n the first case, and
\[ \sum_{k=n+1}^{n} \|y_k\| = h_n \|x_n\| = h_n F(t_n, x_n) \|x_n\| \geq \frac{1}{2} \cdot 4^n \cdot 2^n = 2^{n+1} \]
because \( F(t_n, x_n) \geq F(t_n, x_n) \), in the second case.

Hence the series \( \sum_{n=1}^\infty y_k \) is not absolutely convergent in both cases.

REFERENCES


Reçu par la Réouverture le 33, 3, 1960

INTEGRALS ON QUOTIENT SPACES

By

S. ŚWIERCKOWSKI (WROCŁAW)

NOTATION AND SUMMARY

If \( G \) is a locally compact topological group and \( H \) is a closed subgroup, then every integral \( f \) on the quotient space \( G/H \) is associated with exactly one integral \( \bar{f} \) on \( G \) (cf. formula (2) below). The class of integrals on \( G \) which are of the form \( \bar{f} \) will be characterized in Theorems 1 and 2. It contains the Haar integral if and only if there is an invariant integral on \( G/H \) (Th. 1, Corollary). The integrals \( f \) and \( \bar{f} \) define a pair of Banach spaces \( L^1(G/H) \) and \( L^1(G) \). H. Reiter considered these spaces under the assumption that \( \bar{f} \) is the Haar integral on \( G \) where only in the case where there is an invariant integral on \( G/H \) (cf. [1]). His results will be extended in Theorems 3 and 4 to the general case where \( \bar{f} \) is an arbitrary integral on \( G/H \).

If \( X \) is a locally compact topological space, we shall denote by \( L(X) \) the class of all continuous real-valued functions on \( X \) which vanish outside compact sets. The class of extended Baire functions on \( X \) (cf. [1], [2]; these functions take also infinite values) will be denoted by \( B(X) \). \( L_+(X) \) and \( B_+(X) \) will denote the subclasses of non-negative functions. Every non-negative linear functional \( I \) on \( L(X) \) will be called an integral on \( X \) and we shall sometimes assume that the domain of definition of \( I \) includes \( L_+(X) \) or the class of all \( I \)-summable functions. The class of all integrals on \( X \) will be designated by \( I(X) \). We shall denote by \( s \bar{f} \) the support of a function \( f \) on \( X \), i.e. the set \( \{ x: f(x) \neq 0 \} \).

Now let \( G \) and \( H \) be as in the beginning. Let \( \bar{y} \) denote the coset \( xH \). For any \( f \in L(G) \) we put
\[ \bar{f} = \int f(x) \, d\bar{x}, \]
where \( \int \) is the integral with respect to the left Haar measure in \( H \). It is clear that \( \bar{f} = f \) if \( H = G \) and (see [2], sec. 33A) that \( \int f \, dG/H, \)
where the topology in \( G/H \) is the natural one. For any \( I \epsilon I(G/H) \) and \( f \epsilon L(G) \) we put

\[ \tilde{I}(f) = I(\tilde{f}). \]

Then \( \tilde{I} \epsilon I(G) \) and the mapping \( I \rightarrow \tilde{I} \) maps \( I(G/H) \) into \( I(G) \). Let

\[ \tilde{I}(G/H) = \{ \tilde{I} \epsilon I(G/H) \}. \]

As is well known ([2], sec. 33B), the mapping \( f \rightarrow \tilde{f} \) transforms \( L(G) \) onto \( L(G/H) \). Hence, by (2), the mapping \( I \rightarrow \tilde{I} \) is one-to-one. So we see that the investigation of \( I(G/H) \) may be reduced to that of \( I(G/H) \subset I(G) \). We shall first give some characteristic properties of the class \( \tilde{I}(G/H) \) (Theorem 1) and then we shall estimate the “size” of this class in \( I(G) \) (Theorem 2). For any integral \( I \) on \( G/H \) we shall consider the spaces \( L^1(G/H) \) of all \( I \)-summable and \( \tilde{I} \)-summable functions.

The mapping \( f \rightarrow \tilde{f} \) will be shown to be a bounded linear transformation of \( L^1(G) \) onto \( L^1(G/H) \) with a well-defined kernel \( K \) (Theorem 3). We shall also consider the quotient Banach space \( L^1(G/K) \) with the norm of a code defined as its distance from the origin. Then the mapping \( f \rightarrow \tilde{f} \) defines a norm-preserving isomorphism between the spaces \( L^1(G/K) \) and \( L^1(G/H) \) (Theorem 4).

