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THE DETERMINANT THEORY IN BANACH SPACES

BY

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The theory of determinants in infinitely dimensional Banach spaces is, in some sense, much older than the notion of Banach spaces. In fact, the first determinant theories in some special infinitely dimensional linear space were created by Fredholm [1-4] (in the case of some integral equations) and by von Koch [1-7] (in the case of some infinite systems of linear algebraic equations with infinitely many variables) about 1900 (¹).

The case investigated by von Koch is, I think, much easier because it is the case of spaces of sequences. Roughly speaking, in any space of sequences (such as $\boldsymbol{l}$, $\boldsymbol{l}^p$, $\boldsymbol{c}$, etc.) we have a system of coordinates distinguished in a natural way. Consequently any continuous endomorphism is determined by an infinite square matrix $(a_{ij})_{i,j=1,2,\ldots}$, and we have to define the determinant of this matrix. For instance we can try to define it as the limit

$$\lim_{n\to\infty} \det(a_{ij})_{i,j=1,\ldots,n}.$$

Of course we have to assume some additional hypotheses which assure that this limit exists, and that the determinant so defined has properties analogous to those of the determinant of a finite square matrix. We know also how to define the subdeterminants of an order n: they are determinants of the infinite square matrices obtained from $(a_{ij})_{i,j=1,2,\ldots}$ by omitting n rows and n columns.

The case investigated by Fredholm, i. e. the case of the space of continuous functions on closed interval, was more difficult. In fact, at that time it was not known that there exists a system of coordinates (i. e. a basis) in this space. To-day we know that the system of coordinates exists but none of possible systems of coordinates is natural and any

(¹) For an exposition of those determinant theories, see Lalesco [1], Riesz [1], Hellinger and Toeplitz [1], Riesz and Nagy [1], Smithies [2].

reference to a coordinate system in this space seems to be artificial. In other words, the space of all continuous functions on a closed interval can be interpreted as a space of sequences, but this interpretation is not an adequate method for investigation of this space. The study of a corresponding space of sequences is more complicated than the study of the original space of continuous functions. However, if no coordinate system is distinguished in the space of continuous functions on a closed interval, it is not evident how to define the determinant and subdeterminants of an endomorphism. It is not even evident what notion should be a substitute for the algebraic notion of subdeterminants of a fixed order. Fredholm has overcome these difficulties and has defined the determinant and subdeterminants for every endomorphism A of the form $A = I+T$ where I is the indentity mapping, and T is an integral operation with a continuous kernel. Fredholm's determinant of A is, of course, a number. Fredholm's subdeterminant of an order n (or rather a substitute for the notion of the set of *all* algebraic subdeterminants of order n) is a continuous function of $2n$ variables. Fredholm's definition was closely connected with the integral character of the endomorphism T. Therefore it was not evident how to generalize this definition to the case of an arbitrary Banach space.

On the other hand, the fundamental theorem of the Fredholm theory, called the *Fredholm Alternative*, can be formulated without using the notion of determinant and subdeterminants. This fact has caused many generalizations of the Fredholm Alternative over arbitrary Banach spaces and endomorphisms $A = I+T$ (of course, under some hypotheses on T). All those generalizations were based on another method, without introducing the notion of determinant and subdeterminants of A. However, one thing was lost in all of those generalizations: the formulae for solutions of the linear equation $Ax = x_0$ and the adjoint equation. That is an additional important advantage in the determinant theory of linear equations: the determinant theory yields not only theorems on existence of solutions of linear equations, but also some simple formulae for the solutions.

It happens rather often in Mathematics that some problems are open for a long time and suddenly they are solved independently by several mathematicians at the same time. This has also happened in the case of the theory of determinants in Banach spaces. Nobody had investigated this problem for a long time. Several years ago two different solutions were given independently and almost simultaneously. The first theory of determinants in arbitrary Banach spaces was created by Ruston [1], [2]. This theory was developed and modified by Grothendieck [1-3] who solved the problem independently. Another theory was

created by Leżański [1-2]. This theory was modified and completed by Sikorski [1-8] (2). The two theories are not equivalent. The second theory is more general than the first. The first theory is the most regular case of the second. Also, the languages of both theories are not the same. The first theory is formulated in the language of the theory of tensor product developed by Schatten [1] and Grothendieck [2], [5]. The second theory makes no use of tensor products and is formulated in the language of multilinear functionals.

The purpose of the present paper is to give an account of today's state of the theory of determinants in Banach space. This account will be written in the language of the second theory because this language is simpler, but the case of the first theory will be also included. §§ 1-5 contain a purely algebraic part of the theory, in general linear spaces. The proper theory of determinants in Banach spaces is the subject of §§ 7-16. In particular, §§ 13-14 contain applications to the theory of integral equations and to systems of infinitely many linear algebraic equations with infinitely many variables.

The notation is not always traditional in this paper, e. g. new symbols are used for adjoint transformations, bilinear functionals, finitely dimensional operators, etc. This notation is more convenient than the traditional one because it enables us to perform the calculation in a mechanical way. The virtue of the adopted notation is especially evident in proofs of theorems, but no proofs are given in this paper (they will be the subject of another paper of the author).

A short list of special symbols used and of terms is added at the end of the paper.

§ 1. What should the determinant and subdeterminants be? We have seen that one of the reasons for difficulties in the definition of the determinant and subdeterminants of an endomorphism in an infinitely dimensional space lies in the resignation of the choice of an auxiliary coordinate system in the space. To facilitate the definition, we shall first discuss the finitely dimensional case.

Let X be an m-dimensional space over a field of characteristic zero. Let A be an endomorphism in X, i. e. and additive and homogeneous transformation from X to X. The determinant D_0 of A is a scalar, i. e. an element of the field under consideration. We expect that, similarly, the determinant of an endomorphism in any linear space will also be, if it exists, a scalar.

(2) For generalizations of the both theories over locally convex linear topological spaces, see Altman [1] and Grothendieck [3].

If a system of coordinates is fixed in the m-dimensional space X under consideration, then points $x, y, \ldots$ in X are uniquely determined by their coordinates:

(1) $$x = (v_1, \ldots, v_m), \qquad y = (w_1, \ldots, w_m), \qquad \ldots,$$

and the transformation

$$y = Ax$$

can be written in the form

$$w_i = \sum_{j=1}^{m} a_{i,j} v_j .$$

$a = (a_{i,j})$ is a matrix determining A. The scalars $a_{i,j}$ can be called the *coordinates of* A in the considered system of coordinates in X.

Consider the space conjugate to X, i. e. the space Ξ of all linear forms $\xi, \eta, \ldots$ defined on X. The chosen system of coordinates in X determines uniquely a corresponding system of coordinates in Ξ, and therefore elements $\xi, \eta \in \Xi$ can be interpreted as sequences of its coordinates:

(2) $$\xi = (\varphi_1, \ldots, \varphi_m), \qquad \eta = (\psi_1, \ldots, \psi_m).$$

The value ξx of the functional ξ at the point x is given by the formula

$$\xi x = \sum_{i=1}^{m} \varphi_i v_i .$$

Given any finite sequences $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ of positive integers $\leqslant m$, $n < m$, let $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n}$ be a number defined as follows: if either two of the integers $i_1, \ldots, i_n$ are equal, or two of the integers $j_1, \ldots, j_n$ are equal, then $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n} = 0$; in the opposite case $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n}$ is the determinant of the matrix $(\beta_{i,j})$ where

$$\beta_{i,j} = \begin{cases} a_{i,j} & \text{if none of the equalities } i = i_k,\ j = j_l\ (k, l = 1, \ldots, n) \text{ holds,} \\ 1 & \text{if } i = i_k \text{ and } j = j_k \text{ for an integer } k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

In the second case, $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n}$ is the product of $(-1)^{i_1 + \ldots + i_n + j_1 + \ldots + j_n}$ by the determinant of the matrix obtained from a by omitting the $j_1, \ldots, j_n$-th columns and the $i_1, \ldots, i_n$-th rows.

Thus the set of all $a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n}$ is, roughly speaking, the set of all subdeterminants of a of the *order* n, i. e. the set of all determinants of matrices obtained from a by omitting n columns and n rows.

The set of all the algebraic subdeterminants $a\binom{i}{j}$ of the order 1 determines uniquely a bilinear functional D_1 on $\Xi \times X$, viz. the functional whose value $D_1\binom{\xi}{x}$ at a point $(\xi, x) \in \Xi \times X$ is given by the formula

$$D_1\binom{\xi}{x} = \sum_{i=1}^{m} \sum_{j=1}^{m} \varphi_i a\binom{i}{j} v_j .$$

(see (1), (2)). Similarly, the set of all the algebraic subdeterminants $a\binom{j,l}{i,k}$ of the order 2 determines a four-linear functional D_2 on $\Xi^2 \times X^2$, viz.

$$D_2\binom{\xi, \eta}{x, y} = \sum_{i,k=1}^{m} \sum_{j,l=1}^{m} \varphi_i \psi_k a\binom{i, k}{j, l} v_j w_l$$

(see (1), (2)). More generally, the set of all the algebraic subdeterminants $a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n}$ of an order n $(1 \leqslant n < m)$ determines uniquely a $2n$-linear functional D_n on $\Xi^n \times X^n$:

(3) $$D_n\binom{\xi_1, \ldots, \xi_n}{x_1, \ldots, x_n} = \sum_{i_1, \ldots, i_n = 1}^{m} \sum_{j_1, \ldots, j_n = 1}^{m} \varphi_{1,i_1} \ldots \varphi_{n,i_n} a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} v_{1,j_1} \ldots v_{n,j_n},$$

where $x_r = (v_{r,1}, \ldots, v_{r,m})$ and $\xi_r = (\varphi_{r,1}, \ldots, \varphi_{r,n})$ for $r = 1, \ldots, n$.

The multilinear functionals $D_1, D_2, \ldots$ do not depend on the choice of a system of coordinates in X. They are uniquely determined by the endomorphism A only. The algebraic subdeterminants $a\binom{i}{j}$, $a\binom{i, k}{j, l}, \ldots$ are *coordinates* of $D_1, D_2, \ldots$ respectively, in the assumed system of coordinates in X. Viz.

(4) $$a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} = D_n\binom{e_{i_1}, \ldots, e_{i_n}}{e_{j_1}, \ldots, e_{j_n}},$$

where

$$e_1 = (1, 0, \ldots, 0),$$
$$\ldots\ldots\ldots\ldots$$
$$e_m = (0, \ldots, 0, 1)$$

are the unit vectors of the axes of coordinates in X and Ξ.

Thus we see that the $2n$-linear functional D_n is a substitute for the notion of the set of *all* the algebraic subdeterminants of A of an order n. This substitute is independent of the choice of a system of coordinates in X. Therefore this substitute seems to be adequate for generalization of the notion of subdeterminants of an order n over the case of infinitely dimensional linear spaces.

The above discussion suggests the following conclusion: If we want to define the notion of the determinant and subdeterminants of an endomorphism A in a (infinitely dimensional, in general) linear space X, we must introduce also a conjugate space Ξ, i. e. a space of all, or some, linear functionals on X. We have to expect that the determinant D_0 is a scalar, and that the adequate substitute for the notion of the set of *all* algebraic subdeterminants of an order n is a $2n$-linear functional on $\Xi^n \times X^n$:

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}.$$

Of course, $D_0, D_1, D_2, \ldots$ are not arbitrary, they have to satisfy some indentities or conditions, similar to those satisfied by these multilinear functionals in the finitely dimensional case. They also have to be connected, in a natural way, with the endomorphism A.

We do not expect it to be possible to define $D_0, D_1, D_2, \ldots$ for every endomorphism A, if the space under consideration is infinitely dimensional. It is necessary to distinguish a special, possibly large, class of endomorphisms and to formulate a definition of $D_0, D_1, D_2, \ldots$ for this class only. We expect that the definition of $D_0, D_1, D_2, \ldots$ can be axiomatic or analytic (by giving explicite formulae for $D_0, D_1, D_2, \ldots$). We shall see later that other methods of definitions are possible, e. g., in the case of Banach spaces, we can define D_0 (considered, roughly speaking, as a function of A) as a solution of a differential equation, and we can obtain $D_1, D_2, \ldots$ from D_0 by differentiation. We can also define $D_0, D_1, \ldots$ in infinitely dimensional spaces as a continuous extension of the finitely dimensional case.

We start with an axiomatic definition which is most general. This definition will be quoted in § 3. We must precede the definition by some remarks of an auxiliary character (§ 2).

§ 2. Operators. We shall consider a fixed commutative algebraic field F whose elements will be called *scalars* and denoted by c, λ. Every mapping into F will be called a *functional*.

We shall also consider two fixed linear spaces Ξ and X (infinitely dimensional, in general) over the field F. The letters ξ, η, ζ (with indices, if necessary) will always denote elements of Ξ, and the letters x, y, z — elements of X, unless the contrary is explicitely stated.

We suppose that Ξ and X are *conjugate*, i. e. with every pair (ξ, x) there is associated a scalar, denoted by ξx, in such a way that ξx is a bilinear functional on $\Xi \times X$, i. e.

$$\xi(x_1 + x_2) = \xi x_1 + \xi x_2, \qquad (\xi_1 + \xi_2)x = \xi_1 x + \xi_2 x,$$
$$c(\xi x) = (c\xi)x = \xi(cx),$$

and that the following cancelation laws are satisfied:

(c) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;

(c′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

It follows from the above conditions that every element $\xi \in \Xi$ can be interpreted as a linear functional on X and, conversely, every element $x \in X$ can be interpreted as a linear functional on Ξ. In symbols

(1) $$\Xi \subset X', \qquad X \subset \Xi',$$

where X' and Ξ' denote respectively the linear spaces of all linear functionals on X and Ξ. This remark suggests an extension of the meaning of the symbol

(2) $$\xi x$$

as follows: if $\xi \in X'$ and $x \in X$, then ξx is the value of ξ at x; if $x \in \Xi'$ and $\xi \in \Xi$, then ξx is the value of x at ξ. This extension will sometimes be convenient.

If A is a bilinear functional on $\Xi \times X$, then the value of A at a point $(\xi, x) \in \Xi \times X$ will be denoted by $\xi A x$.

Suppose that A is a bilinear functional on $\Xi \times X$. For every fixed $x \in X$ there exists exactly one element $y \in \Xi'$ such that $\xi y = \xi A x$ for every $\xi \in \Xi$. This element y will be denoted by Ax. By definition,

$$\xi(Ax) = \xi A x \quad \text{for every} \quad \xi \in \Xi.$$

Similarly, for every fixed $\xi \in \Xi$ there exists exactly one element $\eta \in X'$ such that $\eta x = \xi A x$ for every $x \in X$. This element η will be denoted by ξA. By definition,

$$(\xi A)x = \xi A x \quad \text{for every} \quad x \in X.$$

It is easy to see that the mappings

$$y = Ax \quad \text{and} \quad \eta = \xi A$$

are linear transformations of X into Ξ' and of Ξ into X', respectively. The most important case is when they map X into X, and Ξ into Ξ, respectively (see (1)). Then A is said to be an *operator*. The class of all operators will be denoted by $\mathfrak{O}$. Thus every $A \in \mathfrak{O}$ can be simultaneously interpreted as the bilinear functional ξAx on $\Xi \times X$, or as the endomorphism $y = Ax$ in the space X, or as the endomorphism $\eta = \xi A$ in the space Ξ. The three interpretations of $A \in \mathfrak{O}$ will be systematically used in the whole paper. The neutral name "operator" has been adopted in order not to favor any of the three interpretations.

If A_1, A_2 are bilinear functionals on $\Xi \times X$, and one of them is an operator, then we can define the product $A_1 A_2$ by the equality

$$\xi(A_1 A_2)x = (\xi A_1)(A_2 x) \tag{3}$$

using the extended notation (2) to elements ξA_1 and $A_2 x$ on the right side. Viz. if A_1 is an operator, then (3) is the value of the functional $A_2 x \in \Xi'$ at the point $\xi A_1 \in \Xi$. If A_2 is an operator, then (3) is the value of the functional $\xi A_1 \in X'$ at the point $A_2 x \in X$. The product $A_1 A_2$ is also a bilinear functional on $\Xi \times X$.

The most important case is that where both A_1 and A_2 are operators. Then the product $A_1 A_2$ is also an operator. By definition, the operator $A_1 A_2$ interpreted as an endomorphism in X (in Ξ) is the superposition of the endomorphisms A_1 and A_2 (A_2 and A_1), in symbols:

$$(A_1 A_2)x = A_1(A_2 x), \qquad \xi(A_1 A_2) = (\xi A_1)A_2.$$

The last two identities and (3) show that we can omit parentheses in the expressions on both sides, and we can write simply $\xi A_1 A_2 x$, $A_1 A_2 x$, $\xi A_1 A_2$ etc.

It is easy to see that the space $\mathfrak{O}$ of all operators is linear with respect to the natural definition of algebraic operations. It follows from the above consideration that $\mathfrak{O}$ is a linear algebra with respect to the multiplication (3). The algebra $\mathfrak{O}$ has a unit element I. Viz. I is the fundamental bilinear functional

$$\xi I x = \xi x \qquad (\xi \in \Xi, x \in X).$$

By definition,

$$Ix = x \quad \text{and} \quad \xi I = \xi,$$

i. e. I, interpreted as an endomorphism in X or Ξ, is the identity mapping.

The following three conditions are equivalent for every $A \in \mathfrak{O}$:

(i_0) A has an inverse A^{-1} in the algebra $\mathfrak{O}$ (i. e. there exists an element $A^{-1} \in \mathfrak{O}$ such that $AA^{-1} = A^{-1}A = I$);

(i_1) the endomorphism $y = Ax$ is a one-to-one mapping of X onto X, and the endomorphism $\eta = \xi A$ is a one-to-one mapping of Ξ onto Ξ;

(i_2) the endomorphism $y = Ax$ maps X onto X, and the endomorphism $\eta = \xi A$ maps Ξ onto Ξ.

An operator B is said to be a *quasi-inverse* ([3]) of an operator A provided

$$ABA = A \quad \text{and} \quad ABA = B.$$

To explain this notion, let us introduce the following notation:

$$X_1 = \{Ax \colon x \in X\}, \qquad \Xi_1 = \{\xi A \colon \xi \in \Xi\},$$

$$X_2 = \{Bx \colon x \in X\}, \qquad \Xi_2 = \{\xi B \colon \xi \in \Xi\}.$$

Then each of the following conditions is necessary and sufficient for B to be a quasi-inverse of A:

(q) $ABy = y$ for every $y \in X_1$ and $BAx = x$ for every $x \in X_2$;

(q′) $\eta = \eta BA$ for every $\eta \in \Xi_1$ and $\xi = \xi AB$ for every $\xi \in \Xi_2$.

