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ON A PROBLEM OF INTERPOLATION BY PERIODIC FUNCTIONS

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E. Marczewski and C. Ryll-Nardzewski have asked whether there exists such a sequence of real numbers $t_1, t_2, \ldots$ that

(*) *For every sequence $\varepsilon_1, \varepsilon_2, \ldots$, where $\varepsilon_i = \pm 1$, there is a continuous periodic function $f(t)$ with*

$$f(t_i) = \varepsilon_i \quad \textit{for} \quad i = 1, 2, \ldots$$

It is the purpose of this paper to solve this problem in the affirmative; namely we will prove the following theorem:

THEOREM. *If a sequence $t_1, t_2, \ldots$ satisfies the inequalities*

$$(1) \qquad t_1 > 0 \textit{ and } t_{n+1} \geqslant (3+\delta) t_n \quad (n = 1, 2, \ldots; \ \delta > 0),$$

then it has property (*).

I do not know (**P 315**) if the sequence $t_n = 3^n$ has property (*). A class of sequences without property (*) is given in a paper of S. Hartman [1]. The method of the present paper permits us to prove, by easy modification, the existence of sequences $t_1, t_2, \ldots$ satisfying a strengthened version of (*) obtained by allowing the ε_i's to take values from any fixed finite set of real numbers. But we do not know whether any infinite set is admissible [2].

Remark. If $t_1, t_2, \ldots$ has property (*) then for every bounded sequence of real numbers $\varrho_1, \varrho_2, \ldots$ there exists such a function $p(t)$, almost periodic in the sense of Bohr, that

$$p(t_i) = \varrho_i \quad \text{for} \quad i = 1, 2, \ldots$$

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[1] S. Hartman, *On interpolation by almost periodic functions*, this volume, p. 91-101.

[2] Added in proof: A positive answer to this question was recently given by J. Lipiński (to be published in Bulletin de l'Académie Polonaise des Sciences).

This follows easily from the main approximation theorem for almost periodic functions.

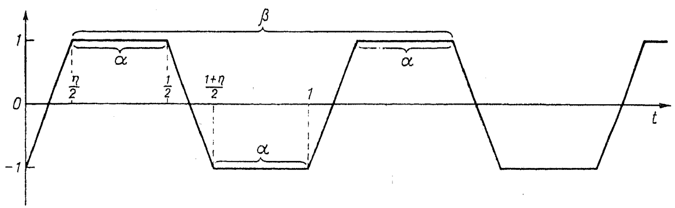
Proof of the Theorem. We suppose that the sequence $t_1, t_2, \ldots$ satisfies (1). Let us choose some positive numbers η, $\varDelta$ and ω such that

$$(2) \qquad \eta < 1, \quad \frac{2\eta}{1-\eta} \leqslant \delta \quad \text{and} \quad t_1 \geqslant \frac{\omega}{2}(3-\eta)\left(1+\frac{\omega}{\varDelta}\right).$$

We adopt the notation

$$(3) \qquad \alpha = \frac{1-\eta}{2}, \quad \beta = \frac{3-\eta}{2}.$$

Let $\varphi(t)$ be a continuous function with period 1 defined in the interval $\langle 0, 1\rangle$ as follows: $\varphi(0) = -1$, φ is linear in $\langle 0, \eta/2\rangle$, $\varphi(t) = 1$ in $\langle \eta/2, \frac{1}{2}\rangle$, φ is linear in $\langle \frac{1}{2}, (1+\eta)/2\rangle$ and $\varphi(t) = -1$ in $\langle (1+\eta)/2, 1\rangle$ (see the figure).



