n the first case, and

$$\sum_{k=k_{n-1}+1}^{k_n} \|y_k\| = h_n \|t_n x_n\|' = h_n F(t_n, x_n) |t_n| \|x_n\|' \geqslant \frac{1}{2} \cdot 4^n \cdot 2^n = 2^{n-1}$$

because $F(t_n, x_n) \geqslant F(\tau_n, x_n)$, in the second case.

Hence the series $\sum_{n=1}^{\infty} y_k$ is not absolutely convergent in both cases.

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INTEGRALS ON QUOTIENT SPACES

BY

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NOTATION AND SUMMARY

If G is a locally compact topological group and H is a closed subgroup, then every integral I on the quotient space G/H is associated with exactly one integral $\tilde{I}$ on G (cf. formula (2) below). The class of integrals on G which are of the form $\tilde{I}$ will be characterized in Theorems 1 and 2. It contains the Haar integral if and only if there is an invariant integral on G/H (Th. 1, Corollary). The integrals I and $\tilde{I}$ define a pair of Banach spaces $L^1(G/H)$ and $L^1(G)$. H. Reiter considered these spaces under the assumption that $\tilde{I}$ is the Haar integral on G, whence only in the case where there is an invariant integral on G/H (cf. [4]). His results will be extended in Theorems 3 and 4 to the general case where I is an arbitrary integral on G/H.

If X is a locally compact topological space, we shall denote by $L(X)$ the class of all continuous real-valued functions on X which vanish outside compact sets. The class of extended Baire functions on X (cf. [1], [2]; these functions take also infinite values) will be denoted by $B(X)$. $L_+(X)$ and $B_+(X)$ will denote the subclasses of non-negative functions. Every non-negative linear functional I on $L(X)$ will be called an *integral on* X and we shall sometimes assume that the domain of definition of I includes $B_+(X)$ or the class of all I-summable functions. The class of all integrals on X will be designated by $\boldsymbol{I}(X)$. We shall denote by S_f the *support* of a function f on X, i. e. the set $\{x: f(x) \neq 0\}$.

Now let G and H be as in the beginning. Let $\bar{x}$ denote the coset xH. For any $f \in L(G)$ we put

$$\bar{f}(\bar{x}) = \int_H f(x\xi)\, d\xi, \tag{1}$$

where $\int_H$ is the integral with respect to the left Haar measure in H. It is clear that $\bar{f}(\bar{x}) = \bar{f}(\bar{y})$ if $\bar{x} = \bar{y}$ and (see [2], sec. 33A) that $\bar{f} \in L(G/H)$, where the topology in G/H is the natural one. For any $I \in \boldsymbol{I}(G/H)$ and $f \in L(G)$ we put

$$\tilde{I}(f) = I(\bar{f}). \tag{2}$$

Then $\tilde{I} \in \boldsymbol{I}(G)$ and the mapping $I \to \tilde{I}$ maps $\boldsymbol{I}(G/H)$ into $\boldsymbol{I}(G)$. Let

$$\tilde{\boldsymbol{I}}(G/H) = \{\tilde{I} \colon I \in \boldsymbol{I}(G/H)\}.$$

As is well known ([2], sec. 33B), the mapping $f \to \bar{f}$ transforms $L(G)$ onto $L(G/H)$. Hence, by (2), the mapping $I \to \tilde{I}$ is one-to-one. So we see that the investigation of $\boldsymbol{I}(G/H)$ may be reduced to that of $\tilde{\boldsymbol{I}}(G/H) \supset \boldsymbol{I}(G)$. We shall first give some characteristic properties of the class $\tilde{\boldsymbol{I}}(G/H)$ (Theorem 1) and then we shall estimate the "size" of this class in $\boldsymbol{I}(G)$ (Theorem 2). For any integral I on G/H we shall consider the spaces $L^1(G/H)$ and $L^1(G)$ of all I-summable and $\tilde{I}$-summable functions. The mapping $f \to \bar{f}$ will be shown to be a bounded linear transformation of $L^1(G)$ onto $L^1(G/H)$ with a well-defined kernel K (Theorem 3). We shall also consider the quotient Banach space $L^1(G)/K$ with the norm of a coset defined as its distance from the origin. Then the mapping $f \to \bar{f}$ defines a norm-preserving isomorphism between the spaces $L^1(G)/K$ and $L^1(G/H)$ (Theorem 4).