Condition (q) means that the mapping $x = By$, considered as a mapping defined only on X_1, is the inverse of the mapping $y = Ax$ considered as a mapping defined only on X_2. Similarly, condition (q′) means that the mapping $\xi = \eta B$, considered as a mapping defined only on Ξ_1, is the inverse of the mapping $\eta = \xi A$ considered as a mapping defined only on Ξ_2.

Let ξ_0, x_0 be fixed. The operator K_0 defined by the formula

$$\xi K_0 x = \xi x_0 \cdot \xi_0 x$$

(i. e. the product of scalars ξx_0 and $\xi_0 x$) is called *one-dimensional* and denoted by $x_0 \cdot \xi_0$. By definition

$$K_0 x = x_0 \cdot \xi_0 x \quad \text{and} \quad \xi K_0 = \xi x_0 \cdot \xi_0.$$

In the last three expressions (and in the sequel, too) the dot replaces parentheses. E. g. $x_0 \cdot \xi_0 x$ is the product of the element x_0 by the scalar $\xi_0 x$, etc.

Every finite sum of one-dimensional operators

$$K = \sum_{i=1}^{m} x_i \cdot \xi_i \tag{4}$$

([3]) For properties of quasi-inverse, see Sikorski [4].

is called a *finitely dimensional operator*. By the definition,

$$\xi K x = \sum_{i=1}^{m} \xi x_i \cdot \xi_i x,$$

$$K x = \sum_{i=1}^{m} x_i \cdot \xi_i x, \qquad \xi K = \sum_{i=1}^{m} \xi x_i \cdot \xi_i .$$

Thus $y = Kx$ ($\eta = \xi K$) maps X (maps Ξ) onto its finitely dimensional subspace, which legitimates the name. Observe further that, for every $A \in \mathfrak{O}$, KA and AK are finitely dimensional, viz.

$$KA = \sum_{i=1}^{m} x_i \cdot \xi_i A, \qquad AK = \sum_{i=1}^{m} A x_i \cdot \xi_i .$$

The class of all finitely dimensional operators will be denoted by $\mathfrak{FO}$.

Let $K \in \mathfrak{FO}$ be represented in the form (1). The scalar

$$\operatorname{tr} K = \sum_{i=1}^{m} \xi_i x_i \tag{5}$$

does not depend on the representation of K in the form (1), and it is called the *trace* of K. Of course, tr is a linear functional on $\mathfrak{FO}$. Observe that if $A \in \mathfrak{O}$ and $K \in \mathfrak{FO}$ is given by (4), then

$$\operatorname{tr} AK = \operatorname{tr} KA = \sum_{i=1}^{m} \xi_i A x_i . \tag{6}$$

§ 3. Determinant systems. By a *determinant system* (4) for an operator A we shall understand every infinite sequence

$$D_0, D_1, D_2, \ldots \tag{1}$$

such that:

(d_1) D_n is $2n$-linear functional on $\Xi^n \times X^n$, the value of D_n at a point $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_n) \in \Xi^n \times X^n$ being denoted by $D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$; in particular, D_0 is a scalar;

(d_2) for $n \geqslant 2$, $D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$ is skew symmetric in $\xi_1, \ldots, \xi_n$ and in $x_1, \ldots, x_n$, i. e. for every permutation $\mathfrak{p} = (p_1, \ldots, p_n)$ of numbers $1, \ldots, n$

$$D_n\begin{pmatrix}\xi_{p_1}, \ldots, \xi_{p_n}\\ x_1, \ldots, x_n\end{pmatrix} = \operatorname{sgn}\mathfrak{p} \cdot D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_{p_1}, \ldots, x_{p_n}\end{pmatrix},$$

(4) Sikorski [4].

where $\operatorname{sgn}\mathfrak{p} = 1$ if the permutation $\mathfrak{p}$ is even, and $\operatorname{sgn}\mathfrak{p} = -1$ if $\mathfrak{p}$ is odd;

(d_3) for $n \geqslant 1$, $D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$ interpreted as a function of one variable ξ_i and one variable x_j only is an operator;

(d_4) there exists an integer $r \geqslant 0$ such that D_r does not vanish identically;

(d_5) the following identities hold for $n \geqslant 0$:

$$(\mathrm{D}_n) \qquad D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n\\ x_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=1}^{n} (-1)^i \xi_0 x_i \cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots, \xi_n\\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix},$$

$$(\mathrm{D}'_n) \qquad D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_n\\ A x_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=1}^{n} (-1)^i \xi_i x_0 \cdot D_n\begin{pmatrix}\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n\\ x_1, \ldots\ldots\ldots, x_n\end{pmatrix}.$$

The smallest integer r such that D_r does not vanish identically is called the *order* of the determinant system (1).

Observe that if X and Ξ are m-dimensional linear spaces over F (see § 1), then $\mathfrak{O}$ is the set of all endomorphisms in X, and every $A \in \mathfrak{O}$ has a determinant system. Viz. define $D_0, D_1, \ldots, D_{m-1}$ as in § 1 (3), and let

$$\begin{aligned} D_m\begin{pmatrix}\xi_1, \ldots, \xi_m\\ x_1, \ldots, x_m\end{pmatrix} &= \det(\xi_i x_j)_{i,j=1,\ldots,m}, \\ D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} &= 0 \quad \text{for} \quad n > m. \end{aligned} \tag{2}$$

The sequence $D_0, D_1, D_2, \ldots$ just defined is a determinant system for A.

The finitely dimensional case just discussed suggests, in the case of arbitrary spaces Ξ and X, that we call

D_0 — the *determinant of* A,

and, for $n > 0$,

D_n — the *subdeterminant of* A of the order n.

This terminology is legitimated because, as we shall see later, D_0, $D_1, D_2, \ldots$ have many properties of the algebraic determinants and subdeterminants. However, the analogy with the algebraic determinants and subdeterminants in the finitely dimensional case is not complete because the determinant system, if it exists, is not uniquely determined by A. In fact, if (1) is a determinant system for A, and $c \neq 0$ is any scalar, then

$$(1') \qquad cD_0, cD_1, cD_2, \ldots$$

is also a determinant system for A. Fortunately, the non-uniqueness is not too great: sequences (1') present all determinant system for A, i. e. the following theorem is true:

THEOREM 1. *The determinant system for A, if it exists, is determined by A uniquely up to a constant factor $\neq 0$* [5].

The fact that the determinant system for A is determined by A up to a factor only, is, of course, a defect of the theory. However, we shall see later that this phenomenon is rather typical for the theory of determinants in infinitely dimensional linear spaces. We shall discuss later the case of operators in Banach spaces. Then the determinant system will be defined by some analytic formulae, but also in this case it will be determined by the operator uniquely up to a factor. Only under some additional hypotheses we shall be able to assign uniquely a determinant system for a class of operators in some Banach spaces (see § 15).

We come back to the definition given at the beginning of this section. We are going to illustrate the definition by a few examples.

If (1) is a determinant system for A and $c \neq 0$, then

$$D_0, c^{-1}D_1, c^{-2}D_2, \ldots$$

is a determinant system for cA.

If (1) is a determinant system for A, and $B \in \mathfrak{O}$ has the inverse B^{-1}, then

$$D_n\begin{pmatrix}\xi_1 B^{-1}, \ldots, \xi_n B^{-1}\\ x_1, \ldots, x_n\end{pmatrix} \qquad (n = 0, 1, 2, \ldots)$$

is a determinant system for AB, and

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ B^{-1}x_1, \ldots, B^{-1}x_n\end{pmatrix} \qquad (n = 0, 1, 2, \ldots)$$

is a determinant system for BA.

The unit I always has a determinant system. In fact, let

$$\theta_0 = 1 \quad \text{and} \quad \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix}\xi_1 x_1, \ldots, \xi_1 x_n\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\\ \xi_n x_1, \ldots, \xi_n x_n\end{vmatrix} \quad \text{for } n = 1, 2, \ldots$$

[5] Sikorski [4].

Then $\theta_0, \theta_1, \theta_2, \ldots$ is a determinant system for I.

It follows immediately from the last two remarks that if A has the inverse A^{-1}, then the formulae

$$(3) \qquad \mathcal{D}_0 = 1, \quad \mathcal{D}_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \theta_n\begin{pmatrix}\xi_1 A^{-1}, \ldots, \xi_n A^{-1}\\ x_1, \ldots, x_n\end{pmatrix} =$$

$$= \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ A^{-1}x_1, \ldots, A^{-1}x_n\end{pmatrix} = \begin{vmatrix}\xi_1 A^{-1}x_1, \ldots, \xi_1 A^{-1}x_n\\ \cdots\cdots\cdots\\ \cdots\cdots\cdots\\ \xi_n A^{-1}x_1, \ldots, \xi_n A^{-1}x_n\end{vmatrix} \quad \text{for } n = 1, 2, \ldots$$

define a determinant system $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$ for A. By Theorem 1, every determinant system (1) of A satisfies in this case the identities

$$(4) \qquad D_n = D_0\mathcal{D}_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

§ 4. Which operators have a determinant system? To answer this question we must introduce the following definition.

An operator A is said to be *Fredholm* if it fulfills the Fredholm Alternative, i. e. there exists an integer $r \geqslant 0$ (called the *order* of A) such that:

(f) the endomorphism $y = Ax$ maps the space X onto its subspace of the codimension r;

(f') the endomorphism $\eta = \xi A$ maps the space Ξ onto its subspace of the codimension r.

In other words, $A \in \mathfrak{A}$ is Fredholm of order r if:

(f_1) the equation $Ax = 0$ has exactly r linearly independent solutions $z_1, \ldots, z_r$;

(f_1') the equation $\xi A = 0$ has exactly r linearly independent solutions $\zeta_1, \ldots, \zeta_r$;

(f_2) the equation $Ax = x_0$ has a solution x if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$;

(f_2') the equation $\xi A = \xi_0$ has a solution ξ if and only if $\xi_0 z_r = 0$ for $i = 1, \ldots, r$.

In particular, A is Fredholm of order 0 if and only if $y = Ax$ is a one-to-one mapping of X onto itself and $\eta = \xi A$ is a one-to-one mapping of Ξ onto itself, i. e. if A has an inverse A^{-1} in the ring $\mathfrak{O}$ (see § 2 (i_0), (i_1)).

It can be easily proved that every operator of the form $I+K$, where K is finitely dimensional, is Fredholm, and that the product A_1A_2 of two Fredholm operators is Fredholm.

Conversely, every Fredholm operator A can be represented in the form $A = (I+K_1)A_1 = A_2(I+K_2)$ where K_1, K_2 are finitely dimensional, and A_1^{-1}, A_2^{-1} exist.

Every Fredholm operator has a quasi-inverse (6).

The problem of existence of determinant systems is solved by the following theorem:

THEOREM 2. *An operator has a determinant system (of order r) if and only if it is Fredholm (of order r)* (7).

§ 5. Determinants and solutions of linear equations. Theorem 2 should be completed by the following more precise theorems which explain the connection between a determinant system for A and solutions of the linear equations

$$\xi A = \xi_0, \qquad Ax = x_0.$$

THEOREM 3 (8). *Suppose that A has a determinant system of an order r. Let $\eta_1, \ldots, \eta_r \in \Xi$ and $y_1, \ldots, y_r \in X$ are such that*

$$D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ x_1, \ldots, x_r\end{pmatrix} \neq 0.$$

Then there exists elements $\zeta_1, \ldots, \zeta_r \in \Xi$ and $z_1, \ldots, z_r \in X$ such that

$$(1)\qquad \zeta_i x = \frac{D_r\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots\ldots, \eta_r\\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad x \in X$$

and

$$(1')\qquad \xi z_i = \frac{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r\\ y_1, \ldots\ldots\ldots\ldots\ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi.$$

The elements $\zeta_1, \ldots, \zeta_r$ are linearly independent and are solutions of the equation

$$(2)\qquad \xi A = 0.$$

The elements $z_1, \ldots, z_r$ are linearly independent and are solutions of the equation

$$(2')\qquad Ax = 0.$$

Conversely, every solution ξ of (2) *is a linear combination of $\zeta_1, \ldots, \zeta_r$, and every solution x of* (2') *is a linear combination of $z_1, \ldots, z_r$.*

(6) For a detailed investigation of Fredholm operators, see e. g. Sikorski [4].

(7) Sikorski [4].

(8) Sikorski [4].

The equation

$$(3)\qquad \xi A = \xi_0$$

has a solution ξ if and only if $\xi_0 z_i = 0$ for $i = 1, \ldots, r$. The equation

$$(3')\qquad Ax = x_0$$

has a solution x if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$.

The bilinear functional B defined by the formula

$$(4)\qquad \xi Bx = \frac{D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ x, y_1, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}}$$

is a quasi-inverse of A. If ξ_0 is orthogonal to all $z_1, \ldots, z_r$, then $\xi_0 B$ is the only solution of (3), *orthogonal to $y_1, \ldots, y_r$. Analogously, if x_0 is orthogonal to $\zeta_1, \ldots, \zeta_r$, then Bx_0 is the only solution of* (3'), *orthogonal to $\eta_1, \ldots, \eta_r$.*

In the case where $r = 0$ Theorem 3 asserts that the homogeneous equations (2), (2') have then only the zero solutions, and

$$(5)\qquad A^{-1} = D_0^{-1} D_1.$$

THEOREM 4 (9). *If A is a Fredholm operator of an order r, $z_1, \ldots, z_r$, $\zeta_1, \ldots, \zeta_r$, satisfy conditions* (f_1), (f_1') *from* § 4, *B is a quasi inverse of A and $c \neq 0$, then the following formulae define a determinant system for A:*

$$(6)\qquad \mathcal{D}_n = 0 \quad \text{for} \quad n = 0, \ldots, r-1,$$

$$(7)\qquad \mathcal{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_r\\ x_1, \ldots, x_r\end{pmatrix} = \begin{vmatrix}\xi_1 z_1, \ldots, \xi_1 z_r\\ \ldots\ldots\ldots\\ \ldots\ldots\ldots\\ \xi_r z_1, \ldots, \xi_r z_r\end{vmatrix} \cdot \begin{vmatrix}\zeta_1 x_1, \ldots, \zeta_1 x_r\\ \ldots\ldots\ldots\\ \ldots\ldots\ldots\\ \zeta_r x_1, \ldots, \zeta_r x_r\end{vmatrix},$$

and for $k = 1, 2, \ldots$

$$(8)\qquad \mathcal{D}_{r+k}\begin{pmatrix}\xi_1, \ldots, \xi_{r+k}\\ x_1, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\cdot\operatorname{sgn}\mathfrak{q}\cdot \begin{vmatrix}\xi_{p_1} Bx_{q_1}, \ldots, \xi_{p_1} Bx_{q_k}\\ \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\\ \xi_{p_k} Bx_{q_1}, \ldots, \xi_{p_k} Bx_{q_k}\end{vmatrix} \cdot \mathcal{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix}$$

(9) Sikorski [4].

where $\sum_{\mathfrak{p},\mathfrak{q}}$ *is extended over all permutations* $\mathfrak{p} = (p_1, \ldots, p_{r+k})$ *and* $\mathfrak{q} = (q_1, \ldots, q_{r+k})$ *of the integers* $1, \ldots, r+k$, *such that*

$$p_1 < p_2 < \ldots < p_k, \quad p_{k+1} < p_{k+2} < \ldots < p_{k+r},$$

$$q_1 < q_2 < \ldots < q_k, \quad q_{k+1} < q_{k+2} < \ldots < q_{k+r}.$$

Observe that in the case $r = 0$ Theorem 4 yields the formula § 3 (4).

Theorem 3 has a great practical value: if we know a determinant system for A, we can completely solve the linear equations (2), (2′), (3), (3′). The found formulae for solutions are an abstract analogue of the well known formulae from Algebra. In the case $r = 0$, formula (4) is an abstract formulation of the Cramer formula. If $r > 0$ and X and Ξ are finitely dimensional, we can additionally suppose that elements $y_1, \ldots, y_r$ and $\eta_1, \ldots, \eta_r$ are unit vectors of the coordinate axes. Then the formulae

$$\xi = \xi_0 B, \quad x = Bx_0,$$

where B is defined by (4), coincide with the classical formulae for a solution of the equations (3) and (3′).

It follows from Theorem 3 that the determinant system for A determines uniquely the operator A.

Theorem 4 has only a theoretical value. If we know a determinant system for A, then we know all solutions of (3) and (3′). Thus the determinant system for A contains the whole knowledge of solutions of (3) and (3′). This fact suggests that perhaps it is possible to express the determinant system for A by means of solutions of (3) and (3′) only. Formulae (6), (7), (8) give an affirmative answer to this question. Theorem 4 has no practical value from the point of view of solving equations (3) and (3′).

§ 6. An analytic case. The main aim of the theory of determinants in Banach spaces is to give some analytic formulae for determinant systems of operators. This will be done in following sections. In this section we would like to quote only one existential theorem which explains the theoretical possibility of the existence of such formulae.

Suppose, for simplicity, that X is a complex Banach space, and Ξ is the space of all complex linear continuous functionals on X. Operators are then continuous complex linear functionals on $\Xi \times X$ which can also be interpreted as continuous endomorphisms in X and Ξ.

Suppose that

$$D_0, D_1, D_2, \ldots$$

is a determinant system for an operator A. It follows from § 3 (d_1), (d_3) that D_n is then an element of the complex Banach space $\mathfrak{O}_n$ of all $2n$-linear complex continuous functionals on $\Xi^n \times X^n$. Of course, $\mathfrak{O}_0$ is the Banach space of all scalars. $\mathfrak{O}$ is a closed subspace of $\mathfrak{O}_1$.

Suppose that $A(\lambda)$ is an analytic mapping from an open region G on the complex plane into the Banach space $\mathfrak{O}$. We have learned in § 4 a criterion for the existence of a determinant system

$$(1) \qquad D_0(\lambda), D_1(\lambda), D_2(\lambda), \ldots$$

for every operator $A(\lambda)$ separately. Now we shall examine the problem under what conditions there exists an analytic function $D_0(\lambda)$ on G and, for $n = 1, 2, \ldots$, an analytic mapping $D_n(\lambda)$ from G into $\mathfrak{O}_n$ such that, for every $\lambda \in G$, the sequence (1) is a determinant system for the operator $A(\lambda)$. Such a sequence (1) will be called an *analytic determinant system for* $A(\lambda)$ *in* G. We shall restrict ourselves to the case where $A(\lambda)^{-1}$ exists for at least one number $\lambda \in G$.

The first necessary condition for the existence of an analytic determinant system (1) is that $A(\lambda)^{-1}$ exists for every $\lambda \in G$, except an isolated set of points $\lambda_1, \lambda_2, \ldots$ In fact, by Theorem 3, $A(\lambda)^{-1}$ exists if $D_0(\lambda) \neq 0$. Since $D(\lambda)$ is analytic, the set of all points $\lambda_1, \lambda_2, \ldots$ such that $D_0(\lambda) = 0$ is isolated.

