4. Problème (P 314). Quelle sont les conditions nécessaires et suffisantes auxquelles doivent satisfaire les mesures μ et ν pour que l'on ait $D_\nu = D_\mu$?

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ON AN APPLICATION OF UNIFORM DISTRIBUTION OF SEQUENCES

BY

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The object of this paper is to prove the following

THEOREM. *Let $\zeta = (\xi, \eta)$ be a point in the complex plane with ξ and η both irrational and let λ_n be the number of points with integral coordinates that lie in the circle*

$$|z - n\zeta| \leqslant 1.$$

We know that $\lambda_n = 2, 3$ or 4. Let k be any one of the numbers 2, 3 or 4 and let

$$f_k(N) = \sum_{\substack{n \leqslant N \\ \lambda_n = k}} 1.$$

Then $\operatorname{Lim} f_k(N)/N$ *exists.*

We divide our proof into two cases, firstly where ξ, η and 1 are linearly independent over the rationals and secondly where they are not so. The result in the first case is contained in a more general theorem of Hartman, concerning an arbitrary bounded Jordan-measurable set (1). However, for the sake of completeness, we shall give here a proof of the result in the first case also.

The arcs in the figure are circular arcs with the four corners of the square as centres. AB is identified with DC, AD with BC.

Let

$$A_2 = A_2^1 \cup A_2^2 \cup A_2^3 \cup A_2^4$$

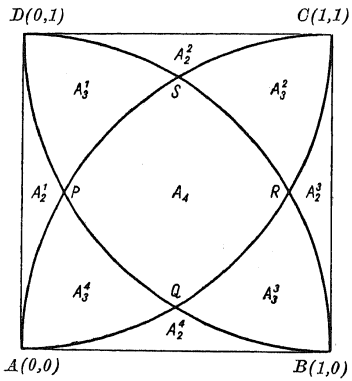
where for example A_2^1 is the region APD minus the closed arcs AP and PD

$$A_3 = A_3^1 \cup A_3^2 \cup A_3^3 \cup A_3^4$$

(1) S. Hartman, *Zur Gitterpunktverteilung bei Verschiebungen von Mengen*, Studia Mathematica 13 (1953), p. 87-93, Satz II.

where for example A_3^1 is the region PDS minus the point D and the closed arc PS, and $A_4 =$ closed region $PQRS$.

Let z_0 be a point in the plane and let (z_0) be the point in the square $ABCD$ that is congruent to z_0. Then the number of points with integral coordinates that lie in the circle $|z-z_0| \leqslant 1$ is 2, 3 or 4 according as (z_0) lies in A_2, A_3 or A_4 respectively. Thus



$$f_k(N) = \sum_{\substack{n\leqslant N\\ \lambda_n = k}} 1 = \sum_{\substack{n\leqslant N\\ (n\zeta)\in A_k}} 1, \quad k = 2,3,4.$$

Case 1. We assume that ξ, η and 1 are linearly independent. Consequently the sequence $(n\zeta)$ is uniformly distributed. Let $\mu(A_k)$ denote the measure of A_k and let $\varepsilon > 0$ be any number. For a given k we can find rectangles R_i and R_j', $1 \leqslant i \leqslant i_0$ and $1 \leqslant j \leqslant j_0$ satisfying the following conditions:

1) All the rectangles are open and pairwise disjoint,
2) The corners of the rectangles have rational coordinates,
3) $\bigcup\limits_i R_i \subset A_k \subset \bigcup\limits_i \bar{R}_i \cup \bigcup\limits_j \bar{R}_j'$,
4) $\sum\limits_j \mu(R_j') < \varepsilon$.

It is clear from the irrationality of ξ and η and from the second condition that the point $(n\zeta)$ cannot lie on the sides of the rectangles R^i and R_j'. Thus

$$\frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in \bigcup_i R_i}} 1 \leqslant \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in A_k}} 1 \leqslant \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in \bigcup_i R_i \cup \bigcup_j R_j'}} 1,$$

i. e.

$$\sum_i \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in R_i}} 1 \leqslant \frac{1}{N} f_k(N) \leqslant \sum_i \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in R_i}} 1 + \sum_j \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in R_j'}} 1.$$

Since the sequence $(n\zeta)$ is uniformly distributed we can choose N_0 such that for $n \geqslant N_0$

$$\left| \frac{1}{N} \sum_{\substack{n\leqslant N\\ (n\zeta)\in R_i}} 1 - \mu(R_i) \right| < \frac{\varepsilon}{i_0}, \quad 1 \leqslant i \leqslant i_0,$$

and

$$\left| \frac{1}{N} \sum_{\substack{n\leqslant N,\\ (n\zeta)\in R_j'}} 1 - \mu(R_j') \right| < \frac{\varepsilon}{j_0}, \quad 1 \leqslant j \leqslant j_0.$$

Therefore for $N \geqslant N_0$

$$\sum_i \mu(R_i) - \varepsilon \leqslant \frac{1}{N} f_k(N) \leqslant \sum_i \mu(R_i) + \varepsilon + \sum_j \mu(R_j') + \varepsilon$$
$$< \sum_i \mu(R_i) + 3\varepsilon < \mu(A_k) + 3\varepsilon.$$

Thus

$$\mu(A_k) - 2\varepsilon \leqslant \frac{1}{N} f_k(N) < \mu(A_k) + 3\varepsilon.$$

Thus

$$\lim_{N\to\infty} \frac{1}{N} f_k(N) = \mu(A_k).$$

This concludes the proof of the theorem in the first case.

