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ON A GENERALIZATION OF THE DVORETZKY-ROGERS THEOREM

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Let X be an F^*-space (for the definition and basic property of an F^*-space see [1], p. 35) with the norm $\|x\|$ (not necessarily homogeneous).

We shall say that the series $\sum_{n=1}^{\infty} x_n$ of elements of X is *unconditionally convergent* if for every sequence $\{\eta_n\}$, where η_n is equal to 1 or 0, the series $\sum_{n=1}^{\infty} \eta_n x_n$ is convergent.

We shall say that the series $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} \|x_n\|$ is convergent (see [5]).

Every absolutely convergent series is also unconditionally convergent. If X is a finite dimensional space and the norm $\| \|$ is homogeneous, then, conversely, every unconditionally convergent series is also absolutely convergent.

A. Dvoretzky and C. A. Rogers [3] have proved that in an arbitrary infinite dimensional B-space X with a homogeneous norm $\| \|$ there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ which is not absolutely convergent.

In this note we prove the following simple generalization of the Dvoretzky-Rogers theorem:

THEOREM. *In every infinite dimensional F^*-space with norm $\| \|$ there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ which is not absolutely convergent.*

Proof. At the beginning we suppose that in X there is a homogeneous norm $\| \|^*$ equivalent to the norm $\| \|$ (1). In this case there is a positive constant A such that $\|x\|^* \leqslant A\|x\|$ if $\|x\|^* < 1$. Indeed, if there is a sequence x_n such that $\|x_n\|^* < 1$ and $\|x_n\|^* > n\|x_n\|$, then the triangle inequality implies

$$\left\|\left[\frac{1}{\|x_n\|^*}\right]x^n\right\| \leqslant \left[\frac{1}{\|x_n\|^*}\right]\|x_n\| \leqslant \left[\frac{1}{\|x_n\|^*}\right]\frac{1}{n}\cdot\|x_n\|^* \leqslant \frac{1}{n},$$

where $[u]$ denotes the integral part of the real number u.

On the other hand,

$$\left\|\left[\frac{1}{\|x_n\|^*}\right]x_n\right\|^* = \left[\frac{1}{\|x_n\|^*}\right]\|x_n\|^* \geqslant \frac{1}{2}$$

because $\|x_n\|^* < 1$.

Hence the norms $\| \|$ and $\| \|^*$ cannot be equivalent.

The Dvoretzky-Rogers theorem implies that there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty}\|x_n\|^* = +\infty$, whence, from the preceding also the series $\sum_{n=1}^{\infty}\|x_n\| = +\infty$.

Now we suppose that there is no equivalent homogeneous norm in X. We shall consider two cases.

Firstly, in X there are "arbitrarily short" straight lines (see [2]), which means that for every $\varepsilon > 0$ there is in X an element $x \neq 0$ such that for every real t the inequality $\|tx\| < \varepsilon$ holds.

In this case we can choose a sequence x_n such that, for arbitrary real t, $\|tx_n\| < 1/2^n$.

Secondly, in X there are no "arbitrarily short" straight lines, which means that there is such an $\varepsilon_0 > 0$ that for an arbitrary $x \epsilon X$ there is such a λ_x that $\|\lambda_x x\| = \varepsilon_0$. By $\|x\|'$ we denote the norm $\|x\|' = \sup_{0<t\leqslant 1}\|tx\|$. The norm $\|x\|'$ is equivalent to the norm $\|x\|$ (see [4]). By $F(t, x)$ we denote the function $F(t, x) = \|tx\|'/|t|\,\|x\|'$.

If, for some $0 < \varepsilon < \varepsilon_0$ that function is bounded on the set $\|x\|' = \varepsilon$, $0 < t \leqslant 1$, then the norm $\|x\|'$, and hence also the norm $\|x\|$, is equivalent to the norm $\|x\|^* = \inf\{t\colon \|x/t\|' = \varepsilon\}$. The norm $\|x\|^*$ is homogeneous and we obtain a contradiction of our supposition. Hence for arbitrary $0 < \varepsilon < \varepsilon_0$, the function $F(t, x)$ is unbounded on the set $\|x\|' = \varepsilon$,

(1) The norms $\| \|$, $\| \|^*$ are called *equivalent* if $\|x_n\|$ tends to zero if and only if $\|x_n\|^*$ tends to zero.

$0 \leqslant t \leqslant 1$. Therefore we can choose sequences $\{x_n\}$ of elements of X and τ_n of such numbers that

1° $\|x_n\|' = 1/2^n$,

2° $F(\tau_n, x_n) > 4^n$.

From the definition of $\|x\|'$ it follows that there are such t_n that $0 < t_n \leqslant \tau_n$, $\|t_n x_n\|' = \|t_n x_n\|$, $\|t_n x_n\|' = \|\tau_n x_n\|'$.