In the set $G - (\lambda_1, \lambda_2, \ldots)$ there exists an analytic determinant system for $A(\lambda)$. Viz. it is given by the sequence

$$(2) \qquad \mathcal{D}_0(\lambda), \mathcal{D}_1(\lambda), \mathcal{D}_2(\lambda), \ldots,$$

where $\mathcal{D}_0(\lambda) = 1$ and, for $n > 0$, $\mathcal{D}_n(\lambda)$ is defined by the formula § 3 (3), where A is replaced by $A(\lambda)$. Every other analytic determinant system for $A(\lambda)$ differs from (2) by a factor which is an analytic function of λ, and does not vanish in $G - (\lambda_1, \lambda_2, \ldots)$.

Now it is evident that an analytic determinant system exists in the whole set G if and only if this analytic factor can be chosen in such a way that its product with $\mathcal{D}_n(\lambda)$ is a holomorphic mapping from the whole domain G into $\mathfrak{O}_n$ $(n = 0, 1, 2, \ldots)$ and, moreover, the products satisfy the condition § 3 (d_4) at each of the points $\lambda_1, \lambda_2, \ldots$ So we get the following theorem:

THEOREM 5. *In order that there exists an analytic determinant system for* $A(\lambda)$ *in the whole region* G *it is necessary and sufficient that* $A(\lambda)^{-1}$ *exist for all* $\lambda \in G$ *except an isolated set of points* $\lambda_1, \lambda_2, \ldots$ *and, for every point* λ_m, *there exists a positive integer* k_m *such that all the mappings* $\mathcal{D}_n(\lambda)$ $(n = 0, 1, 2, \ldots)$ *have at* λ_m *at most a pole of an order* $\leqslant k_m$. *Suppose that* k_m *is the smallest integer with this property, and* $D_0(\lambda)$ *is an analytic complex* *function in G such that* λ_m *is a* k_m*-tuple root of* $D_0(\lambda) = 0$ $(m = 1, 2, \ldots)$, *and* $D_0(\lambda) \neq 0$ *for all* $\lambda \in G - (\lambda_1, \lambda_2, \ldots)$. *Then*

$$D_n(\lambda) = D_0(\lambda) \cdot \mathcal{D}_n(\lambda) \quad (n = 0, 1, 2, \ldots) \tag{3}$$

is an analytic determinant system for $A(\lambda)$ *in* G. *Every other analytic determinant system for* $A(\lambda)$ *in* G *differs from* (3) *by a factor which is an analytic non-vanishing function in* G.

Only the case of

$$A(\lambda) = I + \lambda T$$

(where $T \in \mathfrak{O}$ is fixed) was nearly examined. To formulate the fundamental result, let us denote by $\mathfrak{C}_0\mathfrak{O}$ the closure of the set $\mathfrak{F}\mathfrak{O}$ of all finitely dimensional operators in the Banach space $\mathfrak{O}$, and by $\mathfrak{C}\mathfrak{O}$ — the closed subspace of all compact operators (an operator A is said to be *compact* provided the endomorphism $y = Ax$ is compact). Since $\mathfrak{C}_0\mathfrak{O}$ and $\mathfrak{C}\mathfrak{O}$ are closed ideals in the Banach algebra $\mathfrak{O}$, the quotient spaces $\mathfrak{O}/\mathfrak{C}_0\mathfrak{O}$ and $\mathfrak{O}/\mathfrak{C}\mathfrak{O}$ are Banach algebras. We recall that $\mathfrak{C}_0\mathfrak{O} \subset \mathfrak{C}\mathfrak{O}$. It is an old, unproved conjecture that $\mathfrak{C}_0\mathfrak{O} = \mathfrak{C}\mathfrak{O}$ for every Banach space X. We will discuss this conjecture from the point of view of the theory of determinants in § 15.

THEOREM 6 (10). *Each of the following conditions is both necessary and sufficient for the existence of an analytic determinant system for* $A(\lambda) = I + \lambda T$ *in the whole plane*:

(r_1) $(I + \lambda T)^{-1}$ *exists for all* λ *except an isolated set*; *for every* λ, *the set* $\{(I + \lambda T)^n x : x \in X\}$ *is a closed subspace which is independent of* n *provided* n *is sufficiently large*; *for every* λ, *the set* $\{x : (I + \lambda T)^n x = 0\}$ *is a finitely dimensional subspace which is independent of* n *provided* n *is sufficiently large.*

(r_2) *The element determined by* T *in the quotient algebra* $\mathfrak{O}/\mathfrak{C}_0\mathfrak{O}$ *is quasi-nilpotent* (11).

(r_3) *The element determined by* T *in the quotient algebra* $\mathfrak{O}/\mathfrak{C}\mathfrak{O}$ *is quasi-nilpotent.*

§ 7. Problems to be solved. From now on, X and Ξ will be two conjugate Banach spaces (complex or real). The cancellation laws (c) and (c′) from § 2 will always be replaced by the following stronger condition:

$$|\xi| = \sup_{|x| \leqslant 1} |\xi x|, \qquad |x| = \sup_{|\xi| \leqslant 1} |\xi x|. \tag{n}$$

(10) Ruston [3, 4].

(11) For the definition of quasi-nilpotent elements in Banach algebras, see e. g. Hille and Phillips [1], p. 121.

Consequently, operators are now continuous bilinear functionals on $\Xi \times X$ which can also be interpreted as continuous endomorphisms in the spaces Ξ and X respectively. Condition (n) implies that the norms of the three possible interpretations of an operator A are equal. Their common value will be called the *norm* of the operator A and denoted by $|A|$. The normed space $\mathfrak{O}$ of all operators is a Banach algebra with the unit I.

The inclusions § 2 (1) can be now replaced by the inclusions

$$\Xi \subset X^*, \qquad X \subset \Xi^*,$$

where X^* and Ξ^* denote the Banach spaces of all continuous linear functionals on X and Ξ respectively. The Banach space $\mathfrak{O}$ of all operators is a closed subspace of the Banach space $\mathfrak{O}_1$ of all continuous bilinear functionals on $\Xi \times X$:

$$\mathfrak{O} \subset \mathfrak{O}_1.$$

More exactly, if one of the spaces Ξ, X is reflexive, then so is the remaining one and

$$\Xi = X^*, \qquad X = \Xi^* \quad \text{and} \quad \mathfrak{O} = \mathfrak{O}_1.$$

In the opposite case, at least one of the first two equalities does not hold, and the third one also does not hold, i. e. $\mathfrak{O}$ is a proper subset of $\mathfrak{O}_1$.

We have to give an analytic formula for determinant systems of a rather large class of operators. Our experience in the theory of linear equations in Banach spaces suggests that we should expect that such a definition is possible only in the case of operators of the form

$$A = I + T$$

where the operator T is rather special, rather near to the class of finitely dimensional operators, perhaps it should be a compact operator or an operator of a similar type.

The best we can expect is that we shall be able to distinguish a linear class $\mathfrak{T}$ of operators and to give an analytic formula for determinant systems of operators $A = I + T$, $T \in \mathfrak{T}$. The term "analytic" is not precisely defined. It should mean that all D_n should be analytic functions of the variable operator $T \in \mathfrak{T}$ in a suitable topology in $\mathfrak{T}$, and the functions should be given in an effective way.

Thus we have to solve the following problems:

A) to distuinguish a rather large class $\mathfrak{T}$ of operators,

B) to assign uniquely and effectively to every $T \in \mathfrak{T}$ a determinant system $D_0, D_1, D_2, \ldots$ for $I + T$,

C) to introduce a topology in $\mathfrak{T}$ so that D_n $(n = 0, 1, 2, \ldots)$ are analytic functions of T in this topology.

Moreover, the definition of $D_0, D_1, D_2, \ldots$ should be so that

D) in the case of finitely dimensional spaces, $D_0, D_1, D_2, \ldots$ coincides with the algebraic determinant system for $A = I+T$ defined by § 1 (3) and § 3 (2).

Unfortunately, the situation is more complicated than we have expected. To explain the reason for the difficulty, let us come back to the case where X and Ξ are m-dimensional spaces (see § 1). If a fixed system of coordinates is chosen, then bilinear functionals $K \epsilon \mathfrak{O}_1$ are square matrices $(\varkappa_{i,j})_{i,j=1,\ldots,m}$ and, conversely, every square matrix is an operator. However, every square matrix $(\tau_{i,j})_{i,j=1,\ldots,m}$ can be also interpreted as a functional $\mathcal{T}$ on the linear space $\mathfrak{O}_1$, viz.

$$\mathcal{T}(K) = \sum_{i,j=1}^{m} \tau_{j,i}\varkappa_{i,j} \quad \text{for} \quad K = (\varkappa_{i,j}) \epsilon \mathfrak{O}_1. \tag{1}$$

The last expression is independent on the choice of the system of coordinates in question. Denoting by T the operator determined by the matrix $(\tau_{i,j})$, we can write (1) in the form

$$\mathcal{T}(K) = \operatorname{tr} TK = \operatorname{tr} KT \tag{2}$$

for $K \epsilon \mathfrak{O}_1$, where tr denotes the trace of a finitely dimensional operator (see § 2 (5) and (6)). Formula (2) defines $\mathcal{T}$ in terms of T. The next formula defines T by means of $\mathcal{T}$:

$$\xi T x = \mathcal{T}(x\cdot\xi), \tag{3}$$

where $x\cdot\xi$ denotes the one dimensional operator determined by ξ and x (see § 2, p. 149). We see that there exists a natural one-to-one correspondence

$$\mathcal{T} \leftrightarrow T \tag{4}$$

between operators T and functionals $\mathcal{T}$ on $\mathfrak{O}_1$. The functional $\mathcal{T}$ is said to be the *nucleus* of the operator T. Since T and $\mathcal{T}$ are determined by the same square matrix, they have the same determinant and subdeterminants. However, in Algebra, we are used to speaking only of determinants of operators, not on determinants of their nuclei, because the notion of nucleus does not play any important part in the theory of linear equations in finitely dimensional linear spaces.

Now consider the case where X and Ξ are arbitrary conjugate Banach spaces satisfying condition (n). If $\mathcal{T}$ is any continuous linear functional on the Banach space $\mathfrak{O}_1$ of all bilinear functionals on $\Xi\times X$, then the formula (3) defines a continuous bilinear functional T on $\Xi\times X$. If T is an operator (i. e. if the bilinear functional T can be interpreted as

an endomorphism in Ξ and in X), then $\mathcal{T}$ is said to be a *quasinucleus*, viz. a *quasinucleus of the operator* T. If an operator T has a quasinucleus, i. e. it satisfies (3) for a functional $\mathcal{T}$ on $\mathfrak{O}_1$, then T is said to be *quasinuclear*. If Ξ and X are infinitely dimensional, there are operators which are not quasinuclear, viz. the unit operator I is not quasinuclear. If $\mathcal{T}$ is a quasinucleus of a quasinuclear operator T, i. e. if (3) holds, then (2) holds but only for finitely dimensional operators K. More precisely, $\mathcal{T}$ is a quasinucleus of an operator T if and only if (2) holds for every finitely dimensional operator K.

It follows from this discussion that, if X and Ξ are infinitely dimensional, then instead of the one-to-one correspondence (4) we have only a canonical mapping

$$\mathcal{T} \to T \tag{5}$$

of the space $\mathfrak{O}\mathfrak{N}$ of all quasinuclei into the space $\mathfrak{O}$ of all operators, and this transformation is neither one-to-one, nor a transformation onto $\mathfrak{O}$.

Moreover, unexpectedly, in the case of arbitrary Banach spaces, the notion of the determinant and subdeterminants seems to be connected more closely with quasinuclei than with operators.

To explain the last remark, let us observe that $\mathfrak{O}\mathfrak{N}$ is a Banach space with respect to the ordinary algebraic operations and the ordinary norm of a functional $\mathcal{T}$. We shall see in the next sections that we can introduce a multiplication in $\mathfrak{O}\mathfrak{N}$, so that $\mathfrak{O}\mathfrak{N}$ becomes a Banach algebra. The canonical map (5) is then a ring homomorphism of $\mathfrak{O}\mathfrak{N}$ into $\mathfrak{O}$. If Ξ, X are finitely dimensional, then the algebra $\mathfrak{O}\mathfrak{N}$ has a unit element $\mathcal{I}$ (viz. the functional determined by the unit matrix), and the canonical mapping transforms $\mathcal{I}$ onto I. If Ξ, X are infinitely dimensional, then $\mathfrak{O}\mathfrak{N}$ does not have any unit element, but we can add an abstract unit $\mathcal{I}$ to the algebra $\mathfrak{O}\mathfrak{N}$ and extend the ring homomorphism (5) by the convention

$$\mathcal{I} \to I.$$

Hence, in any case, the (extended) canonical mapping transforms $\mathcal{I}+\mathcal{T}$ onto $I+T$. Writing

$$\mathcal{A} = \mathcal{I}+\mathcal{T}, \qquad A = I+T,$$

we note the fundamental property of the canonical mapping in the form:

$$\mathcal{A} \to A.$$

In the present state of the theory of determinants in Banach spaces we can only uniquely assign, to every quasinucleus $\mathcal{T}$, a sequence

$$D_0(\mathcal{T}), D_1(\mathcal{T}), D_2(\mathcal{T}), \ldots \tag{6}$$

such that

1) for every fixed $\mathcal{T}$, (6) is a determinant system for $A = I+T$;

2) for every fixed n, $D_n(\mathcal{T})$ is an analytic mapping from $\mathfrak{Q}\mathfrak{N}$ into the space $\mathfrak{D}_n$ of all $2n$-dimensional functionals on $\Xi^n \times X^n$.

Simple formulae for (6) will be given in § 8.

It is natural to call (6) the *determinant system of* $\mathcal{A} = \mathcal{I}+\mathcal{T}$. Observe the terminological difference: (6) is a determinant system *of* $\mathcal{A} = \mathcal{I}+\mathcal{T}$, but *for* $A = I+T$. The determinant system (6) is uniquely determined by $\mathcal{T}$, but is not uniquely determined by T. More precisely, every quasinucleus $\mathcal{T}$ uniquely determines a quasinuclear operator T and the determinant system (6) for $I+T$. A given quasinuclear operator T has many different quasinuclei $\mathcal{T}$ (except the finitely dimensional case). The determinant systems (6) of various nuclei $\mathcal{T}$ of a fixed operator T are different in general, but they differ by a scalar factor only, on account of Theorem 1.

The fact that determinant systems are uniquely associated with nuclei, but not with operators, causes the analogy between the theory of determinants in Banach spaces and the theory of determinants in Algebra to be incomplete. This defect of the theory of determinants in Banach spaces is closely connected with the algebraic phenomenon observed in Theorem 1.

Of course, we can try (and we should try) to correct the situation by putting some restrictions on $\mathcal{T}$ and T. Suppose we have distinguished in a natural way, a smaller class $\mathfrak{N}_0 \subset \mathfrak{Q}\mathfrak{N}$ so that the canonical map (5) is one-to-one on $\mathfrak{N}_0$ (we say then, for brevity, that $\mathfrak{N}_0$ has the *uniqueness property*). If an operator T has a pseudonucleus $\mathcal{T}$ in $\mathfrak{N}_0$, then $\mathcal{T}$ is uniquely determined by T, and consequently (6) is uniquely determined by T. Then it would be natural to call (6) the determinant system *of* $I+T$. The class $\mathfrak{N}$ of nuclei defined in § 9 seems to be the smallest sensible class with the uniqueness property. In many concrete examples it has really the uniqueness property. The problem if it always has the uniqueness property is very difficult. It will be discussed in more detail in § 15.

§ 8. The definition of the determinant system of a quasinucleus. The following notation will be useful in the sequel: if $\mathcal{T}$ is any functional on $\mathfrak{D}$, and $A \in \mathfrak{D}$, then we shall write sometimes

$$\mathcal{T}_{\xi x}(\xi A x) \quad \text{instead of} \quad \mathcal{T}(A).$$

This notation [12] is especially convenient in the case where A is defined by a formula containing also other variables $\xi_1, \ldots, x_1, \ldots$

[12] Due to Leżański [1].

For instance, let

$$(1)\qquad \theta_{n,0}\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1 x_1, \ldots, \xi_1 x_n \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \xi_n x_1, \ldots, \xi_n x_n \end{vmatrix}$$

for $n = 1, 2, \ldots$ (see § 3 (2)). The meaning of the symbol

$$(2)\qquad \theta_{n-1,1}\begin{pmatrix}\xi_1, \ldots, \xi_{n-1} \\ x_1, \ldots, x_{n-1}\end{pmatrix} = \mathcal{T}_{\xi_n x_n}\theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix},$$

where $\mathcal{T}$ is a fixed quasinucleus, is clear: If all variables $\xi_1, \ldots, \xi_{n-1}, x_1, \ldots, x_{n-1}$ are fixed, then the equality

$$\xi_n A x_n = \theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix}$$

defines an operator A. The number (2) is the value $\mathcal{T}(A)$ of the functional $\mathcal{T}$ at the point A. Of course, $\mathcal{T}(A)$ depends on $\xi_1, \ldots, \xi_{n-1}, x_1, \ldots, x_{n-1}$ (but not on ξ_n, x_n which are bound variables). It can be proved that, if $\xi_1, \ldots, \xi_{n-2}, x_1, \ldots, x_{n-2}$ are fixed, then (2), considered as a function of ξ_{n-1}, x_{n-1} only, is an operator. Thus the expression

$$\mathcal{T}_{\xi_{n-1}x_{n-1}}\theta_{n-1,1}\begin{pmatrix}\xi_1, \ldots, \xi_{n-1} \\ x_1, \ldots, x_{n-1}\end{pmatrix} = \mathcal{T}_{\xi_{n-1}x_{n-1}}\mathcal{T}_{\xi_n x_n}\theta\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix}$$

is well defined. Continuing this procedure, we define the $2n$-linear functionals $\theta_{n,m}$ on $\Xi^n \times X^n$ $(n, m = 0, 1, 2, \ldots)$:

$$(3)\qquad \theta_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \frac{1}{m!}\mathcal{T}_{\xi_{n+1}x_{n+1}}\ldots\mathcal{T}_{\xi_{n+m}x_{n+m}}\theta_{n+m}\begin{pmatrix}\xi_1, \ldots, \xi_{n+m} \\ x_1, \ldots, x_{n+m}\end{pmatrix}.$$

In particular, $\theta_{0,m}$ are scalars:

$$(4)\qquad \theta_{0,0} = \theta_0 = 1, \qquad \theta_{0,m} = \frac{1}{m!}\mathcal{T}_{\xi_1 x_1}\ldots\mathcal{T}_{\xi_m x_m}\theta_m\begin{pmatrix}\xi_1, \ldots, \xi_m \\ x_1, \ldots, x_m\end{pmatrix} \quad \text{for} \quad m = 1, 2, \ldots$$

THEOREM 7. *For every quasinucleus $\mathcal{T}$, the series*

$$(5)\qquad D_n(\mathcal{T}) = \sum_{m=1}^{\infty}\theta_{n,m} \quad (n = 0, 1, 2, \ldots)$$

converges in the norm in the Banach space $\mathfrak{D}_n$. *It is an analytic mapping from* $\mathfrak{QN}$ *into* $\mathfrak{D}_n$. *The sequence*

(6) $$D_0(\mathcal{T}),\ D_1(\mathcal{T}),\ D_2(\mathcal{T}),\ \ldots$$

is a determinant system for the operator $I+T$, *where* T *is the quasinuclear operator whose quasinucleus is* $\mathcal{T}$.