**THE CLASS \( \tilde{I}(G/H) \)**

For \( f \epsilon L(G) \) and \( a \epsilon G \) we adopt the notation \( f^a(x) = f(ax^{-1}) \). We define the “translation” \( Q^a \) of an integral \( Q \) on \( G \) by \( Q^a(f) = Q(f^a) \). Let \( J \) denote the left invariant Haar integral of \( H \). The modular function \( \delta(\xi) \) of \( J \) is then defined by \( J^\xi = \delta(\xi) J(\xi \epsilon H) \).

**Theorem 1.** The following conditions are equivalent:

- (i) \( Q \epsilon \tilde{I}(G/H) \),
- (ii) \( Q(f) = 0 \) whenever \( f = 0 \) (\( f \epsilon L(G) \)),
- (iii) \( Q^\xi = \delta(\xi) Q \) for each \( \xi \epsilon H \),
- (iv) \( Q(f) = Q(f^g) \), when \( f \epsilon L(G) \).

If \( Q \) is the Haar integral on \( G \) and \( A \) denotes the modular function for \( Q \), then \( Q^a = A(a)Q \) for each \( a \epsilon G \). Thus, by (iii), we infer that the Haar integral belongs to \( \tilde{I}(G/H) \) iff \( A \) and \( \delta \) coincide on \( H \). Hence, by Weil’s condition (cf. [5]) the

\[ Q_{\xi}J_{\xi}(g(x, \xi)) = Q_{\xi}J_{\xi}(g(x, \xi)). \]

Also the well-known formula \( J_{\xi}(h(\xi)) = J_{\xi}(h(\xi^{-1}) \delta(\xi^{-1})) \) will be applied. We have

\[ Q(f) = Q_a[f(a)J_a(g(a))] = Q_{\xi}J_{\xi}[f(a)g(a)], \]

and this, by (iii), is equal to

\[ J_{\xi}(\xi^{-1})Q_{\xi}[f(\xi^{-1})g(a)] = Q_{\xi}[g(a)J_{\xi}[f(\xi^{-1})g(a)]]. \]

(iv) \( \rightarrow \) (i). Suppose that \( f \epsilon L(G) \) and \( \tilde{f} = 0 \). By (iv), it is sufficient to find a function \( g \epsilon L(G) \) such that \( f = gg \) since then \( Q(f) = Q(g) \Rightarrow Q(f) = 0. \) Such a function \( g \) exists because, by Urysohn’s Lemma, there is a function \( a \epsilon L(G/H) \) which is equal to 1 on the bounded set \( \{ \xi : f(\xi) \neq 0 \} \) and there is a function \( g \epsilon L(G) \) such that \( \tilde{g} = a. \)
Proof of theorem 2. The construction of $^*Q$ follows in lemmas A and B.

A. If $Q$ is an integral on $G$ and $p \in L_p(\mathcal{H})$, then the formula

$$Q(f) = \int \chi_H (f(x, \xi)p(x))$$

defines an integral $^*Q \cdot I(\mathcal{G}/\mathcal{H})$.

Proof of A. Since $f(x, \xi)p(x) \in L(\mathcal{G} \times \mathcal{H})$ and $J_P$, $Q_\mathcal{H}$ is an integral on $\mathcal{G} \times \mathcal{H}$ (cf. [3], sec. 163), $^*Q$ is an integral. It remains to verify that $^*Q^* = Q(\mathcal{G})$ for $\mathcal{G} \in I(\mathcal{G}/\mathcal{H})$. If $f \in L(\mathcal{G})$ and $f \in I(\mathcal{G}/\mathcal{H})$, then

$$^*Q^*(f) = \int \chi_H (f(x, \xi)p^*(x)) = Q(f),$$

for an open set of $\xi$'s. This proves that $^*Q(f) > 0$. We have thus shown that $^*Q = Q$.

By A, we have $^*Q \cdot I(\mathcal{G}/\mathcal{H}) (T \times T')$, and thus $^*Q \cdot I(\mathcal{G}/\mathcal{H})$.

THE SPACES $L^p(\mathcal{G})$ AND $L^p(\mathcal{G}/\mathcal{H})$

Let $I$ be an integral on $\mathcal{G}/\mathcal{H}$. We consider the Banach spaces $L^p(\mathcal{G})$ and $L^p(\mathcal{G}/\mathcal{H})$ under the norms

$$\|f\|_I = I(\|f\|), \quad \|g\|_{I(\mathcal{H})} = I(\|g\|).$$

THEOREM 3. The mapping $f \rightarrow \tilde{f}$, when considered on $L^p(\mathcal{G})$, is a bounded linear transformation of this space onto $L^p(\mathcal{G}/\mathcal{H})$. It is the kernel

$$K = \{k \in L^p(\mathcal{G}): \|K\|_{L^p(\mathcal{H})} = 0\},$$

of the closed linear subspace generated by all the functions

$$f(x, \xi) = f(x, \xi) - \delta(\xi) f(x),$$

where $x \in T$ and $\xi \in H$.

