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A REMARK ABOUT CAUCHY'S EQUATION

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It seems rather difficult to raise a new problem concerning the equation

$$(*)\qquad f(x+y) = f(x)+f(y).$$

However, P. Erdös has done it asking if a function which satisfies (*) for *almost every* pair (x, y) of real numbers must be *almost everywhere* equal to a function satisfying this equation for *every* pair (x, y), i. e. to a linear or a Hamel function. Obviously, the first underlined "*almost*" concerns Lebesgue's plane measure and the second refers to the linear measure [1]. Unfortunately, this paper does not contain a reply to this interesting question, since the author has been unable to find one. But let us notice that the problem does not become quite trivial even if the assumption is replaced by the following stronger one:

(i) *The function f fulfils (*) for every pair (x, y) of numbers belonging to a linear set A whose complement has measure zero.*

Then the assertion is positive; moreover: (*) is fulfilled by the function f for *all* x and y. No additional reasoning is needed to prove the more general

THEOREM 1. *If f is a mapping of an Abelian measurable group* (see e. g. [2], p. 257) *G with an invariant σ-finite measure μ into an Abelian group H and if (*) is fulfilled for $x, y \in A \subset G$, $\mu(G \setminus A) = 0$, then f is a homomorphism.*

Proof. Choose $z_1, z_2 \in G$ arbitrarily and let $E(z_1, z_2)$ denote the set of points $(x_1, x_2) \in G \times G$ satisfying the following three conditions:

(ii) $x_1,\ x_2 \in A$,

(iii) $-x_1 \in A - z_1,\ \ -x_2 \in A - z_2$,

(iv) $x_1 - x_2 \in A - z_2,\ \ x_2 - x_1 \in A - z_1$.

It is obvious that almost every point of $G\times G$ fulfils (ii) and (iii). Now, if $B\subset G$ is a μ-measurable set and c_B denotes its characteristic function, then it follows from the measurability of G that $c_B(x_1-x_2)$ is $(\mu\times\mu)$-measurable. Hence, by Fubini's theorem, (iv) is also fulfilled almost everywhere in $G\times G$. So the set $E(z_1,z_2)$ is non-void. Let (x_1,x_2) be one of its elements. Putting $y_1=z_1-x_1$ and $y_2=z_2-x_2$ we have $y_1,y_2\epsilon A$ by (iii) and x_1+y_2, $y_1+x_2\epsilon A$ by (iv). Using also (ii) and the commutativity of G and H we obtain

$$\begin{aligned} f(z_1+z_2) &= f(x_1+y_1+x_2+y_2) = f(x_1+y_2)+f(y_1+x_2) \\ &= f(x_1)+f(y_2)+f(y_1)+f(x_2) = f(x_1+y_1)+f(x_2+y_2) \\ &= f(z_1)+f(z_2). \end{aligned}$$

A slightly better result can be deduced if $\mu(G)=1$ (e. g. for compact groups with normed Haar measure):

THEOREM 2. *If $\mu(G)=1$ is assumed in Theorem 1, then the condition $\mu(G\setminus A)=0$ can be replaced by $\mu(A)>\frac{1}{2}(\sqrt{7}-1)$ without changing the statement about f.*

The proof follows exactly the same lines. One has merely to notice that

$$\begin{aligned} (\mu\times\mu)\big(\{(x_1,x_2)\colon x_1\epsilon A,\ x_2\epsilon A\}\big) \\ = (\mu\times\mu)\big(\{(x_1,x_2)\colon -x_1\epsilon A-z_1,\ -x_2\epsilon A-z_2\}\big) = \big(\mu(A)\big)^2, \end{aligned}$$

and further that on account of Fubini's theorem

$$(\mu\times\mu)\big(\{(x_1,x_2)\colon x_1-x_2\epsilon A-z_2\}\big) = \iint c_{A-z_2}(x_1-x_2)\,\mu(dx_1)\,\mu(dx_2) = \mu(A),$$

and analogously

$$(\mu\times\mu)\big(\{(x_1,x_2)\colon x_2-x_1\epsilon A-z_1\}\big) = \mu(A).$$

Thus, putting $\mu(A)=1-\varepsilon$ $(0<\varepsilon<1)$ we find

$$(\mu\times\mu)\big(E(z_1,z_2)\big) \geqslant 1-2[1-(1-\varepsilon)^2]-2\varepsilon = 2\varepsilon^2-6\varepsilon+1.$$

This is positive (and thus the set E is non-void) if $\varepsilon<\frac{1}{2}(3-\sqrt{7})$.

Theorem 2 can be applied to (*) if the addition $x+y$ is taken mod 1; then it will be sufficient to suppose in (i) that the set $A\subset[0,1)$ is of measure greater than $\frac{1}{2}(\sqrt{7}-1)$.

In the proofs of Theorems 1 and 2 there was but one essential property of the set A which was needed, namely: for any $z_1,z_2\epsilon G$ there are points x_1,x_2,y_1,y_2 of A such that $z_1=x_1+y_1$, $z_2=x_2+y_2$, $x_1+y_2\epsilon A$ and $x_2+y_1\epsilon A$. Let us call a set $B\subset G$ a *small set*, if its complement B' has this property. Thus, if A' is a small set, then f is a homomorphism provided (*) holds for all points of A, without any restriction as to the Abelian groups G and H. If G is a topological group, then every set of 1-st category is small; this can be stated in nearly the same way as has been done for zero-sets in the proofs of the preceding theorems: instead of Fubini's theorem one has but to use its topological analogon, as proved by Sikorski ([3], p. 291, Theorem 3). Hence it appears that the real line or the circle can be decomposed into two small sets (to be small is not an additive property!). For these remarks the author is indebted to E. Marczewski.

The study of the Cauchy equation for groups leads to the question of characterising those Abelian groups which admit non-trivial homomorphisms into the additive group of real numbers. Here is an easy answer:

THEOREM 3. *An Abelian group G admits a non-zero homomorphism into the group R of real numbers if and only if it contains an element of infinite order.*

Proof. From $f(0)=0$ and $f(nx)=nf(x)$ it follows at once that the condition is necessary. Now, if x is of infinite order, fix an irrational λ and put $\chi(nx)=e^{2\pi i\lambda n}$ (n integer). As is well known, a character of a subgroup can always be extended to a character of the whole group. So we may consider χ as defined on G. The circle group K is the direct sum of its torsion part T and of a group L isomorphic to R. If $L\underset{h}{\longleftrightarrow}R$ then putting $h(t)=0$ for $t\epsilon T$ we get a non-zero homomorphism of K into R. Hence $h\chi(z)$ $(z\epsilon G)$ is a homomorphism of G into R and one has $h\chi(x)\neq 0$, since $\chi(x)\notin T$.

REFERENCES

[1] P. Erdös, **P 310**, Colloquium Mathematicum 7 (1960), p. 311.

[2] P. R. Halmos, *Measure Theory*, New York 1950.

[3] R. Sikorski, *On the Cartesian product of metric spaces*, Fundamenta Mathematicae 34 (1947), p. 288-292.

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