The sequence (6) is called the *determinant system* (13) *of* $\mathcal{A}=\mathcal{I}+\mathcal{T}$. It coincides with the sequence (6) from § 7.

The convergence in norm follows from the following inequality (14):

(7) $$|\theta_{n,m}| \leqslant \frac{(n+m)^{(n+m)/2}}{m!}\,|\mathcal{T}|^m.$$

Theorem 7 has a great practical value: Given any quasinucleus $\mathcal{T}$, we can calculate the determinant system (6) for $I+T$ by means of formulae (3), (4) and (5) and solve the linear equations

$$\xi(I+T)=\xi_0, \qquad (I+T)x=x_0$$

applying Theorem 3. Since we have the estimation (7), we can use formulae (3), (4), (5) and the formulae quoted in Theorem 3 to numerical calculation of solutions. As we have stated on p. 142, the additional advantage of the determinant theory of linear equations is that, besides existential theorems on solutions, it yields also formulae for solutions.

§ 9. The algebra of quasinuclei and nuclei. The results from § 8 point out the importance of the space $\mathfrak{QN}$ of all quasinuclei and the canonical mapping

(1) $$\mathcal{T}\to T$$

from $\mathfrak{QN}$ into $\mathfrak{D}$ which, to every $\mathcal{T}\in\mathfrak{QN}$, assigns an operator $\mathcal{T}$ defined by the equality

(2) $$\xi Tx = \mathcal{T}(x\cdot\xi)$$

(see § 7 (3), (5)). The canonical mapping (1) of the Banach space $\mathfrak{QN}$ (with the ordinary norm of functionals $\mathcal{T}\in\mathfrak{QN}$) into the Banach space $\mathfrak{D}$

(13) This definition and Theorem 7 are due to Leżański [1] (see also Sikorski [2]). Ruston [1, 2] and Grothendieck [1, 3] have given other equivalent formulae (expressed in another language) but only for nuclei and nuclear operators defined in §§ 9, 10. That is the characteristic difference between the Leżański theory and the Ruston-Grothendieck theory that the last theory treats of determinant systems of nuclei only.

(14) Leżański [1].

is linear and continuous since

(3) $$|T|\leqslant|\mathcal{T}|.$$

The simplest example of a quasinucleus is as follows:

$$\mathcal{T}_0(A)=\xi_0 A x_0 \quad \text{for all} \quad A\in\mathfrak{D}_1,$$

where ξ_0, x_0 are fixed. This functional $\mathcal{T}_0$ will be denoted by $\xi_0\otimes x_0$ and will be called the *one-dimensional nucleus* determined by ξ_0, x_0. Observe that

$$|\xi_0\otimes x_0| = |\xi_0|\cdot|x_0|.$$

The canonical mapping (1) maps $\xi_0\otimes x_0$ onto the operator $x_0\cdot\xi_0$.

By a *finitely dimensional nucleus* we shall understand any finite sum

(4) $$\mathcal{T}=\sum_{i=1}^{m}\xi_i\otimes x_i$$

of one-dimensional nucleus, i. e. every functional $\mathcal{T}$ on $\mathfrak{D}_1$ of the form

$$\mathcal{T}(A)=\sum_{i=1}^{m}\xi_i A x_i \quad \text{for all} \quad A\in\mathfrak{D}_1,$$

where $\xi_1,\ldots,\xi_m, x_1,\ldots,x_m$ are fixed. It can be proved (15) that the norm of a finitely dimensional nucleus $\mathcal{T}$ is given by the formula

(5) $$|\mathcal{T}|=\inf\sum_{i=1}^{m}|\xi_i|\cdot|x_i|,$$

where "inf" is extended over all possible representations of $\mathcal{T}$ in the form (4). The canonical mapping (1) maps a finitely dimensional nucleus $\sum_{i=1}^{m}\xi_i\otimes x_i$ onto the finitely dimensional operator $\sum_{i=1}^{m}x_i\cdot\xi_i$.

The set $\mathfrak{FN}$ of all finitely dimensional nuclei is a linear subspace of $\mathfrak{QN}$. The closure of $\mathfrak{FN}$ in $\mathfrak{QN}$ is a linear subspace $\mathfrak{N}$ of $\mathfrak{QN}$. The elements of $\mathfrak{N}$ will be called *nuclei*. Every operator T which is the image of a nucleus $\mathcal{T}$ by the canonical mapping (1) is said to be *nuclear*, and $\mathcal{T}$ is said to be a nucleus of T. The linear set of all nuclear operators will be denoted by $\mathfrak{ND}$. The set of all quasinuclear operators will be denoted by $\mathfrak{QD}$.

By definition

(6) $$\mathfrak{FN}\subset\mathfrak{N}\subset\mathfrak{QN}, \qquad \mathfrak{FD}\subset\mathfrak{ND}\subset\mathfrak{QD}\subset\mathfrak{D}.$$

In general, no sign $\subset$ can be here replaced by $=$.

(15) Schatten [2].

It follows from § 7 that quasinuclei are an abstract substitute of the notion of a square matrix considered as a functional on operators. In the theory of matrices, we associate with every matrix a scalar called the trace of the matrix. If the matrix is considered as a functional on operators (see § 7 (1) and (2)), then its trace is the value of the functional on the identity operator I. This fact suggests the following definition:

For every quasinucleus $\mathcal{T}$, the number $\mathcal{T}(I)$ is called the *trace* of $\mathcal{T}$ and denoted by $\mathrm{Tr}\,\mathcal{T}$. Of course, Tr is a linear continuous functional on $\mathfrak{Q}\mathfrak{N}$ and

$$|\mathrm{Tr}(\mathcal{T})| \leqslant |\mathcal{T}|.$$

For instance

$$\mathrm{Tr}\Big(\sum_{i=1}^{m} \xi_i \otimes x_i\Big) = \sum_{i=1}^{m} \xi_i x_i = \mathrm{tr}\Big(\sum_{i=1}^{m} x_i \cdot \xi_i\Big),$$

i. e. the trace of any finitely dimensional nucleus coincides with the trace of the finitely dimensional operator determined by this nucleus.

Observe that the notion of trace, similarly as the notion of determinant, is associated with quasinuclei, not with operators (except the finitely dimensional case). This is a result of the same phenomenon of the dispersion of the notion of square matrix into two notions: quasinucleus and operator.

Square matrices form a linear algebra. So do their substitutes, operators (see § 2), and so also do their other substitutes, quasinuclei. In the space $\mathfrak{Q}\mathfrak{N}$ we can introduce several operations of a multiplicative character.

If $\mathcal{T} \in \mathfrak{Q}\mathfrak{N}$ and $C \in \mathfrak{O}$, then $C\mathcal{T}$ and $\mathcal{T}C$ denote quasinuclei (i. e. functionals on $\mathfrak{O}$) defined by the equalities

$$C\mathcal{T}(A) = \mathcal{T}(AC) \quad \text{and} \quad \mathcal{T}C(A) = \mathcal{T}(CA) \quad \text{for all} \quad A \in \mathfrak{O}_1.$$

The products $C\mathcal{T}$ and $\mathcal{T}C$ satisfy the distributive and associative laws, and

$$|C\mathcal{T}| \leqslant |C|\cdot|\mathcal{T}|, \qquad |\mathcal{T}C| \leqslant |\mathcal{T}|\cdot|C|.$$

Observe the following properties of the canonical mapping (1):

$$C\mathcal{T} \to CT, \qquad \mathcal{T}C \to TC.$$

Hence it follows that $\mathfrak{Q}\mathfrak{O}$ is an ideal in the Banach algebra $\mathfrak{O}$. If $\mathcal{T}$ is a nucleus, so are $C\mathcal{T}$ and $\mathcal{T}C$. Thus $\mathfrak{N}\mathfrak{O}$ is also an ideal in $\mathfrak{O}$.

$\mathfrak{Q}\mathfrak{N}$ is a Banach algebra with respect to each of the following multiplications [16]:

$$(7) \qquad \mathcal{T}_1 \circledcirc \mathcal{T}_2 = T_1\mathcal{T}_2, \qquad \mathcal{T}_1 \circledcirc \mathcal{T}_2 = \mathcal{T}_1 T_2,$$

$$(8) \qquad \mathcal{T}_1 \circ \mathcal{T}_2 = \tfrac{1}{2}(\mathcal{T}_1 \circledcirc \mathcal{T}_2 + \mathcal{T}_1 \circledcirc \mathcal{T}_2)$$

where, of course, T_1, T_2 are operators whose quasinuclei are $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. The canonical mapping (1) is a ring homomorphism with respect to each of the multiplications (7), (8). The multiplication $\mathcal{T}_1 \circ \mathcal{T}_2$ has the following important property:

$$(9) \qquad \mathrm{Tr}(\mathcal{T}_1 \circ \mathcal{T}_2) = \mathrm{Tr}(\mathcal{T}_2 \circ \mathcal{T}_1)$$

which is an extension of the known property of the trace of matrices. The products (7) do not have, in general, this property. Consequently $\mathcal{T}_1 \circ \mathcal{T}_2$ seems to be the most adequate multiplication in $\mathfrak{Q}\mathfrak{N}$. In the sequel we shall always consider $\mathfrak{Q}\mathfrak{N}$ as a Banach algebra with the multiplication (8).

Observe that $\mathfrak{N}$ is a subalgebra of $\mathfrak{Q}\mathfrak{N}$. Moreover, all the products (7), (8) coincide for $\mathcal{T}_1, \mathcal{T}_2 \in \mathfrak{N}$.

§ 10. What operators are quasinuclear? The answer to this question follows immediately from the investigations in § 7. Viz. we have stated that a continuous linear functional $\mathcal{T}$ defined on $\mathfrak{O}_1$ is a quasinucleus of an operator $\mathcal{T}$ if and only if $\mathcal{T}$ is an extension of the following functional defined on the linear set of all finitely dimensional operators:

$$(1) \qquad \mathrm{tr}\, TK \qquad (K \in \mathfrak{F}\mathfrak{O}).$$

Thus, by the Hahn-Banach extension theorem, we get the following statement:

THEOREM 8. *An operator T is quasinuclear if and only if the functional* (1) *is continuous on* $\mathfrak{F}\mathfrak{O}$ [17].

The canonical mapping

$$\mathcal{T} \to T$$

is one-to-one on the set $\mathfrak{F}\mathfrak{N}$ of all finitely dimensional nucleus, and transforms $\mathfrak{F}\mathfrak{N}$ onto the set $\mathfrak{F}\mathfrak{O}$ of all finitely dimensional operators. The

(16) The product $\circledcirc$ was introduced by Leżański [2] (see also Sikorski [2]). The products $\circledcirc$, $\circ$ were introduced by Sikorski [6]. For the case of nucleus, see Grothendieck [2] and Ruston [1, 2].

Observe that the associativity of $\circ$ is not trivial. It follows from a result of Grothendieck [4].

(17) See Sikorski [3].

canonical one-to-one correspondence between $\mathfrak{F}\mathfrak{N}$ and $\mathfrak{F}\mathfrak{O}$ is given by the formula (see § 9, p. 165):

$$\sum_{i=1}^{m} \xi_i \otimes x_i \longleftrightarrow \sum_{i=1}^{m} x_i \cdot \xi_i .$$

Thus we may identify the finitely dimensional operators K with their finitely dimensional nuclei. Consequently the norm in $\mathfrak{O}\mathfrak{N}$ induces another norm in $\mathfrak{F}\mathfrak{N}$, denoted by $\|K\|$ [18]. By § 9 (5),

(2) $$\|K\| = \inf \sum_{i=1}^{m} |\xi_i| \cdot |x_i| ,$$

where "inf" is extended over all representations of K in the form

$$K = \sum_{i=1}^{m} x_i \cdot \xi_i .$$

THEOREM 9. *An operator T is nuclear if and only if there exists a sequence $K_n \in \mathfrak{F}\mathfrak{O}$ such that*

$$|K_n - T| \to 0 \text{ for } n \to \infty, \quad \text{and} \quad \|K_n - K_{n'}\| \to 0 \text{ for } n, n' \to \infty$$

or, equivalently, if it can be represented in the form [19]

$$T = \sum_{n=1}^{\infty} t_n x_n \cdot \xi_n ,$$

where $|x_n| = 1 = |\xi_n|$ *and* $|t_1| + |t_2| + \ldots < \infty$.

Hence it follows that

(3) $$\mathfrak{N}\mathfrak{O} \subset \mathfrak{C}_0\mathfrak{O} \subset \mathfrak{C}\mathfrak{O}$$

(see notation on p. 158). Thus if T is nuclear, then the endomorphisms $y = Tx$ and $\eta = \xi T$ are compact.

There are quasinuclear operators which are not compact and, therefore, not nuclear. Examples of such operators can be given, e. g., in the case [20] where

$$X = \boldsymbol{L}, \quad \Xi = \boldsymbol{M}.$$

---

[18] After the identification just mentioned, $\mathfrak{F}\mathfrak{O}$ is a dense subset of $\mathfrak{N}$. Thus $\mathfrak{N}$ can be also defined as an abstract completion of $\mathfrak{F}\mathfrak{O}$ with respect to the norm $\|\ \|$. This definition is assumed by Grothendieck [1-4] and Ruston [1-2].

[19] For the analogous representation of nuclei, see § 15 (1).

[20] See Sikorski [3], where a simple example due to C. Ryll-Nardzewski is quoted. See also footnote [40], p. 182.

In that case, an operator T is quasinuclear if it is an integral operator, i. e.

$$Tx(s) = \int_0^1 \tau(s,t)x(t)\,dt \quad \text{for} \quad x \in \boldsymbol{L},$$

such that

$$f(s) = \tau(s,\cdot) \in \boldsymbol{M} \quad \text{for almost all } s,$$

and

$$\int_0^1 |f(s)|_{\boldsymbol{M}}\,ds = \int_0^1 \operatorname*{sup\,ess}_t |\tau(s,t)|\,ds < \infty .$$

T is nuclear if the mapping f (from $\langle 0,1\rangle$ into $\boldsymbol{M}$) is Bochner integrable.

On the other hand, in many concrete spaces the both notions: nuclear operator and quasinuclear operator coincide. So it is e. g. in the case where

$$X = \boldsymbol{l}, \quad \Xi = \boldsymbol{m}.$$

In that case, an operator is nuclear (i. e. quasinuclear) if and only if it is determined by an infinite square matrix $(\tau_{i,j})$:

$$Tx = \Big(\sum_{j=1}^{\infty} \tau_{i,j} v_j\Big), \quad x = (v_j) \in \boldsymbol{l},$$

such that

$$\sum_{i=1}^{\infty} \sup_j |\tau_{i,j}| < \infty .$$

The notions of nuclear operator and quasinuclear operator coincide also in the case where

$$X = \boldsymbol{l}^p, \quad \Xi = \boldsymbol{l}^q$$

and

$$X = \boldsymbol{L}^p, \quad \Xi = \boldsymbol{L}^q,$$

$1/p + 1/q = 1$, $1 < p < \infty$. This follows from the more general theorem:

THEOREM 10. *If X is reflexive, then every quasinuclear operator is nuclear* [21].

In the general case, the notion of a quasinuclear operator is closely related to the notion of a compact operator. This follows from the following theorems:

---

[21] Grothendieck [2], Chapter I, p. 134.

THEOREM 11. *If T is a quasinuclear operator, then the endomorphisms $y = Tx$ and $\eta = \xi T$ transform bounded sets into weakly compact sets, and weakly compact sets into compact sets* ([22]).

THEOREM 12. *If T is quasinuclear, then T^2 (and consequently all the powers T^n, $n > 1$) are nuclear, and therefore compact.*

More generally, if T_1, T_2 are quasinuclear, then T_1T_2 is nuclear ([23]).

Many deep theorems on nuclear and quasinuclear operators were proved recently by Grothendieck [2] (in the case where $\Xi = X^*$). We quote only one of them:

THEOREM 13. *In order that an operator T be quasinuclear it is necessary and sufficient that the endomorphism T be the superposition of three linear continuous endomorphisms:*

$$X \overset{U}{\to} M(\Gamma) \overset{I_0}{\to} L(\Gamma) \overset{V}{\to} X, \tag{4}$$

where Γ is a set with a finite measure μ, $M(\Gamma)$ is the Banach space of all bounded μ-measurable functions on Γ, $L(\Gamma)$ is the Banach space of all functions μ-integrable on Γ, and I_0 is the identity mapping from $M(\Gamma)$ into $L(\Gamma)$ ([24]).

T is nuclear if and only if there exists such a decomposition (2) *where μ is an atomic measure.*

§ 11. Other formulae for determinant systems. The formulae we are going to present are another formulation of formula (5) from § 8. This new form is more convenient to explain in which degree the determinant system of $I+T$ is determined by the quasinuclear operator T.

For any operator T, let T_n^m denote the following $2n$-linear functional on $\Xi^n \times X^n$:

$$T_n^m\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \sum \begin{vmatrix} \xi_1 T^{i_1}x_1, \ldots, \xi_1 T^{i_1}x_n \\ \ldots\ldots\ldots\ldots \\ \xi_n T^{i_n}x_1, \ldots, \xi_n T^{i_n}x_n \end{vmatrix}, \tag{1}$$

where the summation is extended over all finite sequences of non-negative integers $i_1, \ldots, i_n$ such that $i_1 + \ldots + i_n = m$ $(m = 0, 1, 2, \ldots, n = 1, 2, \ldots)$. Observe that also

$$T_n^m\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \sum \begin{vmatrix} \xi_1 T^{i_1}x_1, \ldots, \xi_1 T^{i_n}x_n \\ \ldots\ldots\ldots\ldots \\ \xi_n T^{i_1}x_1, \ldots, \xi_n T^{i_n}x_n \end{vmatrix}, \tag{2}$$

([22]) Grothendieck [2], Chapter I, p. 131.

([23]) Grothendieck [2], Chapter I, p. 132-133. A simple proof of the first part of Theorem 2 was given also by Sikorski [5].

([24]) Grothendieck [2], Chapter I, p. 125.

where the summation is extended over the same set of sequences $i_1, \ldots, i_n$.

Note that, for $n = 1$, T_1^m is the m-th power of the operator T.