Let us note that the mapping $f \rightarrow \tilde{f}$ cannot be extended to $B(\mathcal{G})$ because $\tilde{f}(x, \xi) d\xi$ may not exist for some $x \in B(\mathcal{G})$.

THEOREM 4. Let

$$\text{dist}(f, K) = \sup_{k \in I(\mathcal{H})}\|f - k\| = 0,$$

and let $L^p(\mathcal{G})/K$ be the quotient space with the norm of a $K$-cost set $y$ defined as $\text{dist}(f, K)$, where $y$ is any representative of $y$ (cf. [3], sec. 63). Then the mapping $f \rightarrow \tilde{f}$ establishes a norm-preserving isomorphism between the spaces $L^p(\mathcal{G})/K$ and $L^p(\mathcal{G}/\mathcal{H})$ so that

$$\text{dist}(f, K) = \|f\|_{L^p(\mathcal{G}/\mathcal{H})}.$$
In the above theorems we have generalised the results of H. Reiter [4]. He assumed that \( \hat{f} \) is the Haar integral and that there is an invariant measure on \( G/H \). Theorems 3 and 4 include this case since, under the above assumption, the Haar integral belongs to \( \hat{L}(G/H) \) (Th. 1, Corollary).

Proof of Theorem 3. It is known that the mapping \( f \rightarrow \hat{f} \), as defined by (1), transforms \( L_\infty(G) \) onto \( L^\infty(G/H) \) ([2], sec. 33). It extends uniquely to a mapping of \( B_\infty(G) \) onto \( B_\infty(G/H) \). This follows from the fact that (1) is invariant under the formation of limits of monotone sequences and this operation is sufficient to obtain the classes \( B_\infty \). Since also (2) is invariant under these operations, we infer that

\[
\hat{I}(f) = I(\hat{f}) \quad \text{when} \quad f \in B_\infty(G),
\]

where \( \infty \) is allowed as possible value of the integrals. If \( f \in L^\infty(G) \), then both non-negative parts \( f_+ \) and \( f_- \) of \( f \) are \( \hat{I} \)-summable, and thus, by (3), \( f \in L^\infty(G/H) \). It follows that the formula \( \hat{I} = \int f \, d\mu \) defines \( \hat{I} \) as an element of \( L^\infty(G/H) \) (with the usual ambiguity at those points where the summands assume opposite values as \( \infty \) values). We have thus shown that the mapping \( f \rightarrow \hat{f} \) can be extended to \( L^\infty(G) \).

If \( f \) runs over \( B_\infty(G) \), then \( \hat{I}(f) \) runs over \( B_\infty(G/H) \) and if one of these functions is summable, then so is the other, by (3). Hence the transformation \( f \rightarrow \hat{f} \) maps \( L^\infty(G) \) onto \( L^\infty(G/H) \). Consequently \( \hat{I}(f) \) is mapped onto \( L^\infty(G/H) \).

The transformation is bounded because

\[
\|\hat{I}(f)\|_{L^\infty(G/H)} \leq \|I(f)\|_{L^\infty(G)} = \|f\|_{L^\infty(G)}.
\]

Finally, let us show that the kernel \( K \) of this transformation is the closed linear subspace \( N \subset L^\infty(G) \) which is generated by the functions \( \pi(x) \). It is clear that \( \pi = 0 \) and from the continuity of the transformation we infer that \( K \) is closed. Hence \( N \subset K \subset L^\infty(G) \). \( N \) is the class of all bounded linear functionals \( F \) which vanish on \( N \), i. e. the annihilator of \( N \), and if \( F \) is the annihilator of \( X \), then the inclusion \( K \subset N \), which we have to prove, is equivalent to \( N \subset K \subset L^\infty(G) \). Thus we have to verify that if \( f \in N \) and \( k \in K \), then \( F(k) = 0 \). We need the following

**Lemma.** If \( f \in N \), then there are integrals \( I_1 \) and \( I_2 \) on \( G/H \) such that

\[
F = I_1 - I_2
\]

and \( \|I_1 - I_2\| = 1 \).

Remark. The proof given below yields in fact the following stronger result: \( N \) is the class of functionals \( F \) which are of the form \( F = I_1 - I_2 \), where \( I_1 \) and \( I_2 \) are bounded integrals on \( G/H \) such that \( I_1, I_2 \) are finite.

Proof of the Lemma. As is well known ([2], sec. 15A), each bounded functional \( F \) on \( L^\infty(G) \) is expressible as the difference \( F = F^+ - F^- \) of two bounded functionals \( F^+ \) and \( F^- \) on \( L^\infty(G) \) (non-negative functionals), where

\[
F^+(f) = \inf \{F(g) : 0 \leq g \leq f \} \quad \text{when} \quad f \geq 0.
\]

Moreover, if \( F \in N \), then \( F^+ \) and \( F^- \) are bounded on \( G/H \). Indeed, we have

\[
F^+(f) = \inf \{F(g) : 0 \leq g \leq f \} = \inf \{F(g) : 0 \leq g \leq f \}
\]

Thus, by (4), \( F^+ \) satisfies condition (ii) of Theorem 1, and consequently also \( F^- \) satisfies this condition.