THE CLASS $\tilde{\boldsymbol{I}}(G/H)$

For $f \in L(G)$ and $a \in G$ we adopt the notation $f^a(x) = f(xa^{-1})$. We define the "translation" Q^a of an integral Q on G by $Q^a(f) = Q(f^a)$. Let J denote the left invariant Haar integral of H. The modular function $\delta(\xi)$ of J is then defined by $J^\xi = \delta(\xi)J$ $(\xi \in H)$.

THEOREM 1. *The following conditions are equivalent*:

$$Q \in \tilde{\boldsymbol{I}}(G/H), \tag{i}$$

$$Q(f) = 0 \ \textit{whenever}\ \bar{f} = 0 \ (f \in L(G)), \tag{ii}$$

$$Q^\xi = \delta(\xi)Q \ \textit{for each}\ \xi \in H, \tag{iii}$$

$$Q(f\bar{g}) = Q(\bar{f}g), \ \textit{when}\ f, g \in L(G)\ (^1). \tag{iv}$$

If Q is the Haar integral on G and Δ denotes the modular function for Q, then $Q^x = \Delta(x)Q$ for each $x \in G$. Thus, by (iii), we infer that the Haar integral belongs to $\tilde{\boldsymbol{I}}(G/H)$ iff Δ and δ coincide on H. Hence, by Weil's condition (cf. [5]) the

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(1) Here $\bar{f}$ and $\bar{g}$ are defined over G by the formulae $\bar{f}(x) = \bar{f}(\bar{x})$ and $\bar{g}(x) = \bar{g}(\bar{x})$.

COROLLARY. *The Haar integral belongs to* $\tilde{\boldsymbol{I}}(G/H)$ *iff there is an invariant measure on* G/H.

If Q and $^*Q$ are integrals and, for non-negative Baire functions f, $Q(f) = 0$ implies $^*Q(f) = 0$, then we shall write $^*Q \ll Q$. If $^*Q \ll Q$ and $Q \ll {}^*Q$, then Q and $^*Q$ will be called *equivalent* and this will be denoted by $Q \equiv {}^*Q$. Condition (iii) in Theorem 1 implies that $Q \equiv Q^\xi$ for $\xi \in H$. Conversely, we have

THEOREM 2. *If* $Q \in I(G)$ *and* $Q \equiv Q^\xi$ *for every* $\xi \in H$, *then* $Q \equiv {}^*Q$ *for some* $^*Q \in \tilde{\boldsymbol{I}}(G/H)$.

In particular, let Q be the Haar integral on G. Then Theorem 2 implies the existence of an integral I on G/H such that $\tilde{I}$ is equivalent to the Haar integral. The existence of such an integral was shown previously in [3] (the corresponding Baire measure was called *inherited*).

Proof of theorem 1. The equivalence of (i) and (ii) follows by (2). Also the implication (ii) $\to$ (iii) is easily shown. By the definition of δ, we have, for $f \in L(G)$, $\overline{f^\xi} = \delta(\xi)\bar{f}$, whence $Q(f^\xi - \delta(\xi)f) = 0$, by (ii), i. e. $Q^\xi(f) = \delta(\xi)Q(f)$. It remains to show that the implications (iii) $\to$ (iv) $\to$ (ii) hold.

(iii) $\to$ (iv). If $g(x, \xi) \in L(G \times H)$, then we can think of g as being a collection of functions in $L(G)$, each corresponding to a fixed value of ξ. Then, for each ξ, $Q(g)$ is defined and, to be more precise, we shall denote this number by $Q_x(g(x, \xi))$. We adopt a similar convention for the integral J on H, so that $Q_x(g(x, \xi)) \in L(H)$ and $J_\xi(g(x, \xi)) \in L(G)$ (cf. [2], sec. 16B). In the sequel we shall use the well-known equality

$$Q_x J_\xi(g(x, \xi)) = J_\xi Q_x(g(x, \xi)).$$

Also the well-known formula $J_\xi(h(\xi)) = J_\xi(h(\xi^{-1})\delta(\xi^{-1}))$ will be applied. We have

$$\begin{aligned} Q(f\bar{g}) &= Q_x\big(f(x) J_\xi(g(x\xi))\big) = Q_x J_\xi\big(f(x) g(x\xi)\big) \\ &= J_\xi Q_x\big(f(x) g(x\xi)\big) = J_\xi Q_x^{\xi^{-1}}\big(f(x\xi^{-1}) g(x)\big), \end{aligned}$$

and this, by (iii), is equal to

$$\begin{aligned} J_\xi \delta(\xi^{-1}) Q_x\big(f(x\xi^{-1}) g(x)\big) &= Q_x\big(g(x) J_\xi(f(x\xi^{-1})\delta(\xi^{-1}))\big) \\ &= Q_x\big(g(x) J_\xi(f(x\xi))\big) = Q(g\bar{f}). \end{aligned}$$

