and hence, by (3),
\[ |f - k|_0 = \int \frac{|f(x) - k(x)|}{|x|} \, dx \geq |I(f)| = \int |f(x)| \, dx. \]

This proves that \( |f|_0 = \text{dist}(f, K) \). Now, by (3), \( |f|_0 = |g|_{0, H} \) and hence \( |g|_{0, H} = x |g|_{0, H} \).

Proof of (b). By the triangle inequality it suffices to verify that
\[ x|g|_{0, H} - x |g|_{0, H} \geq |g|_{0, H} + |x|_{0, H}. \]

Let \( r, s \in L^1(\mathbb{G}) \) be such that \( r = g, s = h \), where the supports \( S_r, S_s \) satisfy \( S_r \cap S_s = \emptyset \). Then the inequality we wish to prove is equivalent to
\[ \text{dist}(r + t, K) \geq \text{dist}(r, K) + \text{dist}(t, K). \]

It is easily seen that if \( k \in K \), then the restricted functions \( k^n = k |S_nH \) and \( k^n = k |S_nH \) also belong to \( K \), and this implies that
\[ (r + t)^n = \int (|r - x^n|^n + |t - x^n|^n + |k^n - x^n|^n + x^n) \geq (|r - x^n|^n + |t - x^n|^n) \geq \text{dist}(r, K) + \text{dist}(t, K). \]

Hence (5) follows and the proof of Theorem 4 is complete.

REFERENCES


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The multiplication $xy$ cannot be extended onto $X$ in such a way that $X$ is a Banach algebra if and only if for every $\varepsilon > 0$ there exist such $x, y \in R$ that

$$
||x|| < \varepsilon, \quad ||y|| < \varepsilon \quad \text{and} \quad ||xy|| \geq C > 0.
$$

Now we pass to the easy case.

Proof of theorem 1. If the group $G$ is compact, then we may assume that $\mu(G) = 1$. Hence (see [2], p. 156).

$$
||f||_p \leq ||f||_s, \quad 1 \leq p,
$$

for every complex function defined on $G$, and

$$
R = L_p \cap L_1 \subseteq L_p,
$$

We also have (see [1], p. 121–122)

$$
||fg||_p \leq ||f||_p ||g||_p \quad \text{for every} \quad f \in L_1, \quad g \in L_p.
$$

And by (2), (3), and (4) we get

$$
||fg||_p \leq ||f||_p ||g||_p \quad \text{for} \quad f, g \in R.
$$

Hence, by the remark (A), $L_p$ is a Banach algebra, q. e. d.

Proof of theorem 2. At the beginning we assume that $\mu(U) = 1$ for every $t \in G$ (i.e. $G$ is discrete). We shall prove the following remark:

(B) If $1 < p < \infty$, and $L_p$ is a Banach algebra, then the group $G$ is finite.

Indeed, in this case the algebra $L_p$ is a commutative Banach algebra with the unit element (the unity of algebra $L_p$ is the characteristic function of the unit element of the group $G$), and by the Gelfand theory there exists a non-zero multiplicative linear functional $F$. The functional $F$ may be written in the form $F(f) = \int f(t)\gamma(t) dt$, where the function $\gamma$ is a member of $L_p$, $1/p + 1/q = 1$. On the other hand, for the discrete group we have $L_1 \subseteq L_p$. The functional $F$ restricted to $L_1$ may be written in the form $F(f) = \int f(t)\chi(t) dt$ for $f \in L_1$, where $\chi$ is the character of the group $G$ [3]. It follows that $\chi = \gamma \in L_1$. But if $p > 1$, then $q < \infty$, and therefore the group $G$ must be finite, because $||\gamma||_q^2 = ||\chi(t)||^2 = \mu(U) < \infty$.

By (B) and (A') we get the following remark:

(B') If the discrete abelian group $G$ is infinite, then for a given $p$, $1 < p < \infty$, and for every $\varepsilon > 0$, there exist two functions $x, y$ with a compact (i.e. finite) support and a $C > 0$, such that for the norm $||x||_p$, inequalities (1) hold.

Now let $G$ be an arbitrary locally compact abelian group. Let $V$ be any compact neighbourhood of the unit $e$ of $G$. Let $G_0$ be the sub-group of $G$ generated by $V$. The structure of $G_0$ is well known by a certain theorem of Pontriagin (see [2], p. 274). It follows from this theorem that $G_0$ is either compact or it contains such an element $x$ that the subgroup generated by $x$ is discrete and infinite. We shall discuss these two cases.