Let \( I_1, I_2 \) be integrals on \( G/H \) such that \( I_2 = F^+ \) and \( I_1 = F^- \). Since \( F^+ \) and \( F^- \) are bounded, we infer that \( \|I_2\| = \|F^+\| = \|F^-\| = 0 \), i. e. \( I_1 < I_2 \). To prove that \( I_1 < I_2 \) assume that \( I_1(g) = 0 \), where \( g \in B_\infty(G/H) \). There is a function \( f \in B_\infty(G) \) such that \( f \leq g \) and then \( F(f) = 0 \), by (3). It follows that \( I_2(f) = 0 \), and thus, again using (3), \( I_1(g) = 0 \). This proves the lemma.

Suppose now that \( F \in N \) and \( k \in K \), i. e. \( I(k) = 0 \). By the above lemma

\[
F(k) = I_1(k) - I_2(k) = I_1(\hat{k}) - I_2(\hat{k}),
\]

and both \( I_1(\hat{k}) \) and \( I_2(\hat{k}) \) vanish because \( I_1 \) is finite. Our proof is now complete.

Proof of Theorem 4. \( L^\infty(G/H) \) and \( L^\infty(G) \) are isomorphic linear spaces, by the definition of \( K \). We may therefore assume that these spaces are identical, that is to say, a \( K \)-cost with a representative \( f \in L^\infty(G) \) will be identified with \( \hat{f} \). Then we have in \( L^\infty(G/H) \) also the norm taken from \( L^\infty(G) \):

\[
\|\hat{f}\|_{L^\infty(G/H)} = \|f\|_{L^\infty(G)}.
\]

To prove our theorem we must verify that both norms in \( L^\infty(G/H) \) are identical, i. e. that

\[
\|f\|_{L^\infty(G/H)} = \|f\|_{L^\infty(G)} \quad \text{when} \quad f \in L^\infty(G/H).
\]

This is easily seen once we have shown that

\[
\|f\|_{L^\infty(G/H)} = \|g\|_{L^\infty(G/H)} \quad \text{when} \quad g \leq 0 \quad \text{or} \quad g \leq 0,
\]

(a) \( \|f\|_{L^\infty(G/H)} = \|g\|_{L^\infty(G/H)} \) when \( g \geq 0 \) or \( g \leq 0 \),

(b) \( \|g\|_{L^\infty(G/H)} = \|h\|_{L^\infty(G/H)} \) when the supports of \( g \) and \( h \) are disjoint.

Proof of (a). If \( g \in L^\infty(G/H) \), then there is a function \( f \in L^\infty(G) \) such that \( f = g \). Then

\[
\|f\|_{L^\infty(G/H)} = \|f\|_{L^\infty(G)} \quad \text{when} \quad g \in L^\infty(G/H).
\]

If \( k \in K \), then

\[
\int \hat{f}(x) - k(x) \, d\xi = \int f(x) - k(x) \, d\xi = 0.
\]

Colloquium Mathematicum VIII
and hence, by (3),
\[ |f - k|_0 = I \left( \int_{\mathbb{R}} |f(x) - k(x)| \, dx \right) \geq I(f) = \|f\|_0. \]

This proves that \( |f|_0 = \text{dist}(f, K) \). Now, by (3), \( |f|_0 = \|f\|_0 + \eta \) and hence \( \|f\|_0 = \|f\|_0 + \eta \).

Proof of (b). By the triangle inequality it suffices to verify that
\[ x\|g\|_0 + \|h\|_0 \leq x \|g\|_0 + \|h\|_0. \]

Let \( r, s \in L^1(G) \) be such that \( f = g + h \), where the supports \( S_r, S_s \) satisfy \( S_r \cap S_s = \emptyset \). Then the inequality we wish to prove is equivalent to
\[ \text{dist}(r + s, K) \geq \text{dist}(r, K) + \text{dist}(s, K). \]

(5)

It is easily seen that if \( k \in K \), then the restricted functions \( k^n = k|_{S_r} \) and \( k^n = k|_{S_s} \) also belong to \( K \), and this implies that
\[ |r + s - k|_0 = I(|r - k|_0 + |s - k|_0) \approx |r - k|_0 + |s - k|_0 \geq \text{dist}(r, K) + \text{dist}(s, K). \]

Hence (5) follows and the proof of Theorem 4 is complete.

REFERENCES