Let x_n be the sequences chosen in the ways described in the first or the second case. Let $h_n = [1/x_n]$ in the first case, and $h_n = [1/t_n]$ in the second case. We write $k_0 = 0$ and $k_n = h_1 + \ldots + h_n$. Let

$$y_k = \begin{cases} x_n & \text{in the first case,} \\ t_n x_n & \text{in the second case} \end{cases}$$

for $k_{n-1} < k \leqslant k_n$.

In both cases the series $\sum_{k=1}^{\infty} y_k$ is unconditionally convergent but is not absolutely convergent.

Really, let η_k be an arbitrary sequence of zeros and unities. Let m be an arbitrary positive integer. By $n(m)$ we denote $n(m) = \sup_{k_n<m} n$. Now we estimate the rest of the series $\sum_{k=1}^{\infty}\eta_k y_k$:

$$\|R_m\| = \left\|\sum_{k=m+1}^{\infty}\eta_k y_k\right\| = \left\|\sum_{k=m+1}^{k_{n(m)+1}}\eta_k y_k + \sum_{n=n(m)+1}^{\infty}\sum_{k=k_n+1}^{k_{n+1}}\eta_k y_k\right\|$$
$$\leqslant \left\|\sum_{k=m+1}^{k_{n(m)+1}}\eta_k y_k\right\| + \sum_{n=n(m)+1}^{\infty}\left\|\sum_{k=k_n+1}^{k_{n+1}}\eta_k y_k\right\| \leqslant \frac{1}{2^{n(m)-1}}$$

because

$$\left\|\sum_{k=k'}^{k_{n+1}}\eta_k y_k\right\| \leqslant \begin{cases} \sup_{\lambda}\|\lambda x_{n+1}\| \leqslant \dfrac{1}{2^{n+1}} & \text{in the first case,} \\ \|x_{n+1}\|' \leqslant \dfrac{1}{2^{n+1}} & \text{in the second case,} \end{cases}$$

where $k' \geqslant k_n + 1$.

On the other hand,

$$\sum_{k=k_n+1}^{k_{n+1}}\|y_k\| = h_{n+1}\|x_{n+1}\| = \left[\frac{1}{\|x_{n+1}\|}\right]\|x_{n+1}\| \geqslant \frac{1}{2}$$

n the first case, and

$$\sum_{k=k_{n-1}+1}^{k_n} \|y_k\| = h_n \|t_n x_n\|' = h_n F(t_n, x_n) |t_n| \, \|x_n\|' \geqslant \frac{1}{2} \cdot 4^n \cdot 2^n = 2^{n-1}$$

because $F(t_n, x_n) \geqslant F(\tau_n, x_n)$, in the second case.

Hence the series $\sum_{n=1}^{\infty} y_k$ is not absolutely convergent in both cases.

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INTEGRALS ON QUOTIENT SPACES

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NOTATION AND SUMMARY

If G is a locally compact topological group and H is a closed subgroup, then every integral I on the quotient space G/H is associated with exactly one integral $\tilde{I}$ on G (cf. formula (2) below). The class of integrals on G which are of the form $\tilde{I}$ will be characterized in Theorems 1 and 2. It contains the Haar integral if and only if there is an invariant integral on G/H (Th. 1, Corollary). The integrals I and $\tilde{I}$ define a pair of Banach spaces $L^1(G/H)$ and $L^1(G)$. H. Reiter considered these spaces under the assumption that $\tilde{I}$ is the Haar integral on G, whence only in the case where there is an invariant integral on G/H (cf. [4]). His results will be extended in Theorems 3 and 4 to the general case where I is an arbitrary integral on G/H.

If X is a locally compact topological space, we shall denote by $L(X)$ the class of all continuous real-valued functions on X which vanish outside compact sets. The class of extended Baire functions on X (cf. [1], [2]; these functions take also infinite values) will be denoted by $B(X)$. $L_+(X)$ and $B_+(X)$ will denote the subclasses of non-negative functions. Every non-negative linear functional I on $L(X)$ will be called an *integral on* X and we shall sometimes assume that the domain of definition of I includes $B_+(X)$ or the class of all I-summable functions. The class of all integrals on X will be designated by $\boldsymbol{I}(X)$. We shall denote by S_f the *support* of a function f on X, i. e. the set $\{x: f(x) \neq 0\}$.

Now let G and H be as in the beginning. Let $\bar{x}$ denote the coset xH. For any $f \in L(G)$ we put

$$\bar{f}(\bar{x}) = \int_H f(x\xi)\, d\xi, \qquad (1)$$

where $\int_H$ is the integral with respect to the left Haar measure in H. It is clear that $\bar{f}(\bar{x}) = \bar{f}(\bar{y})$ if $\bar{x} = \bar{y}$ and (see [2], sec. 33A) that $\bar{f} \in L(G/H)$,