1° Let $G_0$ be a compact subgroup of $G$. Then $\mu(G_0) < \infty$. On the other hand, $\mu(G_0) \geq \mu(V) > 0$, and we may assume that $\mu(G_0) = 1$. Let $L_p$ be the Banach algebra of the group $G$. We shall consider the subalgebra $L_p \cap L_1$ of all $f \in L_p$ which are constant on the cosets $tG_0, t \in G$. The subalgebra $L_p$ is isometric with the algebra $L_p$, of the discrete group $G/G_0$, and by the remark (B) the number of cosets must be finite, whence the group $G$ must be compact.

2° Let $a \in G_0$, and the subgroup generated by $a$ be discrete. It is to be shown that $L_p$ is not a Banach algebra if $1 < p < \infty$. It may easily be proved that in $G$ there exists a symmetric neighbourhood $U$ of the unit $e$ such that for integer $m, n$

$$
a^m U^2 \cap a^n U^2 = \emptyset \quad \text{if} \quad m \neq n,
$$

where $\emptyset$ denotes the void set.

We may assume that $\mu(U) \leq 1$.

Let $(\alpha_n), (\beta_n), n = 0, \pm 1, \pm 2, \ldots,$ be two finite sequences of complex numbers such that

$$
\left( \sum |\alpha_n|^p \right)^{1/p} < \varepsilon, \quad \left( \sum |\beta_n|^p \right)^{1/p} < \varepsilon, \quad \left( \sum |\gamma_n|^p \right)^{1/p} = 1,
$$

where

$$
\gamma_n = \sum a_{-n} \beta_n,
$$

their existence, for every $\varepsilon > 0$, is proved by the remark (B').

We put

$$
f(t) = \sum a_n \chi_{G_0} \tau_n(t), \quad g(t) = \sum \beta_n \chi_{G_0} \tau_n(t),
$$

where

$$\chi_{G_0} = \begin{cases} 1 & \text{for} \; t \in A, \\ 0 & \text{for} \; t \notin A. \end{cases}
$$

We have using (5)

$$
||f||_p = \left( \int |\sum a_n \chi_{G_0} \tau_n(t) |^p dt \right)^{1/p} = \left( \int |\sum \alpha_n |^p \mu(U)^{1/p} \right)^{1/p} < \varepsilon
$$

and

$$
||g||_p < \varepsilon.
$$
On the other hand

\[ h(t) = f \ast g = \sum_{k} a_{n} \beta_{k} \int_{\mathbb{R}^{n}} \mu_{U} \left( \mathbb{R}^{n} \right) \chi_{a_{n} \mu_{U} \left( \mathbb{R}^{n} \right)}(t) \, dt \]

\[ = \sum_{k} a_{n} \beta_{k} \mu_{U} \left( \mathbb{R}^{n} \right) \mathbb{R}^{n} \right) \chi_{a_{n} \mu_{U} \left( \mathbb{R}^{n} \right)}(t) \, dt \]

\[ = \sum_{k} a_{n} \beta_{k} \mu_{U} \left( \mathbb{R}^{n} \right) \mathbb{R}^{n} \right) \chi_{a_{n} \mu_{U} \left( \mathbb{R}^{n} \right)}(t) \, dt \]

\[ = \sum_{k} a_{n} \beta_{k} \mu_{U} \left( \mathbb{R}^{n} \right) \mathbb{R}^{n} \right) \chi_{a_{n} \mu_{U} \left( \mathbb{R}^{n} \right)}(t) \, dt \]

\[ = \sum_{k} a_{n} \beta_{k} \mu_{U} \left( \mathbb{R}^{n} \right) \mathbb{R}^{n} \right) \chi_{a_{n} \mu_{U} \left( \mathbb{R}^{n} \right)}(t) \, dt \]

But, as is well known, the function \( \varphi(t) = \mu(U \cap U) \) is continuous, and \( \varphi(e) = \mu(U) \). Therefore there exists such a neighbourhood \( V \) of \( e \) that

\[ \varphi(t) \geq \frac{\mu(U)}{2} \quad \text{for} \quad t \in V. \]

Hence

\[ \mu(a^{n}U \cap U) \geq \frac{\mu(U)}{2} \quad \text{for} \quad t \in e^{-n}V. \]

