]
is a complete metric space and in this space the sets \(F_n\) are closed and nowhere dense. Hence, according to Baire's theorem, the set \(F = F - \bigcup_{n=1}^\infty F_n\) is non-empty. Let \(\varphi_0\) belong to \(F\). It is easily seen that \(\text{ess } \Omega_\varphi(\varphi_0, t_n') \geq 1\) for \(n = 1, 2, \ldots\). Since the set \(\{t : \text{ess } \Omega_\varphi(\varphi_0, t) \geq 1\}\) is closed and the set \((t_n)\) is dense in \(T\), we have \(\text{ess } \Omega_\varphi(\varphi_0, t) \geq 1\) for every \(t\) in \(T\), q. e. d.

**Lemma 3.** Let \(B\) be a subspace of \(L^\infty\) such that for every \(\varphi_0\) in \(L^\infty\) there exists a sequence \((\varphi_n)\) in \(B\) satisfying the following conditions:

1. \(\int_{\varphi(t)} f(t) dt = \lim_{n} \int_{\varphi_n(t)} f(t) dt\) for \(f \in L^1\).

2. \((\varphi_n)\) is a weak Cauchy sequence, i.e. for every functional \(\Phi\) in \((L^\infty)^*\) there exists a limit \(\Phi(\varphi_n)\).

Then the space \(B\) is separable.

**Proof.** Suppose a contrary that \(B\) is separable. Let \(T\) be chosen for the separable subspace \(B\) of \(L^\infty\) and \(\varphi_0\) be chosen for \(T\) and that they have the same meaning and properties as in Lemmas 1 and 2. Let \((\varphi_n)\) be a sequence in \(B\) chosen for \(\varphi_0\) satisfying (1) and (2). Since every function \(\varphi\) in \(L^\infty\) is equivalent to a function \(\varphi_0\) the restriction of which is continuous on \(T\), then, according to the Hahn-Banach theorem, for every \(t\) in \(T\) there exists a functional \(\Phi_0\) over \(L^\infty\) such that \(\Phi_0(\varphi_0) = \Phi(t)\) for every \(\varphi\) in \(T\). According to the condition (2) there exists a limit
\[
\lim_{n} \Phi_0(\varphi_n) = \lim_{n} \varphi_n(t) \quad \text{for } t \in T.
\]

On the other hand, as \((\varphi_n)\) is a Cauchy sequence in \(L^\infty\) so
\[
\sup_{n} |\varphi_n(t)| < \infty.
\]

Let \(f\) be an arbitrary function in \(L^1\) vanishing outside the set \(T\). By (1), (3) and (4) according to Lebesgue theorem on integration of sequences of functions, we obtain
\[
\int_{\varphi(t)} f(t) dt = \lim_{n} \int_{\varphi_n(t)} f(t) dt = \lim_{n} \int_{\varphi_0(t)} f(t) dt = \lim_{n} \int_{\varphi(n)} f(t) dt.
\]

Since \(f\) is an arbitrary summable function on \(T\), it follows that \(\varphi_0\) is equivalent to the function \(\lim \rho_\varphi(\cdot)\) belonging to the first Baire class on \(T\), which leads to a contradiction. Lemma 3 is thus proved.

Since \(L^\infty\) is a conjugate space to the space \(L^1\), our Theorem is an immediate consequence of Lemma 3 and the following

**Lemma 4.** Let \(Z\) be a B-space satisfying one of the following conditions:

1. \(\int_{\varphi(t)} f(t) dt = \lim_{n} \int_{\varphi_n(t)} f(t) dt\) for \(f \in L^1\).

2. \((\varphi_n)\) is a weak Cauchy sequence, i.e. for every functional \(\Phi\) in \((L^\infty)^*\) there exists a limit \(\Phi(\varphi_n)\).

Then the space \(Z\) is separable.