The $2n$-linear functionals T_n^m $(m = 0, 1, 2, \ldots)$ are closely related with the $2n$-linear functional

$$\mathcal{D}_n(T) = \mathcal{D}_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1(I+T)^{-1}x_1, \ldots, \xi_1(I+T)^{-1}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_n(I+T)^{-1}x_1, \ldots, \xi_n(I+T)^{-1}x_n \end{vmatrix} \tag{3}$$

(see § 3 (3), where $A = I+T$). Viz. for sufficiently small T (e. g. for $|T| < 1$) the expression $\mathcal{D}_n(T)$ considered as a function of T can be developed in the power series:

$$\mathcal{D}_n = \sum_{m=0}^{\infty} (-1)^m T_n^m. \tag{4}$$

Suppose now that T is quasinuclear and $\mathcal{T}$ is a quasinucleus of T. Let

$$\sigma_n = \mathcal{T}(T^{n-1}) = \operatorname{Tr} \mathcal{T}^n \qquad (n = 1, 2, \ldots) \tag{5}$$

(see § 9). Under the above notation we have the following theorem:

THEOREM 14. *The determinant system of $\mathcal{T}$ satisfies the identities* ([25]):

$$D_0(\mathcal{T}) = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{vmatrix} \sigma_1 & m-1 & 0 & 0 & \ldots & 0 \\ \sigma_2 & \sigma_1 & m-2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_{m-1} & \sigma_{m-2} & \sigma_{m-3} & \ldots & \sigma_1 & 1 \\ \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix}, \tag{6}$$

$$D_n(\mathcal{T}) = \sum_{m=0}^{\infty} \frac{1}{m!} \begin{vmatrix} T_n^0 & m & 0 & 0 & 0 & \ldots & 0 \\ T_n^1 & \sigma_1 & m-1 & 0 & 0 & \ldots & 0 \\ T_n^2 & \sigma_2 & \sigma_1 & m-2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ T_n^{m-1} & \sigma_{m-1} & \sigma_{m-2} & \sigma_{m-3} & \ldots & \sigma_1 & 1 \\ T_n^m & \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix}. \tag{7}$$

The identity (7) is a simple consequence of (4) and (5) and the identity ([26])

$$D_n(\mathcal{T}) = D_0(\mathcal{T}) \cdot \mathcal{D}_n(T) \tag{8}$$

(see § 3 (4)).

([25]) Plemelj's [2] formulae (6), (7) (see also Poincaré [1]) were assumed by Ruston [1, 2] as the definition of $D_0(\mathcal{T})$ and $D_1(\mathcal{T})$ for any nucleus $\mathcal{T}$.

([26]) In the case of arbitrary Banach spaces, this formula was first used by Ruston [3] but it was earlier known in the theory of integral equations. See Plemelj [1, 2], Platrier [1], Hoborski [1], Hurwitz [1].

The following formula is also valid for sufficiently small $\mathcal{T}$, e. g. for $|\mathcal{T}|<1$:

THEOREM 15. *If*

$$\log(\mathcal{I}+\mathcal{T})=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\mathcal{T}^m$$

exists, then (27)

(9) $$D_0(\mathcal{T})=\exp\operatorname{Tr}\log(\mathcal{I}+\mathcal{T}).$$

$\mathcal{I}$ denotes here the abstract unit added to the Banach algebra $\mathfrak{Q}\mathfrak{N}$. Denoting, more suggestively, $D_0(\mathcal{T})$ by $D(\mathcal{I}+\mathcal{T})$, we can write formula (7) in the form

(9′) $$\log D(\mathcal{I}+\mathcal{T})=\operatorname{Tr}\log(\mathcal{I}+\mathcal{T}) \quad \text{for sufficiently small } \mathcal{T}.$$

By Theorem 14, $D_n(\mathcal{T})$ is determined uniquely by the operator T and the numbers (5). By § 7 (2) we have

(10) $$\mathcal{T}(K)=\operatorname{tr}TK \quad \text{for} \quad K\in\mathfrak{F}\mathfrak{O}.$$

Thus the values of any nucleus $\mathcal{T}$ of T are uniquely determined by T on the set of all finitely dimensional operators K. By continuity, the values of $\mathcal{T}$ are also uniquely determined by T on the closure of this set, i. e. on the subspace $\mathfrak{C}_0\mathfrak{O}$. By Theorem 12 and § 10 (2), $T^{m-1}\in\mathfrak{C}_0\mathfrak{O}$ for $n>2$. Thus the numbers

$$\sigma_3,\ \sigma_4,\ \ldots$$

are uniquely determined by the quasinuclear operator T only. They do not depend on the choice of a quasinucleus $\mathcal{T}$ of T (28).

Hence we infer that the determinant system

(11) $$D_0(\mathcal{T}),\ D_1(\mathcal{T}),\ D_2(\mathcal{T}),\ \ldots$$

for $I+T$ is uniquely determined by T and the numbers $\sigma_1=\operatorname{Tr}\mathcal{T}$ and $\sigma_2=\operatorname{Tr}\mathcal{T}^2=\mathcal{T}(T)$ only.

On the other hand, the determinant system (11) is completely determined by the functions

(12) $$\operatorname{Tr}\mathcal{T},\ \operatorname{Tr}\mathcal{T}^2,\ \operatorname{Tr}\mathcal{T}^3,\ \ldots \quad (\mathcal{T}\in\mathfrak{Q}\mathfrak{N}).$$

(27) For the case of arbitrary Banach spaces, see Leżański [2], Sikorski [2], Grothendieck [3]. This identity was earlier known in the determinant theory of integral equations. See e. g. Lalesco [1], p. 114.

(28) Sikorski [5].

In fact, it follows from (9) that (12) determines uniquely $D_0(\mathcal{T})$ for small $\mathcal{T}$ (e. g. for $|\mathcal{T}|<1$). Since $D_0(\mathcal{T})$ is analytic, $D_0(\mathcal{T})$ is the unique analytic extension of the function $\exp\operatorname{Tr}\log(\mathcal{I}+\mathcal{T})$ over the whole $\mathfrak{Q}\mathfrak{N}$. Hence it follows that functions (12) determine uniquely values of the function $D_0(\mathcal{T})$ for all $\mathcal{T}\in\mathfrak{Q}\mathfrak{N}$. However, if we know $D_0(\mathcal{T})$ for all $\mathcal{T}\in\mathfrak{Q}\mathfrak{N}$, we know also all subdeterminants. This follows from Theorem 16 below. To formulate this theorem, let us recall that by the first differential $D_0(\mathcal{T};\mathcal{T}_1)$ of the function $D_0(\mathcal{T})$ ($\mathcal{T},\mathcal{T}_1\in\mathfrak{Q}\mathfrak{N}$) we understand the limit

$$\lim_{\varepsilon\to 0}\frac{D_0(\mathcal{T}+\varepsilon\mathcal{T}_1)-D_0(\mathcal{T})}{\varepsilon}.$$

By induction, the n-th differential $D_0(\mathcal{T};\mathcal{T}_1,\ldots,\mathcal{T}_n)$ is the limit

$$\lim_{\varepsilon\to 0}\frac{D_0(\mathcal{T}+\varepsilon\mathcal{T}_n;\mathcal{T}_1,\ldots,\mathcal{T}_{n-1})-D_0(\mathcal{T};\mathcal{T}_1,\ldots,\mathcal{T}_{n-1})}{\varepsilon}.$$

Of course, $D_0(\mathcal{T};\mathcal{T}_1,\ldots,\mathcal{T}_n)$ is analytic in the variable $\mathcal{T}$, and linear and symmetric in variables $\mathcal{T}_1,\ldots,\mathcal{T}_n$ ($\mathcal{T},\mathcal{T}_1,\ldots,\mathcal{T}_n\in\mathfrak{Q}\mathfrak{N}$).

THEOREM 16 (29). *For* $n=1,2,\ldots$

$$D_n(T)=D_n\begin{pmatrix}\xi_1,\ldots,\xi_n\\ x_1,\ldots,x_n\end{pmatrix}=D_0(T;\ \xi_1\otimes x_1,\ldots,\xi_n\otimes x_n).$$

Of course, to know all the functionals (12) it suffices to know only the functional $\operatorname{Tr}\mathcal{T}$ (for all $\mathcal{T}\in\mathfrak{Q}\mathfrak{N}$) and the multiplication $\mathcal{T}_1\circ\mathcal{T}_2$ in $\mathfrak{Q}\mathfrak{N}$. Thus the determinant and subdeterminants (11) are completely determined by the functional $\operatorname{Tr}\mathcal{T}$ on the Banach algebra $\mathfrak{Q}\mathfrak{N}$.

Note one more theorem of this character:

THEOREM 17 (30). *The determinant* $D_0(\mathcal{T})$ *is the only solution of the differential equation*

$$D_0(\mathcal{T};(I+T)\mathcal{T}_1)=D_0(\mathcal{T})\cdot\operatorname{Tr}\mathcal{T}_1$$

satisfying the initial condition

$$D_0(0)=1.$$

(29) Grothendieck [3]. By Theorem 16, $D_0(\mathcal{T};\mathcal{T}_1,\ldots,\mathcal{T}_n)$ is a natural extension of $D_0(\mathcal{T};\xi_1\otimes x_1,\ldots,\xi_n\otimes x_n)=D_n\begin{pmatrix}\xi_1,\ldots,\xi_n\\ x_1,\ldots,x_n\end{pmatrix}$. Grothendieck [3] considers $D_0(\mathcal{T};\mathcal{T}_1,\ldots,\mathcal{T}_n)$, instead of $D_n\begin{pmatrix}\xi_1,\ldots,\xi_n\\ x_1,\ldots,x_n\end{pmatrix}$, as the substitute of the notion of the set of all algebraic subdeterminants of order n.

(30) Michel and Martin [1] and Sikorski [6].

The following theorem on multiplication of determinants follows immediately from (9), § 9 (6) and some properties ([31]) of logarithm in Banach algebras:

THEOREM 18 ([32]). *For all* $\mathcal{T}_1, \mathcal{T}_2 \epsilon \mathfrak{QN}$

$$D\big((\mathcal{I}+\mathcal{T}_1)(\mathcal{I}+\mathcal{T}_2)\big) = D(\mathcal{I}+\mathcal{T}_1)\cdot D(\mathcal{I}+\mathcal{T}_2).$$

On the other hand,

THEOREM 19 ([33]). *The determinant* $D(\mathcal{I}+\mathcal{T})$ *is the only analytic function of* $\mathcal{T} \epsilon \mathfrak{QN}$ *such that* $D(\mathcal{I}+\mathcal{T})$ *does not vanish identically, and*

$$D\big((\mathcal{I}+\mathcal{T})^2\big) = \big(D(\mathcal{I}+\mathcal{T})\big)^2, \qquad D(\mathcal{I};\mathcal{T}) = \operatorname{Tr}\mathcal{T}.$$

Of course, $D(\mathcal{I};\mathcal{T})$ is the differential

$$D(\mathcal{I};\mathcal{T}) = \lim_{\varepsilon\to 0}\frac{D(\mathcal{I}+\varepsilon\mathcal{T})-D(\mathcal{I})}{\varepsilon} = D_0(0;\mathcal{T}).$$

§ 12. The Fredholm determinant and subdeterminants. The formula (5) from § 8 (i. e. the formulae (6), (7) from § 11) have the property that in the case of finitely dimensional spaces they give the classical algebraic determinant system (see § 1) for the operator $I+T$. However, in the case of linear integral equations examined by Fredholm they do not coincide with the Fredholm determinant and subdeterminants. If T is an integral operator determined by a continuous kernel $\tau(s,t)$, then Fredholm's subdeterminants for $I+T$ are some functions

$$\vartheta_n^*\begin{pmatrix} s_1, \dots, s_n \\ t_1, \dots, t_n \end{pmatrix} \tag{1}$$

of $2n$ real variables $s_1, \dots, s_n, t_1, \dots, t_n$. Of course, from the point of view of Functional Analysis, those functions should be interpreted as $2n$-linear functionals

$$\text{(1')}\quad D_n^*\begin{pmatrix} \xi_1, \dots, \xi_n \\ x_1, \dots, x_n \end{pmatrix} = \int\dots\int \vartheta_n^*\begin{pmatrix} s_1, \dots, s_n \\ t_1, \dots, t_n \end{pmatrix}\xi_1(s_1)\dots\xi_n(s_n)x_1(t_1)\dots x_n(t_n)\,ds_1\dots ds_n\,dt_1\dots dt_n.$$

However, subdeterminants $D_1(\mathcal{T}), D_2(\mathcal{T}), \dots$ are never of this integral form. This will be evident from some formulae quoted in the next section where the case of integral equations is carefully discussed.

([31]) See Michel and Martin [1] and Sikorski [2].

([32]) This theorem is a particular case of a general theorem proved by Michel and Martin [1]. For the case of determinants in Banach spaces, see Leżański [2], Sikorski [1, 2, 6] and Grothendieck [3].

([33]) Leżański [2].

In the case of arbitrary Banach spaces Ξ, X, we can also define some multilinear functionals D_n^* which are a natural generalization of Fredholm's subdeterminants (1), (1'). We assume the following definition:

By the *Fredholm determinant* $D_0^*(\mathcal{T})$ of $\mathcal{A} = \mathcal{I}+\mathcal{T}$, where $\mathcal{T}$ is a quasi-nucleus, we understand the determinant $D_0(\mathcal{T})$:

$$D_0^*(\mathcal{T}) = D_0(\mathcal{T}). \tag{2}$$

By the *Fredholm subdeterminant of order n* of $\mathcal{A} = \mathcal{I}+\mathcal{T}$ $(\mathcal{T} \epsilon \mathfrak{QN})$ we understand the $2n$-linear functional $D_n^*(\mathcal{T})$ on $\Xi^n \times X^n$ defined by the equation

$$D_n^*(\mathcal{T}) = D_n^*\begin{pmatrix} \xi_1, \dots, \xi_n \\ x_1, \dots, x_n \end{pmatrix} = D_n\begin{pmatrix} \xi_1 T, \dots, \xi_n T \\ x_1, \dots, x_n \end{pmatrix} = D_n\begin{pmatrix} \xi_1, \dots, \xi_n \\ Tx_1, \dots, Tx_n \end{pmatrix}, \tag{3}$$

where, for brevity, $D_n^* = D_n^*(\mathcal{T})$, $D_n = D_n(\mathcal{T})$, T being the image of $\mathcal{T}$ by the canonical mapping § 9 (1).

The sequence ([34])

$$D_0^*(\mathcal{T}), D_1^*(\mathcal{T}), D_2^*(\mathcal{T}), \dots \tag{4}$$

is not any determinant system for $I+T$. It satisfies conditions (d_1), (d_2), (d_3), (d_4) from § 5, but it does not satisfy condition (d_5). Instead of (D_n), (D_n') it satisfies the following identities:

$$(\mathrm{D}_n^*)\quad D_{n+1}^*\begin{pmatrix} \xi_0 A, \xi_1, \dots, \xi_n \\ x_0, x_1, \dots, x_n \end{pmatrix} = \sum_{i=0}^{n}(-1)^i\,\xi_0 Tx_i\cdot D_n^*\begin{pmatrix} \xi_1, \dots\dots\dots, \xi_n \\ x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \end{pmatrix},$$

$$(\mathrm{D}_n'^*)\quad D_{n+1}^*\begin{pmatrix} \xi_0, \xi_1, \dots, \xi_n \\ Ax_0, x_1, \dots, x_n \end{pmatrix} = \sum_{i=0}^{n}(-1)^i\,\xi_i Tx_0\cdot D_n^*\begin{pmatrix} \xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n \\ x_1, \dots\dots\dots, x_n \end{pmatrix},$$

where, for brevity, $A = I+T$.

The smallest integer r such that $D_r^*(T)$ is not identically equal to zero is called the *order* of (4).

Although (4) is not any determinant system for

$$A = I+T,$$

([34]) For definition and properties of (4), see Grothendieck [3] and Sikorski [6].

it can be used to solve the linear equations

$$\xi A = x_0, \qquad Ax = x_0.$$

We have the following theorem similar to Theorem 3:

THEOREM 20 [35]. *The order of* $D_0^*(\mathcal{T}), D_1^*(\mathcal{T}), D_2^*(\mathcal{T}), \ldots$ *is equal to the order of* $D_0(\mathcal{T}), D_1(\mathcal{T}), D_2(\mathcal{T}), \ldots$

If r is the order of (4), *then*

$$D_r^* = (-1)^r D_r.$$

Let $\eta_1, \ldots, \eta_r \in \Xi$ *and* $y_1, \ldots, y_r \in X$ *are such that*

$$D_r^*\begin{pmatrix}\eta_1, \ldots, \eta_r\\ x_1, \ldots, x_r\end{pmatrix} \neq 0.$$

Then there exist elements $\zeta_1, \ldots, \zeta_r \in \Xi$ *and* $z_1, \ldots, z_r \in X$ *such that*

$$\zeta_i x = \frac{D_r^*\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots, \eta_r\\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix}}{D_r^*\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad x \in X \tag{5}$$

and

$$\xi z_i = \frac{D_r^*\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r\\ y_1, \ldots\ldots\ldots\ldots, y_r\end{pmatrix}}{D_r^*\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi. \tag{5'}$$

The elements $\zeta_1, \ldots, \zeta_r$ *are linearly independent and are solutions of the equation*

$$\xi A = 0. \tag{6}$$

The elements $z_1, \ldots, z_r$ *are linearly independent and are solutions of the equation*

$$Ax = 0. \tag{6'}$$

Conversely, every solution ξ of (6) *is a linear combination of* $\zeta_1, \ldots, \zeta_r$, *and every solution x of* (6′) *is a linear combination of* $z_1, \ldots, z_r$.

The bilinear functional B^* defined by the formula

$$\xi B^* x = \frac{D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ x, y_1, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix}} \tag{7}$$

is an operator.

[35] Grothendieck [3] and Sikorski [6].

The equation

$$\xi A = \xi_0 \tag{8}$$

has a solution ξ if and only if $\xi_0 z_i = 0$ *for* $i = 1, \ldots, r$. *Viz. the element*

$$\xi = \xi_0 - \xi_0 B^*$$

is a solution of (8).

The equation

$$Ax = x_0 \tag{8'}$$

has a solution x if and only if $\zeta_i x_0 = 0$ *for* $i = 1, \ldots, r$. *Viz. the element*

$$x = x_0 - B^* x_0$$

is a solution of (8′).

Observe also that

$$D_n^*(\mathcal{T}) = \sum_{m=0}^{\infty} \theta_{n,m}^*, \tag{9}$$

where

$$\theta_{n,m}^*\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \theta_{n,m}\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = \theta_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix} \tag{10}$$

are the Fredholm analogoues of the $2n$-linear functionals $\theta_{n,m}$ defined in § 8 (4).

§ 13. An integral model. To illustrate the theory of determinants in Banach spaces, let us assume additionally in this section that Ξ and X are two Banach spaces of some measurable functions defined on a set Γ with a measure μ. Integrals extended over the whole space Γ will be written, for brevity, in the form $\int f(t)\,dt$ instead of $\int_\Gamma f(t)\,d\mu(t)$, and similarly for multiple integrals.