By (5) the functions \( \varphi_{k}(t) = \varphi(a^{n}t) \) have disjoint supports and by (6), (7), and (8) we have

\[ \| h \|_{p} = \left\| \sum_{k} \varphi_{k}(t) \right\|_{p} = \left\{ \left( \int \left| \sum_{k} \varphi_{k}(t) \right|^{p} \, dt \right)^{1/p} \right\}
\]

\[ = \left\{ \sum_{k} \int \left| \varphi_{k}(t) \right|^{p} \, dt \right\}^{1/p} \]

\[ \geq \left\{ \sum_{k} \int \left| \varphi_{k}(t) \right|^{p} \, dt \right\}^{1/p} \]

\[ \geq \left\{ \sum_{k} \int \left( \varphi_{k}(t) \right)^{p} \, dt \right\}^{1/p} \]

\[ \geq \left\{ \sum_{k} \int \left( \varphi(t) \right)^{p} \, dt \right\}^{1/p} \]

\[ = \left\{ \int \left( \varphi(t) \right)^{p} \, dt \right\}^{1/p} \]

\[ = \left\{ \int \left( \varphi(t) \right)^{p} \, dt \right\}^{1/p} = \frac{\mu(U)}{2} \mu(V)^{1/p} > 0. \]

Hence, by (A'), \( L_{p} \) is not a Banach algebra, q. e. d.

In case \( p = 2 \) the proof of theorem 2 may be obtained in another way, where the form of the group \( G \) need not be discussed. To this aim let us assume that the space \( L_{2} \) of locally compact abelian group \( G \) is a Banach algebra with convolution. By Boel's inequality (see [3], p. 45, formula (6.1)) we get

\[ V \frac{\left\| f \right\|_{2}}{2} \geq \left\| \frac{f_{1}}{2} \right\|_{2} \geq \frac{\left\| f \right\|_{2}}{2}, \]

where \( f_{1} = f_{2} + \ldots + f_{n} \),

which holds for every function from \( L_{2} \) satisfying \( f(t^{-1}) = f(t) \), \( t \in G \), e. g. for a characteristic function of a compact symmetric neighborhood of \( e \). Hence algebra \( L_{2} \) is not radical, and there exists a non-zero multiplicative linear functional \( F \). The functional \( F \) may be written in the form \( F(f) = \int f(t) \varphi(t) \, dt \), where \( \varphi \) is a member of \( L_{1} \). By the multiplicativity of \( F \) and the Fubini theorem we have

\[ F(fg) = \int \varphi(t) \int f(t) \varphi(t) \, dt \, ds = \int \varphi(t) \varphi(s) \, dt \, ds \]

\[ = \int f(t) \varphi(t) \, dt \int \varphi(s) \, ds \]

for every \( f, g \in L_{1} \).

It follows that

\[ \varphi(p) \neq \varphi(p) \varphi(t) \quad \text{e. a.} \quad (\mu \times \mu). \]

Since \( \varphi \) is not a. e. equal to zero, there is a set \( C \subset G \) such that

\[ \int_{C} \varphi(t) \, dt = 0 \quad \text{and} \quad \int_{C} \varphi \varphi \, dt < \infty. \]

Let \( Q \subset G \) be any set of \( \sigma \)-finite measure \( \mu \) such that the \( (\sigma \text{-finite}) \) support of \( \varphi \) is contained in \( Q \). Then, by Fubini's theorem and by (9),

\[ \int_{C} \varphi \varphi \, dt < \infty. \]

So we get \( \mu(Q) = \int_{C} \varphi \varphi \, dt \int \varphi(t) \, dt. \) Hence \( \mu(Q) < \infty \), and the group \( G \) is compact.
A PROOF OF A THEOREM OF ŽELAZKO ON $l^p$-ALGEBRAS

by

K. URBANIK (WROCŁAW)

Let $G$ be a locally compact Abelian topological group. For each $p \geq 1$, we define the space $l^p(G)$ as the space of all measurable complex-valued functions $f$ on $G$ such that $|f|^p$ is integrable with respect to the Haar measure $m$ on $G$. Obviously, $l^p(G)$ is a Banach space under the norm

$$
\|f\|_p = \left( \int |f(x)|^p m(dx) \right)^{1/p}.
$$

In the sequel we shall denote by $fg$ the convolution of functions $f$ and $g$, i.e.

$$
(fg)(x) = \int f(y) g(xy^{-1}) m(dx) \quad (a \ast b).
$$

In this note we shall give a simple proof of the following theorem, proved by W. Żelazko in paper [3]:

If, for a number $p > 1$, $l^p(G)$ is a topological ring under the convolution multiplication, then $G$ is a compact group.

Proof. Let $R$ be the extension of $L_1(G)$ to a topological ring with a unit element (see [1], p. 158). The norm in $R$, which is an extension of the norm in $l^p(G)$, will henceforth be denoted by $\| \|_p$. It is well known that the norm

$$
\|f\| = \sup_{a \in R, |a|_{l^p} = 1} |fg|_p
$$

makes $R$ a normed ring (see [1], p. 168).

First we shall prove that the ring $l^p(G)$ admits a non-trivial continuous homomorphism into the complex field. To prove this it is sufficient to show that $l^p(G)$ contains an element which does not belong to the radical of $R$. 