Let us fix (arbitrarily) the sequence $\varepsilon_1, \varepsilon_2, \ldots$ ($\varepsilon_i = \pm 1$). We shall prove the existence of such a sequence $\lambda_1, \lambda_2, \ldots$ that

$$(4) \quad \omega \leqslant \lambda_i \leqslant \omega+\varDelta, \quad \varphi(t_j/\lambda_i) = \varepsilon_j \quad \text{for} \quad j = 1, \ldots, i;\ i = 1, 2, \ldots$$

This will be a proof of our theorem since then obviously

$$\varphi(t_i/\limsup_{n\to\infty} \lambda_n) = \varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots$$

But first we shall prove the existence of such sequences

$$\varDelta_0 = \varDelta, \varDelta_1, \varDelta_2, \ldots \text{ and } \omega_0 = \omega, \omega_1, \omega_2, \ldots$$

that the following conditions are satisfied for $n = 1, 2, \ldots$:

(A_n) $\omega \leqslant \omega_{n-1} < \omega_{n-1}+\varDelta_{n-1} \leqslant \omega+\varDelta$;

(B_n) for every $\lambda \in \langle \omega_{n-1}, \omega_{n-1}+\varDelta_{n-1}\rangle$ there is $\varphi(t_i/\lambda) = \varepsilon_i$ for $i = 1, \ldots, n-1$;

(C_n) $\dfrac{t_n}{\omega_{n-1}} - \dfrac{t_n}{\omega_{n-1}+\varDelta_{n-1}} \geqslant \beta$.

Propositions (A_1) and (B_1) are obvious and (C_1) follows from the third inequality of (2) and from (3). Let us suppose that (A_n), (B_n) and (C_n) are established; we have to prove (A_{n+1}), (B_{n+1}) and (C_{n+1}).

The geometrical sense of (C_n) is the following:

(C'_n) When λ moves in the interval $\langle \omega_{n-1}, \omega_{n-1}+\varDelta_{n-1}\rangle$, then a fraction β of the period of the function $\varphi(t/\lambda)$ moves over the point t_n.

We will prove that one can chose such $\varDelta_n$ and ω_n that

(A) $\omega_{n-1} \leqslant \omega_n < \omega_n+\varDelta_n \leqslant \omega_{n-1}+\varDelta_{n-1}$;

(B) for every $\lambda \in \langle \omega_n, \omega_n+\varDelta_n\rangle$ there is $\varphi(t_n/\lambda) = \varepsilon_n$;

(C) $\dfrac{t_n}{\omega_n} - \dfrac{t_n}{\omega_n+\varDelta_n} = \alpha$.

In fact (C'_n) implies on account of the definition of φ that (B) holds for some interval $\langle \omega_n, \omega_n+\varDelta_n\rangle$ which is contained in $\langle \omega_{n-1}, \omega_{n-1}+\varDelta_{n-1}\rangle$ and fulfills (C).

Clearly (A_n) & (A) $\Rightarrow$ (A_{n+1}) and (B_n) & (B) $\Rightarrow$ (B_{n+1}); it remains to prove (C_{n+1}). Since by (C) .

$$\varDelta_n = \frac{\alpha\omega_n^2}{t_n - \alpha\omega_n}$$

one has

$$(5) \qquad \beta\omega_n\left(1+\frac{\omega_n}{\varDelta_n}\right) = \beta\omega_n\left(1+\frac{\omega_n(t_n-\alpha\omega_n)}{\alpha\omega_n^2}\right) = \frac{\beta}{\alpha}t_n.$$

Now applying (1), the second inequality of (2) and (3) we have

$$(6) \qquad \frac{\beta}{\alpha}t_n = \left(3+\frac{2\eta}{1-\eta}\right)t_n \leqslant t_{n+1}.$$

From (5) and (6) follows

$$\beta\omega_n\left(1+\frac{\omega_n}{\varDelta_n}\right) \leqslant t_{n+1},$$

which implies (C_{n+1}).

The statements (A_n) and (B_n) being proved for $n = 1, 2, \ldots$, we infer the existence of a sequence $\lambda_1, \lambda_2, \ldots$ as required in (4), q. e. d.

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