Suppose that ξx is defined by the formula

$$\xi x = \int \xi(t) x(t)\,dt. \tag{1}$$

An operator K is said to be an *integral operator* provided it is of the form

$$\xi K x = \iint \xi(s)\varkappa(s, t) x(t)\,ds\,dt. \tag{2}$$

where $\varkappa(s, t)$ is a measurable function called the *kernel* of K. For instance, every finitely dimensional $K = \sum_{i=1}^{m} x_i \cdot \xi_i$ is an integral operator, the kernel $\varkappa(s, t)$ being defined by the formula $\varkappa(s, t) = \sum_{i=1}^{m} x_i(s)\cdot \xi_i(t)$.

A quasinucleus $\mathcal{T}$ is said to be an *integral quasinucleus* provided there exists a measurable function $\tau(t,s)$ (called the *kernel* of $\mathcal{T}$), such that

$$\mathcal{T}(I)=\int\tau(s,s)\,ds \tag{3}$$

and

$$\mathcal{T}(K)=\int\tau(t,s)\varkappa(s,t)\,ds\,dt \tag{4}$$

for every integral operator K with a kernel $\varkappa(s,t)$ (36). Then $\mathcal{T}$ is the quasinucleus of the integral operator T whose kernel is $\tau(s,t)$.

The integral quasinucleus $\mathcal{T}$ determines the determinant system

$$D_0(\mathcal{T}),D_1(\mathcal{T}),D_2(\mathcal{T}),\ldots \tag{5}$$

and the corresponding Fredholm system

$$D_0^*(\mathcal{T}),D_1^*(\mathcal{T}),D_2^*(\mathcal{T}),\ldots \tag{6}$$

for the operator $I+T$, i. e. for the linear integral equations

$$x(s)+\int\tau(s,t)x(t)\,dt=x_0(s),$$

$$\xi(t)+\int\xi(s)\tau(s,t)\,ds=\xi_0(s).$$

To write some integral formula for (5), it is convenient to introduce a formal expression $\delta(s,t)$ which is a substitute of the Dirac delta distribution, and enables us to write the unit operator I in an integral form. Viz. we define axiomatically $\delta(s,t)$ by the equations

$$\int\delta(s,t)x(t)\,dt=x(s),\qquad\int\xi(s)\delta(s,t)\,ds=\xi(t), \tag{7}$$

$$\iint\delta(s,t)\tau(t,s)\,ds\,dt=\mathcal{T}(I). \tag{8}$$

If all points in Γ have positive measures, then of course there exists a function $\delta(s,t)$ satisfying this identity. In the general case, formulae (7) should be interpreted quite formally as an integral notation for the unit operator I. Similarly we interpret the following formulae which are a consequence of notation (7), (8):

$$\iint\xi(s)\delta(s,t)x(t)=\xi x,$$

$$\iiint\delta(s,t)\tau(t,r)\tau(r,s)\,dr\,ds\,dt=\iint\tau(t,r)\tau(r,t)\,dr\,dt,\qquad\text{etc.}$$

(36) For examples of integral quasinuclei, see Leżański [1].

THEOREM 21 (37). *If $\mathcal{T}$ is an integral quasinucleus, then the following integral formula holds for the $2n$-linear functionals $\theta_{n,m}$ defined in* § 8 (4):

$$\theta_{n,m}\begin{pmatrix}\xi_1,\ldots,\xi_n\\x_1,\ldots,x_n\end{pmatrix}$$

$$=\int\ldots\int\vartheta_{n,m}\begin{pmatrix}s_1,\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}\xi_1(s_1)\ldots\xi_n(s_n)x_1(t_1)\ldots x_n(t_n)\,ds_1\ldots ds_n\,dt_1\ldots dt_n,$$

where

$$\vartheta_{n,m}\begin{pmatrix}s_1.\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}$$

$$=\frac{1}{m!}\int\ldots\int\begin{vmatrix}\delta(s_1,t_1),\ldots,&\delta(s_1,t_n),&\delta(s_1,r_1),\ldots,&\delta(s_1,r_m)\\\cdots&\cdots&\cdots&\cdots\\\delta(s_n,t_1),\ldots,&\delta(s_n,t_n),&\delta(s_n,r_1),\ldots,&\delta(s_n,r_m)\\T(r_1,t_1),\ldots,&T(r_1,t_n),&T(r_1,r_1),\ldots,&T(r_1,r_m)\\\cdots&\cdots&\cdots&\cdots\\T(r_m,t_1),\ldots,&T(r_m,t_n),&T(r_m,r_1),\ldots,&T(r_m,r_m)\end{vmatrix}dr_1\ldots dr_m.$$

Consequently

$$D_n(\mathcal{T})=D_n\begin{pmatrix}\xi_1,\ldots,\xi_n\\x_1,\ldots,x_n\end{pmatrix} \tag{9}$$

$$=\int\ldots\int\vartheta_n\begin{pmatrix}s_1,\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}\xi_1(s_1)\ldots\xi_n(s_n)x_1(t_1)\ldots x_n(t_n)\,ds_1\ldots ds_n\,dt_1\ldots dt_n,$$

where

$$\vartheta_n\begin{pmatrix}s_1,\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}=\sum_{m=0}^{\infty}\vartheta_{n,m}\begin{pmatrix}s_1,\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}. \tag{10}$$

THEOREM 22 (38). *If $\mathcal{T}$ is an integral quasinucleus, then the following formula holds for the $2n$-linear functionals $\theta_{n,m}^*$ defined in* § 12 (10):

$$\theta_{n,m}^*\begin{pmatrix}\xi_1,\ldots,\xi_n\\x_1,\ldots,x_n\end{pmatrix}$$

$$=\int\ldots\int\vartheta_{n,m}^*\begin{pmatrix}s_1,\ldots,s_n\\t_1,\ldots,t_n\end{pmatrix}\xi_1(s_1)\ldots\xi_n(s_n)x_1(t_1)\ldots x_n(t_n)\,ds_1\ldots ds_n\,dt_1\ldots dt_n,$$

(37) Sikorski [6].
(38) Sikorski [6].

where

$$\vartheta^*_{n,m}\begin{pmatrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{pmatrix}$$

$$= \frac{1}{m!}\int \ldots \int \begin{vmatrix} T(s_1, t_1), \ldots, T(s_1, t_n), & T(s_1, r_1), \ldots, & T(s_1, r_m) \\ \cdots & \cdots & \cdots \\ T(s_n, t_1), \ldots, T(s_n, t_n), & T(s_n, r_1), \ldots, & T(s_n, r_m) \\ T(r_1, t_1), \ldots, T(r_1, t_n), & T(r_1, r_1), \ldots, & T(r_1, r_m) \\ \cdots & \cdots & \cdots \\ T(r_m, t_1), \ldots, T(r_m, t_n), & T(r_m, r_1), \ldots, & T(r_m, r_m) \end{vmatrix} dr_1 \ldots dr_m.$$

Consequently

$$(11) \quad D^*_n(\mathcal{T}) = D^*_n\begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n \end{pmatrix}$$

$$= \int \ldots \int \vartheta^*_n\begin{pmatrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{pmatrix} \xi_1(s_1) \ldots \xi_n(s_n) x_1(t_1) \ldots x_n(t_n)\, ds_1 \ldots ds_n\, dt_1 \ldots dt_n,$$

where

$$(12) \quad \vartheta^*_n\begin{pmatrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{pmatrix} = \sum_{m=0}^{\infty} \vartheta^*_{n,m}\begin{pmatrix} s_1, \ldots, s_n \\ t_1, \ldots, t_n \end{pmatrix}.$$

Formulae (9) and (11) should be understood only formally, as another way of writing the exact formulae for $D_n(\mathcal{T})$ and $D^*_n(\mathcal{T})$ in §§ 8 and 12. The convergence of series (10), (12) should also be understood to be the convergence in norm of the corresponding multilinear functionals represented formally by the kernels under consideration.

However in the series (12) the symbol $\delta(s, t)$ does not appear. $\vartheta^*_{n,m}$ are well-defined functions. Consequently the convergence of (12) can sometimes be understood as a convergence of functions in a suitable space of functions, and the meaning of (11) is ordinary. E. g. in the case investigated by Fredholm, when $T(s, t)$ is a continuous function, $\vartheta^*_{n,m}$ are also continuous functions and the series (12) converges uniformly to a continuous function ϑ^*_n. This is Fredholm's original subdeterminant of order n, mentioned in § 12 (1).

If $\delta(s, t)$ can be interpreted as a function, also $\vartheta_{n,m}$ are functions and the convergence in (10) can be understood sometimes in another sense, e. g. as the pointwise convergence. So it is in the case of matrices investigated in the next section.

§ 14. The matrix model (39). Now let Γ be the set of all positive integers with the trivial measure:

$$\mu(\Gamma') = \text{the number of elements in } \Gamma'$$

defined for all sets $\Gamma' \subset \Gamma$. All integrals will now be replaced by convergent series.

Elements of Ξ and X are now infinite sequences

$$(1) \quad \begin{aligned} \xi &= (\varphi_1, \varphi_2, \ldots), & \eta &= (\psi_1, \psi_2, \ldots), & \ldots \\ x &= (v_1, v_2, \ldots), & y &= (w_1, w_2, \ldots), & \ldots \end{aligned}$$

and the product ξx is now the ordinary scalar product of sequences ξ, x:

$$\xi x = \sum_{i=1}^{\infty} \varphi_i v_i,$$

the series being absolutely convergent.

Suppose additionally that the sequences

$$e_1 = (1, 0, 0, \ldots)$$
$$e_2 = (0, 1, 0, \ldots)$$
$$\cdots\cdots\cdots$$

form a basis in Ξ and in X.

Every operator K is now uniquely represented by an infinite square matrix $\varkappa = (\varkappa_{i,j})$ (the kernel of K, in the terminology from § 13). By definition,

$$\varkappa_{i,j} = e_i K e_j.$$

The equality $y = Kx$ is an abbreviation for the system of linear equations

$$w_i = \sum_{j=1}^{\infty} \varkappa_{i,j} v_j \quad (i = 1, 2, \ldots),$$

and the equality $\eta = \xi K$ is an abbreviation for

$$\psi_j = \sum_{i=1}^{\infty} \varphi_i \varkappa_{i,j} \quad (j = 1, 2, \ldots)$$

(see the notation (1)).

The correspondence

$$K \leftrightarrow \varkappa \quad (K \in \mathfrak{O})$$

(39) Theorems quoted in § 14 were proved by Koch [1-7] under much stronger hypotheses. See also Riesz [1]. In the case where $X = \boldsymbol{l}$ and $\Xi = \boldsymbol{m}$ or $\boldsymbol{c}_0$ the results mentioned in § 14 were obtained by Leżański [1].

is one-to-one. Consequently we can identify operators K with corresponding matrices $\varkappa$, writing $K = (\varkappa_{i,j})$.

In particular, the unit operator I is represented by the matrix $\delta = (\delta_{ij})$, where

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

The matrix $\delta_{i,j}$ is the function $\delta(s, t)$ from § 13.

By a *matrix quasinucleus* we understand any quasinucleus $\mathcal{T}$ on $\mathfrak{D}_1$ of the form

$$(2) \qquad \mathcal{T}(K) = \sum_{i,j=1}^{\infty} \tau_{j,i}\varkappa_{i,j} \quad \text{for} \quad K = (\varkappa_{i,j}) \in \mathfrak{D}_1,$$

where $\tau = (\tau_{i,j})$ is an infinite square matrix (the kernel of $\mathcal{T}$, in the terminology of § 13). The definition (2) is the definition (3), (4) from § 13. More exactly, (2) coincides with condition § 13 (4). Condition § 13 (3) is automatically satisfied, since by (2)

$$(3) \qquad \operatorname{Tr}\mathcal{T} = \mathcal{T}(I) = \sum_{i,j=1}^{\infty} \tau_{j,i}\,\delta_{i,j} = \sum_{i=1}^{\infty} \tau_{i,i}.$$

The class of all matrix quasinuclei will be denoted by $\mathfrak{M}\mathfrak{N}$. The correspondence

$$\mathcal{T} \leftrightarrow \tau \qquad (\mathcal{T} \in \mathfrak{M}\mathfrak{N})$$

is one-to-one. Consequently we can identify matrix quasinuclei $\mathcal{T}$ with corresponding matrices τ, writing $\mathcal{T} = (\tau_{i,j})$, $(\tau_{i,j}) \in \mathfrak{M}\mathfrak{N}$, etc. Observe that

$$\tau_{i,j} = \mathcal{T}(e_j \cdot e_i).$$

It follows from (3) that the trace of a matrix quasinucleus $\mathcal{T} = (\tau_{i,j})$ is the algebraic trace of the matrix $(\tau_{i,j})$.

If $\mathcal{T} = (\tau_{i,j})$ is a matrix quasinucleus, then the matrix $\tau = (\tau_{i,j})$ can be also interpreted as an operator ([40]). Viz. τ is the matrix (the kernel) of the quasinuclear operator T which is the image of $\mathcal{T}$ by the canonical mapping (see § 9 (1)). Hence it follows that the canonical mapping

$$\mathcal{T} \to T$$

([40]) Such an operator is quasinuclear but, in general, it is not compact and consequently it is not nuclear. A simple example of an operator of this kind in the space $X = \boldsymbol{l} \times \boldsymbol{c}_0$ was communicated to me by Pełczyński and Szlenk: $T(x, y) = (0, x)$.

is one-to-one on the class $\mathfrak{M}\mathfrak{N}$ of matrix quasinuclei. Consequently we can identify $\mathcal{T}$ with T:

$$(4) \qquad \mathcal{T} = (\tau_{i,j}) = T \qquad (\mathcal{T} \in \mathfrak{M}\mathfrak{N}).$$

However this identification is not extended over the norms of $(\tau_{i,j})$: the norm of $(\tau_{i,j})$ interpreted as an element of $\mathfrak{Q}\mathfrak{N}$ is not equivalent, in general, to the norm of $(\tau_{i,j})$ interpreted as an element of $\mathfrak{D}$.

Since the extended canonical mapping transforms the abstract unit $\mathcal{I}$, added to the algebra $\mathfrak{Q}\mathfrak{N}$, onto the unit operator I, the identification (4) suggests to identify $\mathcal{I}$ with I, i. e. with the unit matrix $\delta = (\delta_{i,j})$. Consequently the determinant $D(\mathcal{I}+\mathcal{T})$ (i. e. $D_0(\mathcal{T})$ — see § 11, p. 172) will be called the *determinant of the matrix*

$$(5) \qquad \alpha = \delta + \tau = (\delta_{i,j} + \tau_{i,j}).$$

and denoted by $\det \alpha$ or by

$$\begin{vmatrix} \alpha_{1,1}, & \alpha_{1,2}, & \dots \\ \alpha_{2,1}, & \alpha_{2,2}, & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

The class of all infinite square matrices α of the form (5) (i. e. of all the matrices α such that $\alpha - \delta \in \mathfrak{M}\mathfrak{N}$) will be denoted by $\mathfrak{M}$. Thus, with every $\alpha \in \mathfrak{M}$, we have uniquely associated a number $\det \alpha$.

Of course, the determinant system $D_0(\mathcal{T}), D_1(\mathcal{T}), D_2(\mathcal{T}), \dots$ will be also called the *determinant system of the matrix* α. The series § 13 (10), (12) defining $D_n(\mathcal{T})$ $(n = 0, 1, 2, \dots)$ converges pointwise. In particular, Theorem 21 yields the following formula for $\det(\delta+\tau)$, $\tau \in \mathfrak{M}\mathfrak{N}$:

$$(6) \qquad \det(\delta+\tau) = \sum_{m=0}^{\infty} \theta_{0,m},$$

where $\theta_{0,0} = 1$ and, for $m = 1, 2, \dots$,

$$(7) \qquad \theta_{0,m} = \sum_{i_1 < \dots < i_m} \begin{vmatrix} \tau_{i_1,i_2}, & \dots, & \tau_{i_1,i_m} \\ \dots & \dots & \dots \\ \tau_{i_m,i_1}, & \dots, & \tau_{i_m,i_m} \end{vmatrix}.$$

The formulae (6), (7) define the determinant for the system of linear equations

$$(8) \qquad \sum_{j=1}^{\infty} a_{ij} v_j = w_j \qquad (j = 1, 2, \dots),$$

and for the adjoint system of linear equations

$$\sum_{i=1}^{\infty} \varphi_i a_{ij} = \psi_j \qquad (j = 1, 2, \ldots) \tag{8'}$$

(see (1)), where $a = \delta + \tau$.

The product $\alpha\beta$ of any matrices $\alpha, \beta \in \mathfrak{M}$ exists and belongs to $\mathfrak{M}$. It follows from Theorem 18 that

THEOREM 23. *For any* $\alpha, \beta \in \mathfrak{M}$,

$$\det \alpha\beta = \det \alpha \cdot \det \beta .$$

If $a \in \mathfrak{M}$, then the j-th column of a, i. e. the sequence $(a_{1,j}, a_{2,j}, \ldots)$, is an element of X (viz. it is the element $(I+T)e_j$). Any matrix a_x obtained from a by replacing the j-th column by terms of a sequence $x \in X$ belongs also to $\mathfrak{M}$. Moreover $\det a_x$ is linear function of $x \in X$. Similarly, if $a \in \mathfrak{M}$, then the i-th row of a, i. e. the sequence $(a_{i,1}, a_{i,2}, \ldots)$ is an element of Ξ (viz. it is the element $e_i(I+T)$). Any matrix $_\xi a$ obtained from a by replacing the i-th row by a sequence $\xi \in \Xi$ belongs also to $\mathfrak{M}$. Moreover $\det {}_\xi a$ is a linear function of $\xi \in \Xi$.

We express shortly the two last remarks in the form of the following theorem, analogous to a known theorem on determinants of finite square matrices:

THEOREM 24. *The determinant* $\det a$ *is a linear function of any of the columns and rows of the matrix* $a \in \mathfrak{M}$.

The next theorem is the analogue to the known theorem on skew symmetry of determinants of finite square matrices:

THEOREM 25. *If two rows or two columns are commuted in a matrix* $a \in \mathfrak{M}$, *then* $\det a$ *changes its sign.*

Let $a = (a_{i,j}) \in \mathfrak{M}$ and let $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ be two finite sequences of positive integers. Define the numbers $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n}$ in the same manner as in the finitely dimensional case (see § 1, p. 144), viz.: if either two of the integers $i_1, \ldots, i_n$ are equal, or two of the integers $j_1, \ldots, j_n$ are equal, then $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n} = 0$; in the opposite case $a\binom{j_1, \ldots, j_n}{i_1, \ldots, i_n}$ is the determinant of the matrix $(\beta_{i,j}) \in \mathfrak{M}$ where

$$\beta_{i,j} = \begin{cases} a_{i,j} & \text{if none of the equalities } i = i_k,\ j = j_l \\ & \qquad\qquad (k, l = 1, \ldots, n) \text{ holds,} \\ 1 & \text{if } \ i = i_k \text{ and } j = j_k \text{ for an integer } k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem below asserts that $a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n}$ have the same meaning as the analogous symbols in the finitely dimensional case (see § 1, p. 144-145), i. e. they are algebraic subdeterminants of order n of the infinite square matrix $a \in \mathfrak{M}$.