(iv) $\to$ (ii). Suppose that $f \in L(G)$ and $\bar{f} = 0$. By (iv), it is sufficient to find a function $g \in L(G)$ such that $f = f\bar{g}$ since then $Q(f) = Q(f\bar{g}) = Q(\bar{f}g) = 0$. Such a function g exists because, by Urysohn's Lemma, there is a function $d \in L(G/H)$ which is equal to 1 on the bounded set $\{\bar{x} \colon f(x) \neq 0\}$ and there is a function $g \in L(G)$ such that $\bar{g} = d$.

Proof of theorem 2. The construction of $^*Q$ follows in lemmas A and B.

A. *If Q is an integral on G and $p \in L_+(G)$, then the formula*

$$_pQ(f) = J_\xi Q_x\big(f(x\xi)p(x)\big)$$

defines an integral $_pQ \in \tilde{I}(G/H)$.

Proof of A. Since $f(x\xi)p(x) \in L(G\times H)$ and $J_\xi Q_x$ is an integral on $G\times H$ (cf. [2], sec. 16B), $_pQ$ is an integral. It remains to verify that $_pQ^\sigma = \delta(\sigma)_pQ$ for $\sigma \in H$. If $f \in L(G)$ and $\sigma \in H$, then

$$_pQ^\sigma(f) = J_\xi Q_x\big(f(x\xi\sigma^{-1})p(x)\big) = Q_x\Big(J_\xi\big(f(x\xi\sigma^{-1})\big)p(x)\Big)$$
$$= Q_x\Big(\delta(\sigma)J_\xi\big(f(x\xi)\big)\,p(x)\Big) = \delta(\sigma)_pQ(f).$$

B. Let $Q \in I(G)$ satisfy $Q^\xi \equiv Q$ for $\xi \in H$ and let $0 \neq p \in L_+(G)$. We consider the integral $_pQ$ which was defined in A and also all such integrals with p replaced by a translation p^t $(t \in G)$. We denote these integrals by $_tQ$. *Then there is a set $T \subset G$ such that the expression*

$$^*Q(f) = \sum_{t\in T} {_tQ}(f)$$

defines an integral $^*Q \equiv Q$, Moreover, $^*Q \in \tilde{I}(G/H)$.

Proof of B. Let T be any subset of G which is minimal with respect to the property $S_pTH = G$ (S_p is the support of p). If φ is the natural mapping of G onto G/H, then our condition means that the open sets $\varphi(S_pt)$, $t \in T$, form a minimal covering of G/H.

We show first that

(*) *$T \cap CH$ is finite when $C \subset G$ is a compact set.*

Consider the compact set $\bar{S}_p$ (closure of S_p). The set $\bar{S}_pC$ is compact and hence $\varphi(\bar{S}_pC)$ is compact. Since $\varphi(S_pt)$, $t \in T$, are open and their union covers G/H, $\varphi(\bar{S}_pC)$ can be covered by a finite union of these sets. Thus, by the minimality of T, there cannot be infinitely many sets $\varphi(S_pt)$ contained in $\varphi(\bar{S}_pC)$. Hence the relation $t \in T\cap CH$ holds for a finite number of t's at most.

To prove that $^*Q$ is an integral it is enough to show that, for each $f \in L(G)$, the sum defining $^*Q$ has only a finite number of non-zero terms. Indeed, if $_tQ(f) \neq 0$, then, by A, $f^\xi p^t \neq 0$ for some $\xi \in H$, $t \in T$. Then there is an x such that $x\xi^{-1} \in S_f$ and $xt^{-1} \in S_p$ and, this implies that $t \in S_p^{-1}S_fH$. Since the closure of $S_p^{-1}S_f$ is compact, we see by (*), that only a finite number of elements of T can satisfy this condition.

