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ON THE IMPOSSIBILITY OF EMBEDDING OF THE SPACE L IN CERTAIN BANACH SPACES

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In this note we shall prove the following

THEOREM. *The space L of all absolutely summable real-valued functions defined on the interval $[0,1]$ is not isomorphic to a subspace of a separable B-space X^* conjugate to the B-space X, as well as to a subspace of a B-space with an unconditional basis.*

The impossibility of the embedding of the space L in a conjugate separable B-space was first proved by Gelfand (see [4], p. 265). Recently a similar proof was given by Dieudonné [2]. The arguments of Gelfand and Dieudonné are based on the representation of linear operators in L by kernels. Our proof is quite different and is based on the fact that for every perfect set $T\subset[0,1]$ of a positive Lebesgue measure there is a bounded measurable function which is equivalent to no function belonging to the first Baire class on T. The alternative proofs that the space L has no unconditional basis are given in [5] and [6].

Remark. All results in this paper remain valid if we replace the space L by the space $L(Q,\mu)$, where μ is a non-purely atomic measure defined on the σ-field of all Borel sets in a compact metric space Q, and $L(Q,\mu)$ denotes the space of all absolutely summable real-valued functions f defined on Q, under the norm $\|f\|=\int_Q|f(q)|\mu(dq)$.

By L^∞ we shall denote the space of all real-valued essentially bounded functions φ defined on the interval $[0,1]$ with the norm $\|\varphi\|=\operatorname{ess\,sup}_{t\in[0,1]}|\varphi(t)|$. In the sequel by measure of a set $T\subset[0,1]$ we shall mean the Lebesgue measure of this set and we shall denote it by mes T.

LEMMA 1. *Let E be a separable subspace of L^∞. Then there is a perfect set T with positive measure such that every function φ in E is equivalent to a function φ, the restriction of which is continuous on T.*

Proof. Let (x_n) be a dense denumerable set in E and $0 < \varepsilon < 1$. According to the Lusin theorem we define a decreasing sequence of perfect sets (T_n) such that $\operatorname{mes} T_n < \operatorname{mes} T_{n+1} - \varepsilon 2^{-n-1}$ and the function x_n is equivalent to a function restriction of which is continuous on T_n for $n = 1, 2, \ldots$ Let us put $T = \bigcap_{n=1}^{\infty} T_n$. We omit the easy proof that T is a required set.

LEMMA 2. *Let T be a perfect set in $[0, 1]$ with positive measure. Then there exists a bounded measurable real function φ_0 such that $\varphi_0(t) = 0$ for $t \notin T$ and φ_0 is not equivalent to any function belonging to the first Baire class on T.*

Proof. Let us put, for a measurable function φ,

$$\Omega_T(\varphi, t) = \sup \overline{\lim_n} |\varphi(t_n') - \varphi(t_n'')|,$$

where the supremum is extended on all sequences (t_n') and (t_n'') in T with $\lim_n t_n' = \lim_n t_n'' = t$,

$$\operatorname{ess} \Omega_T(\varphi, t) = \inf \Omega_T(\psi, t),$$

where the infimum is extended on all functions ψ equivalent to φ.

It is well known that if the function φ belongs to the first Baire class on a perfect set T, then for every $\varepsilon > 0$ the set $\{t \in T : \Omega_T(\varphi, t) \geqslant 1\}$ is closed and nowhere dense in T. Hence, to prove our Lemma it is sufficient to show that there exists a bounded measurable function φ_0 vanishing outside the set T and such that $\operatorname{ess} \Omega_T(\varphi_0, t) \geqslant 1$ for every t in T.

Let (t_n) be a dense denumerable set in T. We shall consider the following sets of functions:

$$F = \{\varphi \in L^\infty : \|\varphi\| \leqslant 1 \text{ and } \varphi(t) = 0 \text{ for } t \notin T\},$$

$$F_n = \{\varphi \in F : \operatorname{ess} \Omega_T(\varphi, t_n) \geqslant 1\} \quad (n = 1, 2, \ldots).$$

The set F under the metric

$$\varrho(\varphi', \varphi'') = \int_0^1 \frac{|\varphi'(t) - \varphi''(t)|}{1 + |\varphi'(t) - \varphi''(t)|} dt$$

is a complete metric space and in this space the sets F_n are closed and nowhere dense. Hence, according to Baire's theorem, the set $F_0 = F - \bigcup_{n=1}^{\infty} F_n$ is non-empty. Let φ_0 belong to F_0. It is easily seen that $\operatorname{ess} \Omega_T(\varphi_0, t_n) \geqslant 1$ for $n = 1, 2, \ldots$ Since the set $\{t : \operatorname{ess} \Omega_T(\varphi_0, t) \geqslant 1\}$ is closed and the set (t_n) is dense in T, we have $\operatorname{ess} \Omega_T(\varphi_0, t) \geqslant 1$ for every t in T, q. e. d.