THEOREM 26. *The numbers* $a\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n}$ *are coordinates of the* $2n$*-linear functional* $D_n(\mathcal{T})$ $(a = \delta + \tau,\ \mathcal{T} = \tau = (\tau_{ij}) \in \mathfrak{M}\mathfrak{N})$, *i. e. the following formulae hold:*

$$a\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} = D_n\begin{pmatrix} e_{i_1}, \ldots, e_{i_n} \\ e_{j_1}, \ldots, e_{j_n} \end{pmatrix},$$

where, for brevity, $D_n = D_n(\mathcal{T})$. *Consequently*

$$D_n\begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n \end{pmatrix} = \sum_{i_1, \ldots, i_n = 1}^{\infty} \sum_{j_1, \ldots, j_n = 1}^{\infty} \varphi_{1,i_1}, \ldots, \varphi_{n,i_n} a\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} v_{1,j_1} \ldots v_{n,j_n},$$

where $x_r = (v_{r,1}, v_{r,2}, \ldots)$, $\xi_r = (\varphi_{r,1}, \varphi_{r,2}, \ldots)$ *for* $r = 1, \ldots, n$.

Observe that the series § 13 (10) in Theorem 21 converges pointwise, and we have the identities:

$$a\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} = \vartheta_n\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} = \sum_{m=0}^{\infty} \vartheta_{n,m}\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix}, \tag{9}$$

$$\vartheta_{n,m}\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} = \sum_{r_1 < r_2 < \ldots < r_m} \begin{vmatrix} \delta_{i_1,j_1}, & \ldots, & \delta_{i_1,j_n}, & \delta_{i_1,r_1}, & \ldots, & \delta_{i_1,r_m} \\ \cdots & & & & & \cdots \\ \delta_{i_n,j_1}, & \ldots, & \delta_{i_n,j_n}, & \delta_{i_n,r_1}, & \ldots, & \delta_{i_n,r_m} \\ \tau_{r_1,j_1}, & \ldots, & \tau_{r_1,j_n}, & \tau_{r_1,r_1}, & \ldots, & \tau_{r_1,r_m} \\ \cdots & & & & & \cdots \\ \tau_{r_m,j_1}, & \ldots, & \tau_{r_m,j_n}, & \tau_{r_m,r_1}, & \ldots, & \tau_{r_m,r_m} \end{vmatrix}. \tag{10}$$

In particular, $a\binom{j}{i}$ is the analogue of the subdeterminant (of the order 1) obtained by omitting the j-th column and the i-th row. We have the following formula for the development of a determinant by a row or by a column:

THEOREM 27. *For any positive integers i_0 and j_0,*

$$\det a = \sum_{j=1}^{\infty} a_{i_0,j}\, a\binom{j}{i_0} = \sum_{i=1}^{\infty} a_{i,j_0}\, a\binom{j_0}{i}.$$

Theorem 3 yields the Cramer formulae for solutions of (8) and (8′):

THEOREM 28. *If* $\det a \neq 0$, *then the only solutions of* (8) *and* (8′) *are:*

$$v_j = \frac{\begin{vmatrix} a_{1,1}, \ldots, a_{1,j-1}, w_1, a_{1,j+1}, \ldots \\ a_{2,1}, \ldots, a_{2,j-1}, w_2, a_{2,j+1}, \ldots \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{vmatrix}}{\begin{vmatrix} a_{1,1}, a_{1,2}, \ldots \\ a_{2,1}, a_{2,2}, \ldots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{vmatrix}} \qquad (j = 1, 2, \ldots),$$

$$\varphi_i = \frac{\begin{vmatrix} a_{1,1}, & a_{1,2}, & \ldots \\ \cdots & \cdots & \cdots \\ a_{i-1,1}, & a_{i-1,2}, & \ldots \\ \psi_1, & \psi_2, & \ldots \\ a_{i+1,1}, & a_{i+1,2}, & \ldots \\ \cdots & \cdots & \cdots \end{vmatrix}}{\begin{vmatrix} a_{1,1}, a_{1,2}, \ldots \\ a_{2,1}, a_{2,2}, \ldots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{vmatrix}} \qquad (i = 1, 2, \ldots).$$

Every matrix $\tau = (\tau_{i,j})$ can be formally written as the sum of an infinite series of matrices, each of which contains one column (or row) of τ and zeros elsewhere. For this purpose, let $x_j = (\tau_{1,j}, \tau_{2,j}, \ldots)$ and $\xi_i = (\tau_{i,1}, \tau_{i,2}, \ldots)$. If τ is considered as an operator T, then we can write formally

$$T = \sum_{j=1}^{\infty} x_j \cdot e_j, \qquad T = \sum_{i=1}^{\infty} e_i \cdot \xi_i. \tag{11}$$

If τ is considered as a quasinucleus $\mathcal{T}$, then we can write formally

$$\mathcal{T} = \sum_{j=1}^{\infty} e_j \otimes x_j, \qquad \mathcal{T} = \sum_{i=1}^{\infty} \xi_i \otimes e_i. \tag{12}$$

In general, neither the series (11) converge (in norm) in the space $\mathfrak{O}$, nor the series (12) converge (in norm) in the space $\mathfrak{O}\mathfrak{N}$. Of course, the convergence of one of the series (12) implies the convergence of the corresponding series (11) (see the norm inequality (3) at the beginning of § 9). In some cases, one of the series (12) converges. So is e. g. in the case $X = \boldsymbol{l}$, $\Xi = \boldsymbol{c}_0$.

THEOREM 29. *If* $\tau = (\tau_{i,j}) \in \mathfrak{M}$, *and one of the series* (12) *converges to* $\mathcal{T} = \tau$ *is the space* $\mathfrak{O}\mathfrak{N}$, *then the matrix* $a = \delta + \tau$ *has the property*

$$\det a = \lim_{n\to\infty} \begin{vmatrix} a_{1,1}, \ldots, a_{1,n} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ a_{n,1}, \ldots, a_{n,n} \end{vmatrix}.$$

This property was suggested on p. 141 as one of the possible definitions of the notion of the determinant of an infinite square matrix.

§ 15. The poblem of uniqueness. In §§ 8 and 11 we have solved the problems A), B), C), D) from § 7 with one modification: the determinant system is uniquely associated with quasinuclei $\mathcal{T}$ (i. e. with $\mathcal{A} = \mathcal{I} + \mathcal{T}$) but not with operators $A = I + T$. However we have given analytic, effective formulae for determinant systems for $A = I + T$, $T \in \mathfrak{O}\mathfrak{O}$, and these formulae are indeed relevant. In the case of finitely dimensional spaces they coincide with the algebraic formulae, and in the case of infinite square matrices they lead to a theory of determinants analogous, in all details, to the theory of determinants of finite square matrices.

The theory just presented has only one defect: the determinant system for $A = I + T$ is not uniquely determined by A itself [41]. We have learned that the reason for this defect lies in the dispersion (in the case of infinitely dimensional space) of the notion of square matrix into two notions: operator and its quasinucleus. At the end of § 7 we have proposed a method for omitting the phenomenon of non-uniqueness: to restrict the class $\mathfrak{O}\mathfrak{N}$ of all quasinuclei to a smaller class $\mathfrak{N}_0$ having the uniqueness property (i. e. such that the canonical mapping $\mathcal{T} \to T$ is one-to-one on $\mathfrak{N}_0$), and to associate with $A = I + T$ the determinant system of $\mathcal{A} = \mathcal{I} + \mathcal{T}$ where $\mathcal{T}$ is the only quasinucleus of T in $\mathfrak{N}_0$. This method has been applied in § 14 where we have assumed $\mathfrak{N}_0 = \mathfrak{M}\mathfrak{N} =$ the class of all matrix quasinuclei. For every matrix $a \in \mathfrak{M}$, there exists infinitely many quasinuclei $\mathcal{T}$ such that $\mathcal{T} \to T$, $T = (\tau_{i,j}) = a - \delta$, but

[41] Observe that this non-uniqueness appears also in the determinant theory of integral equations with non-continuous kernels.

there exists only one matrix quasinucleus $\mathcal{T}$ satisfying these conditions. Therefore we have been able to assign uniquely, to every $a \epsilon \mathfrak{M}$, a determinant system of a. The hypothesis that Γ is the set of all natural integers is not essential in § 14. Therefore the results from § 14 can be generalized to the case where Γ is any countable or uncountable set of indices.

Roughly speaking, the method from § 14 can be applied to any spaces Ξ, X having bases. In fact, suppose that Ξ and X have adjoint countable basis. Then we can consider Ξ and X as some spaces of sequences, viz. all sequences of coordinates of $\xi \epsilon \Xi$ and $x \epsilon X$ with respect to the bases in question. We can now introduce the notion of matrix nucleus and the corresponding set $\mathfrak{M}$ of matrices. With every operator whose matrix is in $\mathfrak{M}$ we can assign, uniquely and effectively, a determinant system by the method described in § 14. Since main results from § 14 remain valid in the case of any uncountable set Γ, we can also generalize the above procedure over the case of spaces with uncountable bases.

Unfortunately the problem of the existence of a basis in every (separable) Banach space is not solved, and therefore the method just described cannot be applied in the case of arbitrary Banach spaces. Moreover the choice of bases in Ξ and X is not always convenient in practice (the difficulty of characterizing the corresponding spaces of sequences!) and such the procedure does not seem to be natural in many concrete cases.

The above discussion proves, however, the importance of the problem of existence of basis for the theory of determinants in Banach spaces. § 14 also reconfirms our remark on p. 141 that the case of spaces of sequences is much easier than the general case.

If Ξ and X are arbitrary Banach spaces, we know that the class $\mathfrak{FN}$ of all finitely dimensional nuclei has the uniqueness property. However this class is too small for our purpose because it leads only to operators $A = I+K$ where K is finitely dimensional. The smallest closed subspace in $\mathfrak{QN}$, which contains $\mathfrak{FN}$, is the class $\mathfrak{N}$ of all nuclei.

The problem arises whether $\mathfrak{N}$ always has the uniqueness property.

Before giving any answer to this problem, let us discuss first the situation in the case where $\mathfrak{N}$ has the uniqueness property.

We have then a full solution of problems A), B), C), D) from § 7, if we assume $\mathfrak{T} = \mathfrak{NO}$ but with the topology induced by the norm in $\mathfrak{N}$. We can assign uniquely to every operator $A = I+T$ $(T \epsilon \mathfrak{NO})$ a determinant system by analytic formulae. It is natural to call it the determinant system *of* A. The restriction to nuclear operators T is not very strong. On the one hand, in many concrete cases it is not any restriction because the notion of quasinuclear and nuclear operators coincide (see e. g. Theorem 10). On the other hand, even if $\mathfrak{QN}$ is larger than $\mathfrak{NO}$, the both notions are rather close (see Theorem 12).

The phenomenon of dispersion of notions does not appear because it is natural to identify then nuclear operators with their nuclei. This fact simplifies the theory. $\mathfrak{NO}$ is a Banach space with respect to the norm from $\mathfrak{N}$.

The only defect of the determinant theory is radically removed. We get a theory which is completely analogous to the case of finitely dimensional spaces.

The adopted definition of determinant system is, in some sense, the only natural extension (over $A = I+T$, $T \epsilon \mathfrak{NO}$) of the notion of algebraic determinant systems in finitely dimensional spaces. In fact, by purely algebraic means we can define uniquely the determinant system of any operator $I+T$ where T is finitely dimensional. This follows from the fact that we can represent X and Ξ as Cartesian products $X = X_0 \times \times X_1$, $\Xi = \Xi_0 \times \Xi_1$, such that X_0, Ξ_0 are finitely dimensional and conjugate, and T operates only on X_0, Ξ_0. Thus the determinant system of $I+T$ is the algebraic determinant system of its finitely dimensional restriction to X_0, Ξ_0. The defined algebraic system of $I+T$ does not depend continuously on T with respect to the ordinary norm $|T|$ of the operator T (except the case where X and Ξ are finitely dimensional). However it is a continuous function of T with respect to the norm $\|T\|$ induced in $\mathfrak{FO}$ by the norm in $\mathfrak{N}$ (see § 10, p. 168). Since $\mathfrak{FN}$ is dense in $\mathfrak{NO}$ with respect to the norm induced by $\mathfrak{N}$, the determinant system of $I+T$, $T \epsilon \mathfrak{NO}$, is the only continuous extension of the algebraic determinat system of $I+T$, $T \epsilon \mathfrak{FO}$ ([42]).

Thus we see that if $\mathfrak{N}$ has the uniqueness property and we restrict ourselves to operators $A = I+T$ with T nuclear, the theory becomes more regular and more simple.

In many concrete cases we know that $\mathfrak{N}$ has the uniqueness property. We do not know any example where $\mathfrak{N}$ does not have this property. However we cannot prove that $\mathfrak{N}$ always has the uniqueness property.

The problem of uniqueness was examined recently by Grothendieck [2] in the case where $\Xi = X^*$. The main result is given by the following theorem:

THEOREM 30. $\mathfrak{N}$ *has the uniqueness property if and only if* $\mathfrak{C}_0\mathfrak{O} = \mathfrak{CO}$, *i. e. if every compact endomorphism in* X *is the limit in norm of a sequence of finitely dimensional endomorphisms* ([43]).

([42]) The definition of $D_n(\mathcal{T})$ as an extension, by continuity, of $D_n(\mathcal{T}')$ where $\mathcal{T}'$ is finitely dimensional was the starting point of the theory of Ruston [1, 2].

([43]) Grothendieck [2], Chapter I, § 5, 1, p. 165.

Unfortunately, the problem whether $\mathfrak{C}_0\mathfrak{O} = \mathfrak{C}\mathfrak{O}$ for every Banach space is an old, very difficult, unsolved problem of Functional Analysis. Consequently it seems to be hopeless to find any answer to the question of whether $\mathfrak{N}$ always has the uniqueness property.

Observe that if X has a basis, then $\mathfrak{N}$ has the uniqueness property. This is one proof more for the statement that in spaces of sequences the theory of determinants is simpler.

We can try to omit the phenomenon of non-uniqueness also in the following way.

We have seen in § 11 that any determinant system for $A = I+T$ is uniquely determined by T and the numbers $\sigma = \operatorname{Tr}\mathcal{T} = \mathcal{T}(I)$, $\sigma_2 = \operatorname{Tr}\mathcal{T}^2 = \mathcal{T}(T)$, $\mathcal{T}$ being a quasinucleus of T.

In general, σ_2 is not uniquely determined by T, but it is very easy to restrict the class of quasinuclear operators so that it will be uniquely determined by T. For this purpose it suffices to take into consideration only those quasinuclear operators which are a limit in norm of a sequence of finitely dimensional operators, i. e. operators $T \in \mathfrak{Q}\mathfrak{O} \cap \mathfrak{C}_0\mathfrak{O}$. The last class contains the class $\mathfrak{N}\mathfrak{O}$ of nuclear operators but, in general, it is wider than $\mathfrak{N}\mathfrak{O}$.

The number σ_1 is never uniquely determined by T if no additional hypotheses on admitted quasinucleus of T are assumed. σ_1 is the trace of $\mathcal{T}$ but, we know from the theory of determinants of finite or infinite square matrices (see § 14), σ_1 should be the trace of the operator T. Consequently we have to investigate operators for which it is possible to define, in a natural way, the notion of trace.

We have earlier defined the trace of finitely dimensional operators (§ 2 (2)). Unfortunately, the trace is not a continuous functional on $\mathfrak{F}\mathfrak{O}$ (with respect to the ordinary norm of an operator). Therefore it cannot be extended by continuity over a larger class of operators.

However, we can try to define the notion of trace as follows: Suppose that the canonical mapping $\mathcal{T} \to T$ has the following property:

(u) if $\mathcal{T}_0$ is a nucleus of the zero operator, then $\operatorname{Tr}\mathcal{T}_0 = 0$.

Then all nuclei $\mathcal{T}$ of a fixed nuclear operator T have the same trace. Their common value will be called the *trace* of T and denoted by $\operatorname{tr} T$.

Of course the uniqueness property of $\mathfrak{N}$ implies the condition (u). On the other hand, it is evident that for our purpose it suffices to assume only the hypothesis (u). For if T is nuclear, then the numbers σ_1 and σ_2 are uniquely determined by T. Thus the determinant system for $I+T$ does not depend on the choice of any nucleus of T, it is uniquely determined by T only.

Unfortunately, the hypothesis of whether condition (u) always holds is as difficult as the uniqueness problem of $\mathfrak{N}$. Grothendieck [2] has proved that both problems are equivalent.

However, for some special nuclear operators T it is possible to define the trace of T in another way, e. g. as the sum of eigenvalues of T (44). So we can get the uniqeness of the definition of the determinant system for $I+T$ for some special nuclear operators T.

Note the following result of Grothendieck [2]. Every nucleus $\mathcal{T}$ can be written in the form

(1) $$\mathcal{T} = \sum_{i=1}^{\infty} t_i\, \xi_i \otimes x_i,$$

where

(2) $$|\xi_i| = 1 = |x_i| \quad \text{and} \quad \sum_{i=1}^{\infty} |t_i| < \infty.$$

Moreover

(3) $$|\mathcal{T}| = \inf \sum_{i=1}^{\infty} |t_i|,$$

where "inf" is extended over all representations of nucleus $\mathcal{T}$ in the form (1), (2).

For $0 < p \leqslant 1$, denote by $\mathfrak{N}_p$ the class of all nuclei $\mathcal{T}$ which can be represented in the form (1) where

(4) $$|\xi_i| = 1 = |x_i| \quad \text{and} \quad \sum_{i=1}^{\infty} |t_i|^p < \infty.$$

Let

(5) $$|\mathcal{T}|_p = \inf \sum_{i=1}^{\infty} |t_i|^p,$$

where "inf" is extended over all representations of $\mathcal{T} \in \mathfrak{N}_p$ in the form (1), (4). Of course, $\mathfrak{N}_1 = \mathfrak{N}$ and $|\mathcal{T}|_1 = |\mathcal{T}|$. For $0 < p < 1$, $\mathfrak{N}_p$ is a proper linear subset of $\mathfrak{N}$ (except the case where X is finitely dimensional). $\mathfrak{N}_p$ is a metric linear space but not a Banach space since

$$|\mathcal{T}_1 + \mathcal{T}_2|_p \leqslant |\mathcal{T}_1|_p + |\mathcal{T}_2|_p \quad \text{and} \quad |c\mathcal{T}|_p = |c|^p |\mathcal{T}|_p$$

for $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T} \in \mathfrak{N}_p$.