**Proof.** Suppose a contrary that \(Z\) is separable. Let \(T\) be chosen for the separable subspace \(Z\) of \(L^\infty\) and \(\varphi_0\) be chosen for \(T\) and that they have the same meaning and properties as in Lemmas 1 and 2. Let \((\varphi_n)\) be a sequence in \(Z\) chosen for \(\varphi_0\) satisfying (1) and (2). Since every function \(\varphi\) in \(L^\infty\) is equivalent to a function \(\varphi_0\) the restriction of which is continuous on \(T\), then, according to the Hahn-Banach theorem, for every \(t\) in \(T\) there exists a functional \(\Phi_0\) over \(L^\infty\) such that \(\Phi_0(\varphi_0) = \Phi(t)\) for every \(\varphi\) in \(T\). According to the condition (2) there exists a limit
\[
\lim_{n} \Phi_0(\varphi_n) = \lim_{n} \varphi_n(t) \quad \text{for } t \in T.
\]

On the other hand, as \((\varphi_n)\) is a Cauchy sequence in \(L^\infty\) so
\[
\sup_{n} |\varphi_n(t)| < \infty.
\]

Let \(f\) be an arbitrary function in \(L^1\) vanishing outside the set \(T\). By (1), (3) and (4) according to Lebesgue theorem on integration of sequences of functions, we obtain
\[
\int_{\varphi(t)} f(t) dt = \lim_{n} \int_{\varphi_n(t)} f(t) dt = \lim_{n} \int_{\varphi_0(t)} f(t) dt = \lim_{n} \int_{\varphi(n)} f(t) dt.
\]

Since \(f\) is an arbitrary summable function on \(T\), it follows that \(\varphi_0\) is equivalent to the function \(\lim \rho_\varphi(\cdot)\) belonging to the first Baire class on \(T\), which leads to a contradiction. Lemma 3 is thus proved.

Since \(L^\infty\) is a conjugate space to the space \(L^1\), our Theorem is an immediate consequence of Lemma 3 and the following
(a) $Z$ is separable and it is conjugate to a $B$-space $X$ ($Z = X^*$);
(b) $Z$ has an unconditional basis $(e_n)$ \(^{(1)}\).

Then for every subspace $Y$ of $Z$ there exists a separable subspace $E_Y$ in $Y^*$ such that for every $y^*_n$ in $Y^*$ there is a sequence $(y^*_n)$ in $E_Y$ such that \((1')\) $y^*(y) = \lim y^*_n(y)$ for every $y$ in $Y$.

\(2')\) the sequence $(y^*_n)$ is a weak Cauchy sequence in $Y^*$, i.e. for every $y^{**}$ in $Y^{**}$ there exists a limit $\lim y^{**}(y^*_n)$.

**Proof.** It is sufficient to restrict our attention to the case where $Y$ is equal to the whole space $Z$. Indeed, suppose that $E_Y$ is a required subspace in this case. Then for arbitrary subspace $Y$ of $Z$ we put $E_Y = R E_x$, where $R$ is the operation of restriction of linear functionals over $Z$ to the linear functionals over $Y$, i.e. for every $y^*$ in $Y^*$, $R y^* = y^*$, where $y^*(y) = y^*(y)$ for every $y$ in $Y$.

Now suppose that $Y = Z$ and consider separately the cases (a) and (b).

(a) In this case we put $E_Y = \hat{X}$, where $\hat{X}$ denotes the image under the natural embedding of the space $X$ in its second conjugate space $X^{**} = = Z^*$. The fact that the subspace $\hat{X}$ fulfills the conditions (1') and (2') follows immediately from a result of Gantmacher and Smulyan in [3], which states that if $X^*$ is separable, then for every $z^{**}$ in $X^{**}$ there is a sequence $(z_n)$ in $X$ such that $z^{**}(z_n) = \lim z^*(z_n)$ for every $z^*$ in $X^*$.

(b) Let $(e_n)$ be a sequence in $Z^*$ which consists with the basis $(e_n)$ the biorthonormal system, i.e. $e_n^*(e_m) = \delta_{nm}$ ($n, m = 1, 2, \ldots$). We put $E_x = \operatorname{Lin}(e_n)$ (the symbol $\operatorname{Lin}(e_n)$ denotes the smallest closed linear manifold containing the sequence $(e_n)$). Obviously $E_x$ is separable. Let $y^*_n$ be an arbitrary functional in $Z^*$. We put

$$y^*_n = \sum_{n=1}^{\infty} y^*_n(e_n) e_n^* \quad (n = 1, 2, \ldots)$$

Since $(e_n)$ is a basis in $Z$, the sequence $y^*_n$ satisfies the condition (1').

Since the basis $(e_n)$ is unconditional, the series $\sum y^*_n(e_m) e_n^*$ is weakly unconditionally convergent that is $\sigma^*$-unordered (see [1], p. 73, Theorem 1). Hence $(y^*_n)$ is a weak Cauchy sequence.

**REFERENCES**


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