LEMMA 3. *Let E be a subspace of L^∞ such that for every φ_0 in L^∞ there exists a sequence (φ_n) in E satisfying the following conditions:*

(1) $$\int_0^1 \varphi_0(t) f(t)\, dt = \lim_n \int_0^1 \varphi_n(t) f(t)\, dt \quad \text{for} \quad f \in L;$$

(2) *the sequence (φ_n) is a weak Cauchy sequence, i. e. for every functional Φ in $(L^\infty)^*$ there exists a limit $\lim_n \Phi(\varphi_n)$.*

Then the space E is unseparable.

Proof. Suppose a contrario that E is separable. Let T be chosen for the separable subspace E of L^∞ and φ_0 be chosen for T and that they have the same meaning and properties as in Lemmas 1 and 2. Let (φ_n) be a sequence in E chosen for φ_0 and satisfying (1) and (2). Since every function φ in E is equivalent to a function $\tilde{\varphi}$, the restriction of which is continuous on T, then, according to the Hahn-Banach theorem, for every t in T there exists a functional Φ_t over L^∞ such that $\Phi_t(\varphi) = \tilde{\varphi}(t)$ for every φ in E. According to the condition (2) there exists a limit

(3) $$\lim_n \Phi_t(\varphi_n) = \lim_n \tilde{\varphi}_n(t) \quad \text{for} \quad t \in T.$$

On the other hand, as (φ_n) is a Cauchy sequence in L^∞ so

(4) $$\sup_n \|\varphi_n\| < \infty.$$

Let f be an arbitrary function in L vanishing outside the set T. By (1), (3) and (4) according to Lebesgue theorem on integration of sequences of functions, we obtain

$$\int_T f(t)\varphi_0(t)\, dt = \int_0^1 f(t)\varphi_0(t)\, dt = \lim_n \int_0^1 f(t)\varphi_n(t)\, dt = \int_T \lim_n \tilde{\varphi}_n(t) \cdot f(t)\, dt.$$

Since f is an arbitrary summable function on T, it follows that φ_0 is equivalent to the function $\lim_n \tilde{\varphi}_n(\cdot)$ belonging to the first Baire class on T, which leads to a contradiction.

Lemma 3 is thus proved.

Since L^∞ is a conjugate space to the space L, our Theorem is an immediate consequence of Lemma 3 and the following

LEMMA 4. *Let Z be a B-space satisfying one of the following conditions:*

(a) *Z is separable and it is conjugate to a B-space X ($Z = X^*$);*
(b) *Z has an unconditional basis (e_n)* (1).

Then for every subspace Y of Z there exists a separable subspace E_Y in Y^* such that for every y_0^* in Y^* there is a sequence (y_n^*) in E_Y such that

(1′) *$y_0^*(y) = \lim_n y_n^*(y)$ for every y in Y,*

(2′) *the sequence (y_n^*) is a weak Cauchy sequence in Y^*, i. e. for every y^{**} in Y^{**} there exists a limit, $\lim_n y^{**}(y_n^*)$.*

Proof. It is sufficient to restrict our attention to the case where Y is equal to the whole space Z. Indeed, suppose that E_Z is a required subspace in this case. Then for arbitrary subspace Y of Z we put $E_Y = RE_Z$, where R is the operation of restriction of linear functionals over Z to the linear functionals over Y, i. e. for every z^* in Z^*, $Rz^* = y^*$, where $y^*(y) = z^*(y)$ for every y in Y.

Now suppose that $Y = Z$ and consider separately the cases (a) and (b).

(a) In this case we put $E_Z = \hat{X}$, where $\hat{X}$ denotes the image under the natural embedding of the space X in its second conjugate space $X^{**} = Z^*$. The fact that the subspace $\hat{X}$ fulfills the conditions (1′) and (2′) follows immediately from a result of Gantmacher and Smulyan in [3], which states that if X^* is separable, then for every x_0^{**} in X^{**} there is a sequence (x_n) in X such that $x_0^{**}(x^*) = \lim_n x^*(x_n)$ for every x^* in X^*.

(b) Let (e_n^*) be a sequence in Z^* which consists with the basis (e_n) the biorthonormal system, i. e. $e_n^*(e_m) = \delta_n^m$ $(n, m = 1, 2, \ldots)$. We put $E_Z = \mathrm{Lin}(e_n^*)$ (the symbol $\mathrm{Lin}(x_n)$ denotes the smallest closed linear manifold containing the sequence (x_n)). Obviously E_Z is separable. Let y_0^* be an arbitrary functional in Z^*. We put

$$y_n^* = \sum_{m=1}^{n} y_0^*(e_m)e_m^* \qquad (n = 1, 2, \ldots).$$

Since (e_n) is a basis in Z, the sequence y_n^* satisfies the condition (1′). Since the basis (e_n) is unconditional, the series $\sum_{n=1}^{\infty} y_0^*(e_n)e_n^*$ is weakly unconditionally convergent that is ω^*-unordered (see [1], p. 73, Theorem 1). Hence (y_n^*) is a weak Cauchy sequence.

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