THEOREM 31. *If* $0 < p \leqslant 2/3$, *the canonical mapping is one-to-one on* $\mathfrak{N}_p$ (45).

(44) See Grothendieck [4].
(45) Grothendieck [2], Chapter II, p. 18.

Observe also that if we resignate from the condition D) (§ 7, p. 160), we can very easily assign uniquely, to every operator $A = I+T$ ($T \in \mathfrak{Q}\mathfrak{D}$), a determinant system by some simple analytic formulae. Viz. it follows from § 11 that the expression

$$(6) \qquad D_n^{(2)}(T) = D_n(\mathcal{T})\cdot\exp(-\mathrm{Tr}\mathcal{T}+\tfrac{1}{2}\mathrm{Tr}\mathcal{T}^2) \qquad (n = 0, 1, 2, \ldots)$$

does not depend on the choice of the quasinucleus $\mathcal{T}$ of T, it depends only on T. The sequence

$$(7) \qquad D_0^{(2)}(T),\ D_1^{(2)}(T),\ D_2^{(2)}(T), \ldots$$

is a determinant system for $I+T$.

If T is nuclear (or, more generally, if T is quasinuclear and $T \in \mathfrak{C}_0\mathfrak{D}$), then we can replace formula (6) by the following ones:

$$(8) \qquad D_n^{(1)}(T) = D_n(\mathcal{T})\cdot\exp(-\mathrm{Tr}\mathcal{T}) \qquad (n = 0, 1, 2, \ldots).$$

The expression (3) does not depend on the choice of the nucleus $\mathcal{T}$ of T, and the sequence

$$(9) \qquad D_0^{(1)}(T),\ D_1^{(1)}(T),\ D_2^{(1)}(T), \ldots$$

is a determinant system for $I+T$.

However, in the case of finitely dimensional spaces, the sequences (7) and (9) do not coincide with the algebraic determinant system for $I+T$. They do not have all the properties of the determinant system $D_0(\mathcal{T}), D_1(\mathcal{T}), D_2(\mathcal{T}), \ldots$ defined in §§ 8 and 11 (e. g. Theorem 18 on multiplication of determinants does not hold) but, of course, they can be applied to solve linear equations by means of Theorem 3.

Observe that $D_n^{(2)}(T)$ can be represented in the form

$$(10) \qquad D_n^{(2)}(T) = \sum_{m=0}^{\infty} \frac{1}{m!}\theta_{n,m}^{(2)}(T),$$

where $\theta_{n,m}^{(2)}(T)$ are given by the formulae § 11 (6) (7) with the following modification: the symbols σ_1 and σ_2 are everywhere replaced by 0. For $n > 2$, the number σ_n is uniquely determined by T only, and it should be called the trace of T^m.

Similarly, if T is nuclear (or, more generally, T is quasinuclear and $T \in \mathfrak{C}_0\mathfrak{D}$), then

$$(11) \qquad D_n^{(1)}(T) = \sum_{m=0}^{\infty} \frac{1}{m!}\,\theta_{n,m}^{(1)}(T),$$

where $\theta_{n,m}^{(1)}(T)$ are given by the formulae § 11 (6) (7) with the following modification: the symbol σ_1 is everywhere replaced by 0. For $n > 1$, the number σ_n is uniquely determined by T only, and it should be called the trace of T^m. $\theta_{n,m}^{(1)}(T)$ can be also defined by formula § 8 (3) with the following modification: in the determinant θ_{n+m} (on the right side of § 8 (3)) the expressions $\xi_{n+1}x_{n+1}, \ldots, \xi_{n+m}x_{n+m}$ should be replaced by 0 ([46]).

In the case where $X = \Xi =$ a Hilbert space, the notions of a nuclear operator and a quasinuclear operator coincide. An operator is nuclear if and only if it is the product (superposition) of two Hilbert-Schmidt operators. We recall that T is a Hilbert-Schmidt operator if and only if, interpreting X as a space $\boldsymbol{L}^2(\Gamma, \mu)$, T is an integral operator with a kernel $\tau(s, t)$ such that

$$(12) \qquad |||T||| = \sqrt{\int\int |\tau(s,t)|^2\,ds\,dt} < \infty.$$

In the case where $X = \Xi =$ a Hilbert space, the series (11) converges for all Hilbert-Schmidt operators T and gives a determinant system for $A = I+T$, i. e. for the integral equations

$$(13) \qquad \begin{aligned} x(s) + \int \tau(s, t)x(t)\,dt &= x_0(s), \\ \xi(t) + \int \xi(s)\tau(s, t)\,ds &= \xi_0(t) \end{aligned}$$

with a kernel $\tau(s, t)$ satisfying (12). So we obtain Carleman's determinant theory ([47]) of the integral equations (13). More precisely, Carleman's original determinant and subdeterminants coincide with the expressions

$$D_n^{(1)*}(T) = D_n^{(1)}\begin{pmatrix} \xi_1 T, \ldots, \xi_n T \\ x_1, \ldots, x_n \end{pmatrix} = D_n^{(1)}\begin{pmatrix} \xi_1, \ldots, \xi_n \\ Tx_1, \ldots, Tx_n \end{pmatrix}$$

(see the analogous formula in § 12).

The above example shows that sometimes it is useful to investigate the determinant systems (7) i (9) instead of the determinant system defined in § 8.

§ 16. Determinants and eigenvalues. Suppose now that X, Ξ are complex Banach spaces. Let $T \in \mathfrak{Q}\mathfrak{D}$ and let $\mathcal{T}$ be a quasinucleus of T. Thus $D(\mathcal{I}+\lambda\mathcal{T})$ (see the notation in § 11, p. 172) is the determinant for the operator $I+\lambda T$. It follows immediately from § 8 (7) that

([46]) Following an idea of Hilbert [1] in the determinant theory of integral equations. An analogous modification should be performed in formulae quoted in Theorems 21 and 22 in the integral model of the theory.

([47]) Carleman [1] (see also Hille and Tamarkin [1, 2] and Smithies [1], [2]). The general determinant theory in abstract Hilbert spaces is the subject of papers: Sikorski [7, 8]. See also Fuglede and Kadison [1, 2].

THEOREM 32. *$D(\mathcal{I}+\lambda\mathcal{T})$ is an entire function (of the complex variable λ) of an order $\leqslant 2$.*

Let $\lambda_1, \lambda_2, \ldots$ be the sequence formed of all roots λ of the equation $D(\mathcal{I}+\lambda\mathcal{T}) = 0$ (every root repeated according to its multiplicity). Since $D(\mathcal{I}+\lambda_i\mathcal{T}) = 0$, the equation $(I+\lambda_i T)x = 0$ has a non trivial solution (see Theorem 3), i. e. the number

$$\nu_j = -\frac{1}{\lambda_j}$$

is an eigenvalue of the operator T.

It follows from Theorem 32 that

$$\sum_j |\nu_j|^q = \sum_i \frac{1}{|\lambda_j|^q} < \infty \tag{1}$$

for every $q > 2$, and

$$D(\mathcal{I}+\lambda\mathcal{T}) = \exp(\alpha\lambda+\beta\lambda^2)\prod_j\big((1+\lambda\nu_j)\exp(-\lambda\nu_j+\tfrac{1}{2}\lambda^2\gamma_j^2)\big). \tag{2}$$

The problem of convergence of the series (1) for $q \leqslant 2$ has not been examined for arbitrary quasinucleus $\mathcal{T}$. It was examined by Grothendieck [2] in the case where $\mathcal{T}$ is a nucleus ($\Xi = X^*$). Note the main results:

THEOREM 33 (48). *If $\mathcal{T}$ is a nucleus, then*

$$\sum_j |\nu_j|^2 < \infty \tag{3}$$

and consequently

$$D(\mathcal{I}+\lambda\mathcal{T}) = \exp(\lambda \operatorname{Tr}\mathcal{T})\prod_j\big((1+\lambda\nu_j)\exp(-\lambda\nu_j)\big). \tag{4}$$

If, moreover,

$$\sum_j |\nu_j| < \infty, \tag{5}$$

then

$$D(\mathcal{I}+\lambda\mathcal{T}) = \exp(a\lambda)\prod_j(1+\lambda\nu_j), \tag{6}$$

where

$$a = \operatorname{Tr}\mathcal{T} - \sum_j \nu_j. \tag{7}$$

(48) Grothendieck [2], Chapter II, p. 17-20.

If $\mathcal{T} \in \mathfrak{N}_p$ $(0 < p \leqslant 1)$, then $D(\mathcal{I}+\lambda\mathcal{T})$ is an entire function of an order $\leqslant r$, where

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{2},$$

and of the type 0 for $p \leqslant 2/3$. Moreover

$$\Big(\sum_i |\nu_i|^r\Big)^{p/r} \leqslant |\mathcal{T}|_p. \tag{8}$$

If $\mathcal{T} \in \mathfrak{N}_p$, where $0 < p \leqslant 2/3$, then (5) holds, and $a = 0$. Thus

$$D(\mathcal{I}+\lambda\mathcal{T}) = \prod_j (1+\lambda\nu_j), \tag{9}$$

$$\operatorname{Tr}\mathcal{T} = \sum_j \nu_j. \tag{10}$$

The problem whether $a = 0$ (see (7)) for every $\mathcal{T} \in \mathfrak{N}$ satisfying (5) is not solved.

The next theorem treats the case where T has no non-zero eigenvalue.

THEOREM 34 (49). *The following conditions are equivalent:*

(i) *$D(\mathcal{I}+\lambda\mathcal{T})$ does not vanish for any λ;*

(ii) *$\operatorname{Tr}\mathcal{T}^n = 0$ for $n = 2, 3, \ldots$;*

(iii) $\lim\limits_{n\to\infty} |\mathcal{T}^n|^{1/n} = 0$;

(iv) $\lim\limits_{n\to\infty} |T^n|^{1/n} = 0$.

If $T \in \mathfrak{N}_p$, where $0 < p < 2/3$, then each of the conditions (i)-(iv) *implies*

$$\operatorname{Tr}\mathcal{T} = 0.$$

The problem whether the last part of Theorem 34 is valid without the hypothesis $0 < p \leqslant 2/3$ is not solved.

List of symbols

(49) Grothendieck [2], Chapter I, p. 117, and Chapter II, p. 19-20.

List of terms

REFERENCES

M. Altman [1] *The Fredholm theory of linear equations in locally convex linear topological spaces*, Bulletin de l'Académie Polonaise des Sciences, Classe III. 2 (1954), p. 267-269.

T. Carleman [1] *Zur Theorie der linearen Integralgleichungen*, Mathematische Zeitschrift 9 (1921), p. 196-217.

I. Fredholm [1] *Sur une classe de transformations rationnelles*, Comptes Rendus, Paris, 134 (1902), p. 219-222.

— [2] *Sur une classe d'équations fonctionnelles*, ibidem 134 (1902), p. 1561-1564.

— [3] *Sur une nouvelle méthode pour résolution du problème de Dirichlet*, Kong. Vetenskaps-Akademiens Förh., Stockholm 1900, p. 39-46.

— [4] *Sur une classe d'equations fonctionnelles*, Acta Mathematica 27 (1903), p. 365-390.

B. Fuglede and R. V. Kadison [1], *On determinants and a property of the trace in finite factors*, Proceedings of the National Academy of Sciences of USA 37 (1951), p. 425-431.

— [2] *Determinant theory in finite factors*, Annals of Mathematics 55 (1952), p. 520-530.

A. Grothendieck [1] *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liée à cette notion*, Comptes Rendus, Paris, 233 (1951), p. 1556-1558.

— [2] *Produits tensoriels topologiques et espaces nucléaires*, Memoires of the American Mathematical Society 1955.

— [3] *La théorie de Fredholm*, Bulletin de la Société Mathématique de France 84 (1956), p. 319-384.

— [4] *The trace of some operators*, Studia Mathematica 20 (1961), p. 141-143.

— [5] *Résumé de la théorie métrique des produits tensoriel topologiques*, Boletim da Sociedade de Matemática de São Paulo 8 (1956), p. 1-79.

E. Hellinger und O. Toeplitz [1] *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, Encyklopädie der mathematischen Wissenschaften, 2(3)2 (1923/27), p. 1335-1601.

D. Hilbert [1] *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig-Berlin 1912.

E. Hille and R. S. Phillips [1] *Functional analysis and semi-groups*, Providence 1957.

E. Hille and J. D. Tamarkin [1] *On the characteristic values of linear integral equations*, Proceedings of the National Academy of Sciences 14 (1928), p. 911-914.

— [2] *On the characteristic values of linear integral equations*, Acta Mathematica 57 (1931), p. 1-76.

A. Hoborski [1] *Über zwei Sätze von Platrier*, Archiv der Mathematik und Physik (3) 23 (1914), p. 297-302.

W. A. Hurwitz [1] *Note on the Fredholm determinant*, Bulletin of the American Mathematical Society (2) 20 (1914), p. 406-408.

H. von Koch [1] *Sur les determinants infini et les équations différentielles linéaires*, Acta Mathematica 16 (1892), p. 217-295.

— [2] *Sur les équations différentielles linéaires à coefficients rationnels*, Comptes Rendus de l'Académie des Sciences, Paris, 106 (1893), p. 91-93.

— [3] *Sur les intégrales régulières des équations différentielles linéaires*, Acta Mathematica 18 (1894), p. 337-419.

— [4] *Sur la convergence des déterminants d'ordre infini*, Bihang till Kongl. Svenska Vetenskaps-Academiens Handlingar 22 (1896), p. 1-31.

— [5] *Sur quelques points de la théorie des déterminants infinis*, Acta Mathematica 24 (1900), p. 89-122.

— [6] *Sur la convergence des déterminants infinis*, Rendiconti dei Circolo Matematico di Palermo 28 (1909), p. 255-266.

— [7] *Sur les systèmes d'une infinité d'équations linéaires à une infinité d'inconnues*, Comptes Rendus du Congrès de Stockholm 1910, p. 43-61.

T. Lalesco [1] *Introduction à la théorie des équations intégrales*, Paris 1912.

T. Leżański [1] *The Fredholm theory of linear equations in Banach spaces*, Studia Mathematica 13 (1953), p. 244-276.

— [2] *Sur les fonctionnelles multiplicatives*, ibidem 14 (1953), p. 13-23.

A. D. Michel and R. S. Martin [1] *Some expansions in vector space*, Journal des Mathématiques Pures et Appliquées 13 (1934), p. 69-91.

Ch. Platrier [1] *Sur les mineurs de la fonction déterminante de Fredholm et sur les systèmes d'équations intégrales linéaires*, ibidem 9 (1913), p. 233-304.

J. Plemelj [1] *Über die Anwendung der Fredholmschen Funktionalgleichung in der Potentialtheorie*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Abt. IIa, 1903.

— [2] *Zur Theorie der Fredholmschen Funktionalgleichung*, Monatshefte für Mathematik und Physik 15 (1904), p. 93-128.

H. Poincaré [1] *Remarques diverses sur l'équation de Fredholm*, Acta Mathematica 33 (1910), p. 57-86.

F. Riesz [1] *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris 1952.

F. Riesz et B. Sz.-Nagy [1] *Leçons d'analyse fonctionnelle*, Budapest 1953.

A. F. Ruston [1] *On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space*, Proceedings of the London Mathematical Society (2) 53 (1951), p. 109-124.

— [2] *Direct product of Banach spaces and linear functional equations*, ibidem (3) 1 (1951), p. 327-384.

— [3] *Formulae of Fredholm type for compact linear operators on a general Banach spaces*, ibidem (3) 3 (1953), p. 368-377.

— [4] *Operators with a Fredholm theory*, Journal of London Mathematical Society 29 (1954), p. 318-326.

R. Schatten [1] *A theory of cross spaces*, Annals of Mathematical Studies 1950.

— [2] *The cross-space of linear transformations*, Annals of Mathematics 47 (1946), p. 73-84.

R. Sikorski [1] *On multiplication of determinants in Banach spaces*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 1 (1953), p. 219-221.

— [2] *On Leżański's determinants of linear equations in Banach spaces*, Studia Mathematica 14 (1953), p. 24-48.

— [3] *On determinants of Leżański and Ruston*, ibidem 16 (1957), p. 99-112.

— [4] *Determinant systems*, ibidem 18 (1959), p. 161-186.

— [5] *On Leżański endomorphisms*, ibidem 18 (1959), p. 187-188.

— [6] *Remarks on Leżański's determinants*, ibidem 20 (1961), p. 145-161.

— [7] *The determinant theory of the Carleman type*, Bulletin de l'Académie Polonaise des sciences, Série des sci math., phys. et astr. 8 (1960), p. 685-689.

— [8] *On the Carleman determinants*, Studia Mathematica 20 (1961), p. 327-346.

F. Smithies [1] *The Fredholm theory of integral equations*, Duke Mathematical Journal 8 (1941), p. 107-130.

— [2] *Integral equations*, Cambridge 1958.

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ON THE IMPOSSIBILITY OF EMBEDDING OF THE SPACE L IN CERTAIN BANACH SPACES

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In this note we shall prove the following

THEOREM. *The space L of all absolutely summable real-valued functions defined on the interval* $[0,1]$ *is not isomorphic to a subspace of a separable B-space X^* conjugate to the B-space X, as well as to a subspace of a B-space with an unconditional basis.*

The impossibility of the embedding of the space L in a conjugate separable B-space was first proved by Gelfand (see [4], p. 265). Recently a similar proof was given by Dieudonné [2]. The arguments of Gelfand and Dieudonné are based on the representation of linear operators in L by kernels. Our proof is quite different and is based on the fact that for every perfect set $T\subset[0,1]$ of a positive Lebesgue measure there is a bounded measurable function which is equivalent to no function belonging to the first Baire class on T. The alternative proofs that the space L has no unconditional basis are given in [5] and [6].

Remark. All results in this paper remain valid if we replace the space L by the space $L(Q,\mu)$, where μ is a non-purely atomic measure defined on the σ-field of all Borel sets in a compact metric space Q, and $L(Q,\mu)$ denotes the space of all absolutely summable real-valued functions f defined on Q, under the norm $\|f\| = \int_Q |f(q)|\,\mu(dq)$.

By L^∞ we shall denote the space of all real-valued essentially bounded functions φ defined on the interval $[0,1]$ with the norm $\|\varphi\| = \operatorname{ess\,sup}_{t\in[0,1]}|\varphi(t)|$. In the sequel by measure of a set $T\subset[0,1]$ we shall mean the Lebesgue measure of this set and we shall denote it by $\operatorname{mes} T$.

LEMMA 1. *Let E be a separable subspace of L^∞. Then there is a perfect set T with positive measure such that every function φ in E is equivalent to a function φ, the restriction of which is continuous on T.*