Now let us show that $^*Q \equiv Q$. Since $Q \equiv Q^\xi$ for $\xi \in H$, it easily follows from our construction that $^*Q \ll Q$. To show that $Q \ll {^*Q}$, we have to show that

(**) $Q(f) > 0$ *implies* $^*Q(f) > 0$ *if* $f \in B_+(G)$.

Since G is covered by the open sets $S_pt\xi$ $(t \in T,\ \xi \in H)$, the support of any Baire function can be covered by a countable union of such sets. It follows that it is enough to prove (**) under the assumption that the support S_f is contained in one of the sets $S_pt\xi$. Moreover, we can assume that f is bounded. We have that $fp^{t\xi}$ is positive on S_f, whence $Q(fp^{t\xi}) > 0$. Let us fix the element t for which this inequality holds. Since p is uniformly continuous and f is bounded, we infer that $Q(fp^{t\xi})$ is a continuous function of ξ. Thus it is positive on a certain open subset of H and, by $Q \equiv Q^\xi$, we have

$$Q^{\xi^{-1}}(fp^{t\xi}) = Q(f^{\xi^{-1}}p^t) = Q_x\big(f(x\xi)p^t(x)\big) > 0$$

for an open set of ξ's. This proves that $^*Q(f) > 0$. We have thus shown that $^*Q \equiv Q$.

By A, we have $_tQ \in \tilde{I}(G/H)$ $(t \in T)$, and thus $^*Q \in \tilde{I}(G/H)$.

THE SPACES $L^1(G)$ AND $L^1(G/H)$

Let I be an integral on G/H. We consider the Banach spaces $L^1(G)$ and $L^1(G/H)$ under the norms

$$\|f\|_G = \tilde{I}(|f|), \qquad \|g\|_{G/H} = I(|g|).$$

THEOREM 3. *The mapping $f \to \bar{f}$, when considered on $L^1(G)$, is a bounded linear transformation of this space onto $L^1(G/H)$. Its kernel*

$$K = \{k \in L^1(G) : \|\bar{k}\|_{G/H} = 0\}$$

is the closed linear subspace generated by all the functions

$$n(x) = f^\xi(x) - \delta(\xi)f(x),$$

where $f \in L(G)$ and $\xi \in H$.

Let us note that the mapping $f \to \bar{f}$ cannot be extended to $B(G)$ because $\int_H f(x\xi)\,d\xi$ may not exist for some $f \in B(G)$.

THEOREM 4. *Let*

$$\mathrm{dist}\{f, K\} = \mathrm{g.l.b.}\{\|f-k\|_G : k \in K\},$$

and let $L^1(G)/K$ be the quotient space with the norm of a K-coset y defined as $\mathrm{dist}\{f, K\}$, where f is any representative of y (cf. [2], sec. 6B). *Then the mapping $f \to \bar{f}$ establishes a norm-preserving isomorphism between the spaces $L^1(G)/K$ and $L^1(G/H)$ so that*

$$\mathrm{dist}\{f, K\} = \|\bar{f}\|_{G/H}.$$

In the above theorems we have generalized the results of H. Reiter [4]. He assumed that $\tilde{I}$ is the Haar integral and that there is an invariant measure on G/H. Theorems 3 and 4 include this case since, under the above assumption, the Haar integral belongs to $\tilde{\boldsymbol{I}}(G/H)$ (Th. 1, Corollary).

Proof of theorem 3. It is known that the mapping $f\to\bar{f}$, as defined by (1), transforms $L_+(G)$ onto $L^+(G/H)$ ([2], sec. 33). It extends uniquely to a mapping of $B_+(G)$ onto $B_+^1(G/H)$. This follows from the fact that (1) is invariant under the formation of limits of monotone sequences and this operation is sufficient to obtain the classes B_+ from the classes L_+. Since also (2) is invariant under these operations, we infer that

$$\tilde{I}(f)=I(\bar{f})\quad \text{when}\quad f\in B_+(G), \tag{3}$$

where ∞ is allowed as a possible value of the integrals. If $f\in L^1(G)$, then both non-negative parts, f_1 and f_2, of f ($f=f_1-f_2$, $f_i\geqslant 0$) are $\tilde{I}$-summable, and thus, by (3), $\bar{f}_i\in L_+^1(G/H)$. It follows that the formula $\bar{f}=\bar{f}_1-\bar{f}_2$ defines $\bar{f}$ as an element of $L^1(G/H)$ (with the usual ambiguity at those points where the summands assume opposite infinities as values). We have thus shown that the mapping $f\to\bar{f}$ can be extended to $L^1(G)$.