Case 2. ξ, η and 1 are not linearly independent over the rationals. In this case we proceed as follows:

Let $L = \{l_i \mid 1 \leqslant i \leqslant i_0\}$ be a finite set of line-segments l_i all contained in the unit square $ABCD$ such that no two of them intersect. By $\mu(l)$ we mean the length of the line segment l and by $\mu(L)$ we mean $\sum\limits_{l\in L} \mu(l)$. We say that a sequence z_n of points in the complex plane is *uniformly distributed modulo L* if the following conditions are satisfied:

(i) The point (z_n) in the unit square $ABCD$ congruent to z_n lies on some $l \in L$.

(ii) If l' is a subsegment of l and if

$$f(N) = \sum_{\substack{n\leqslant N\\ (z_n)\in \text{some } l'}} 1,$$

then

$$\lim_{N\to\infty} \frac{1}{N} f(N) = \frac{1}{\mu(L)} \sum_{l\in L} \mu(l').$$

We prove the following theorem first:

THEOREM A. *Let ξ and η be two irrational numbers such that there exist integers p, p' and q such that*

$$\eta = \frac{p\xi + p'}{q}, \quad (p, p', q) = 1.$$

Then the sequence $(n\xi, n\eta)$ is uniformly distributed modulo some finite set L of line segments in the unit square $ABCD$.

In the proof of this theorem we shall need the following two lemmas the proofs of which are immediate:

LEMMA 1. *Let (a_n, β_n) be a sequence uniformly distributed modulo L and let θ and θ' be two real numbers. Then the sequence $(a_n+\theta, \beta_n+\theta')$ is also uniformly distributed modulo some L'.*

LEMMA 2. *Let z_n and z_n' be two sequences uniformly distributed modulo L and L' respectively. Suppose further that $\mu(L) = \mu(L')$ and that $l \in L$, $l' \in L' \Rightarrow l \cap l' = \emptyset$; then the sequence $z_1, z_1', z_2, z_2', \ldots$ is uniformly distributed modulo $L \cup L'$.*

We can now prove Theorem A.

Without loss of generality we can assume ξ and η to be both positive. We first consider the special case where $p' = 0$. Let A be the point (ξ, η) and P be the point (q, p) and let λ be the half ray through the origin and the points A and P. Since $(p, q) = 1$, P is the first lattice point other than the origin on λ. All the points nA lie on λ. The distance of A from the origin is $\xi q^{-1}\sqrt{p^2+q^2}$ and the distance of P from the origin is $\sqrt{p^2+q^2}$. The ratio of these distance being irrational, the points nA are uniformly distributed modulo the segment OP. The result of Theorem A, in the pecial case that we are considering now, easily follows.

Suppose now that $p' \neq 0$. Let $0 \leqslant r < q$ and let us first consider the sequence nA where $n \equiv r (\operatorname{mod} q)$. We will prove that this sequence is uniformly distributed modulo some finite set L_r of line segments in the unit square $ABCD$. The sequence in question is $\{((mq+r)\xi, (mq+r)\eta) | m \geqslant 0\}$ and so by lemma 1 it would be enough to prove the uniform distribution of the sequence $\{(mq\xi, mq\eta) | m \geqslant 0\}$ or of the sequence $\{(mq\xi, mp\xi) | m \geqslant 0\}$ modulo some L. But the uniform distrebution of this last sequence modulo some L follows from the special casthat we have considered above.

It can be seen that (owing to the irrationality of ξ) from $l \in L_r$, $l' \in L_{r'}$ and $r \neq r'$ follows $l \cap l' = \emptyset$ and that $\mu(L_r) = \mu(L_{r'})$. Using the obvious generalisation of lemma 2 we infer that the sequence $(n\xi, n\eta)$ is uniformly distributed modulo $L = \bigcup_{0 \leqslant r < q} L_r$.

This concludes the proof of Theorem A.

Having proved Theorem A we can now give the proof of our theorem in the case where ξ, η and 1 are not linearly independent over the rationals. Thus there exist inetgers p, p', q such that $(p, p', q) = 1$ and

$$\eta = \frac{p\xi + p'}{q};$$

consequently, the sequence $n\zeta = (n\xi, n\eta)$ is uniformly distributed modulo some L. Now

$$f_k(N) = \sum_{\substack{n \leqslant N \\ \lambda_n = k}} 1 = \sum_{\substack{n \leqslant N \\ (n\zeta) \in A_k}} 1 = \sum_{\substack{n \leqslant N \\ (n\zeta) \in \bigcup_{l \in L} (l \cap A_k)}} 1$$

and so

$$\lim_{N \to \infty} \frac{1}{N} f_k(N) = \frac{1}{\mu(L)} \sum_{l \in L} \mu(l \cap A_k).$$

This completes the proof of our main theorem.

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