If f runs over $B_+(G)$, then $\bar{f}$ runs over $B_+(G/H)$ and if one of these functions is summable, then so is the other, by (3). Hence the transformation $f\to\bar{f}$ maps $L_+^1(G)$ onto $L_+^1(G/H)$. Consequently $L^1(G)$ is mapped onto $L^1(G/H)$.

The transformation is bounded because

$$\|\bar{f}\|_{G/H}=I(|\bar{f}|)\leqslant I(|\bar{f}_1|+|\bar{f}_2|)=\tilde{I}(|f|)=\|f\|_G.$$

Finally, let us show that the kernel K of this transformation is the closed linear subspace $N\subset L^1(G)$ which is generated by the functions $n(x)$. It is clear that $\bar{n}=0$ and from the continuity of the transformation we infer that K is closed. Hence $N\subset K$. If $N^\perp$ denotes the class of all bounded linear functionals F which vanish on N, i. e. the annihilator of N, and if $K^\perp$ is the annihilator of K, then the inclusion $K\subset N$, which we have to prove, is equivalent to $N^\perp\subset K^\perp$. Thus we have to verify that if $F\in N^\perp$ and $k\in K$, then $F(k)=0$. We need the following

LEMMA. *If $F\in N^\perp$, then there are integrals I_0 and I_1 on G/H such that*

$$F=\tilde{I}_0-\tilde{I}_1$$

and $I_0, I_1\ll I$.

Remark. The proof given below yields in fact the following stronger result: $N^\perp$ is the class of functionals F which are of the form $F=\tilde{I}_0-\tilde{I}_1$, where I_0, I_1 are bounded integrals on G/H such that $I_0, I_1\ll I$.

Proof of the Lemma. As is well known ([2], sec. 15A), each bounded functional F on $L^1(G)$ is expressible as the difference $F=F^+-F^-$ of two bounded integrals (non-negative functionals), where

$$F^+(f)=\operatorname{lub}\{F(g)\colon 0\leqslant g\leqslant f\}\quad \text{when}\quad f\geqslant 0. \tag{4}$$

Moreover, if $F\in N^\perp$, then $F^+, F^-\in\tilde{\boldsymbol{I}}(G/H)$. Indeed, we have $F(f^\xi-\delta(\xi)f)=0$, when $f\in L(G)$, $\xi\in H$, and hence $F^\xi=\delta(\xi)F$. Thus, by (4), F^+ satisfies condition (iii) of Theorem 1, and consequently also $F^-=F^+-F$ satisfies this condition.

Let I_0, I_1 be integrals on G/H such that $\tilde{I}_0=F^+$ and $\tilde{I}_1=F^-$. Since F^+ and F^- are bounded, we infer that $\|f\|_G=0$ implies $F^\pm(f)=0$, i. e. $\tilde{I}_j\ll\tilde{I}$ ($j=0,1$). To prove that $I_j\ll I$ assume that $I(g)=0$, where $g\in B_+(G/H)$. There is a function $f\in B_+(G)$ such that $\bar{f}=g$, and then $\tilde{I}(f)=0$, by (3). It follows that $\tilde{I}_j(f)=0$, and thus, again using (3), $I_j(g)=0$. This proves the lemma.

Suppose now that $F\in N^\perp$ and $k\in K$, i. e. $I(|\bar{k}|)=0$. By the above lemma

$$F(k)=\tilde{I}_0(k)-\tilde{I}_1(k)=I_0(\bar{k})-I_1(\bar{k}),$$

and both $I_j(\bar{k})$ vanish because $I_j\ll I$. Our proof is now complete.

Proof of theorem 4. $L^1(G)/K$ and $L^1(G/H)$ are isomorphic linear spaces, by the definition of K. We may therefore assume that these spaces are identical, that is to say, a K-coset with a representative $f\in L^1(G)$ will be identified with $\bar{f}$. Then we have in $L^1(G/H)$ also the norm taken from $L^1(G)/K$:

$$_K\|\bar{f}\|_{G/H}=\operatorname{dist}\{f,K\}.$$

To prove our theorem we must verify that both norms in $L^1(G/H)$ are identical, i. e. that

$$_K\|g\|_{G/H}=\|g\|_{G/H}\quad \text{when}\quad g\in L^1(G/H).$$

This is easily seen once we have shown that

(a) $_K\|g\|_{G/H}=\|g\|_{G/H}$, *when $g\geqslant 0$ or $g\leqslant 0$,*

(b) $_K\|g+h\|_{G/H}={}_K\|g\|_{G/H}+{}_K\|h\|_{G/H}$, *when the supports of g and h are disjoint.*

Proof of (a). If $g\in L_\pm^1(G/H)$, then there is a function $f\in L_\pm^1(G)$ such that $\bar{f}=g$. Then $_K\|g\|_{G/H}=\operatorname{dist}\{f,K\}$. If $k\in K$, then

$$\int_H|f(x\xi)-k(x\xi)|\,d\xi\geqslant\Big|\int_H\big(f(x\xi)-k(x\xi)\big)\,d\xi\Big|=|\bar{f}(\bar{x})|$$

and hence, by (3),

$$\|f-k\|_G = I\Big(\int_H |f(x\xi)-k(x\xi)|\,d\xi\Big) \geqslant I(|\bar{f}|) = \tilde{I}(|f|) = \|f\|_G.$$

This proves that $\|f\|_G = \operatorname{dist}\{f, K\}$. Now, by (3), $\|f\|_G = \|g\|_{G/H}$ and hence $\|g\|_{G/H} = {}_K\|g\|_{G/H}$.

Proof of (b). By the triangle inequality it suffices to verify that

$${}_K\|g+h\|_{G/H} \geqslant {}_K\|g\|_{G/H} + {}_K\|h\|_{G/H}.$$

Let $r, t \epsilon L^1(G)$ be such that $\bar{r} = g$, $\bar{t} = h$, where the supports S_r, S_t satisfy $S_r H \cap S_t H = \emptyset$. Then the inequality we wish to prove is equivalent to

$$\operatorname{dist}\{r+t, K\} \geqslant \operatorname{dist}\{r, K\} + \operatorname{dist}\{t, K\}. \tag{5}$$

It is easily seen that if $k \epsilon K$, then the restricted functions $k^{(r)} = k\,|\,S_r H$ and $k^{(t)} = k\,|\,S_t H$ also belong to K, and this implies that

$$\|r+t-k\|_G = \tilde{I}(|r-k^{(r)}|+|t-k^{(t)}|+|k-k^{(r)}-k^{(t)}|)$$
$$\geqslant \|r-k^{(r)}\|_G + \|t-k^{(t)}\|_G \geqslant \operatorname{dist}\{r, k\} + \operatorname{dist}\{t, K\}.$$

Hence (5) follows and the proof of Theorem 4 is complete.

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ON THE ALGEBRAS L_p OF LOCALLY COMPACT GROUPS

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Let G be locally compact group, and μ its left invariant Haar measure. Let L_p be the Banach space of complex functions defined on G, for which

$$\|f\|_p^p = \int |f(t)|^p\,d\mu(t) < \infty.$$

It is well known that L_1 is a Banach algebra if multiplication is defined as the convolution

$$f*g(t) = \int f(t\tau^{-1})\,g(\tau)\,d\mu(\tau).$$

It is also known that if the group G is compact, then the space L_2 is also a Banach algebra with the same multiplication (see [1], p. 156). Here I shall prove that this theorem and the converse theorem hold for all $p>1$. More precisely I shall prove

THEOREM 1. *If the locally compact group G is compact, then for every p, $1 \leqslant p \leqslant \infty$, the space L_p is a Banach algebra under convolution.*

THEOREM 2. *If for a locally compact abelian group the space L_p is a Banach algebra under convolution, and $1 < p < \infty$, then the group G is compact.*

The following simple remark is useful in the proofs:

Let X be a Banach space with the norm $\|x\|$, and R a dense linear subspace, which is at the same time an algebra with the multiplication xy. Then

(A) *X is a Banach algebra with the same multiplication if and only if there exists such a number $C>0$ that*

$$\|xy\| \leqslant C\,\|x\|\,\|y\| \quad \textit{for every} \quad x, y \,\epsilon\, R.$$

Or, what is equivalent,