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REPRESENTATION AND DISTRIBUTIVITY OF BOOLEAN ALGEBRAS*

BY

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A Boolean algebra is, by definition, a non-empty set $\mathfrak{A}$ of elements (denoted here by letters $A, B, \ldots$) on which there are defined three operations $A \cup B$, $A \cap B$, $-A$ called the *join*, the *meet* and the *complement* of A, B respectively. These operations are characterized by a set of axioms. Many sets of axioms are known (1) but we shall not quote here any of them. We recall only that every set of axioms ensures that the Boolean operations $\cup$, $\cap$, $-$ have, roughly speaking, the same properties as the set-theoretical operations on sets: union, intersection and complement (relative to a fixed space).

We mention below the most typical examples of Boolean algebras:

1° The class $\mathfrak{A}(X)$ of all subsets of a space X, the Boolean operations being the set-theoretical ones.

2° Subalgebras $\mathfrak{F}$ of $\mathfrak{A}(X)$, i. e. non-empty subclasses of $\mathfrak{A}(X)$ closed with respect to the set-theoretical union $A \cup B$, intersection $A \cap B$ and complementation $-A = X - A$. Boolean algebras of this kind are called *fields of sets*.

3° The *two-element Boolean algebra*, i. e. the algebra $\mathfrak{A}(X)$ where X is a one-element set.

4° *Quotient algebras* $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ is a field of subsets of a space X, and $\mathfrak{I}$ is an ideal of sets, i. e. a non-empty additive and hereditary class of subsets of X. The algebra $\mathfrak{F}/\mathfrak{I}$ is obtained by identification of sets whose symmetric difference belongs to $\mathfrak{I}$. The element of $\mathfrak{A}/\mathfrak{I}$ determined by a set $A \in \mathfrak{F}$ is denoted by $[A]$. The Boolean operations in $\mathfrak{F}/\mathfrak{I}$ are defined by the equalities:

$$[A] \cup [B] = [A \cup B], \quad [A] \cap [B] = [A \cap B], \quad -[A] = [-A].$$

* This paper is a lecture held by the author in Amsterdam, Poznań and Wrocław in 1959.

(1) For bibliographical notes, see Sikorski [23], p. 2.

The Boolean algebra $\mathfrak{F}/\mathfrak{J}$ is sometimes called the *algebra of sets* $\epsilon\ \mathfrak{F}$ *modulo the ideal* $\mathfrak{J}$.

There are also other types of Boolean algebras examined in practice, such that their definition has nothing in common with the notion of field of sets. For instance, in Mathematical Logic one investigates the Boolean algebra obtained from the set of all formulas of a formalized theory by identification of equivalent formulas.

However, every Boolean algebra is, by its axiomatic definition, something like a field of sets. Elements of any Boolean algebra are analogues of subsets of a fixed space. The Boolean operations $\cup$, $\cap$, $-$ are analogues of the set-theoretical ones. Every Boolean algebra contains the *zero element* (or: *empty element*) $\wedge$ and the *unit element* (or: *full element*) $\vee$ which are Boolean analogues of the empty set and the whole space. The relation $\subset$ defined as

$$A \subset B \text{ iff } A \cup B = B$$

is the Boolean analogue of the set-theoretical inclusion, etc.

The connection between the notions of Boolean algebras and fields of sets is much deeper. On one hand, every field of sets is a Boolean algebra. On the other hand, the converse statement is also true in some sense: Every Boolean algebra can be represented as a field of sets, that is, for every Boolean algebra $\mathfrak{A}$ there exist a space $\mathcal{X}$ and an isomorphism of $\mathfrak{A}$ into $\mathfrak{A}(\mathcal{X})$, i. e. a one-to-one mapping h of $\mathfrak{A}$ into $\mathfrak{A}(\mathcal{X})$ which transforms the Boolean operations onto the corresponding set-theoretical operations:

$$(1) \qquad h(A \cup B) = h(A) \cup h(B), \qquad h(A \cap B) = h(A) \cap h(B), \qquad h(-A) = -h(A).$$

More exactly, M. H. Stone ([26] and [27]) has proved that, for every Boolean algebra $\mathfrak{A}$, there exists a totally disconnected compact topological space $\mathcal{X}$ such that $\mathfrak{A}$ is isomorphic to the field of all clopen ([2]) subsets of $\mathcal{X}$. The space $\mathcal{X}$ is determined by $\mathfrak{A}$ uniquely up to homeomorphism and is called the *Stone space* of $\mathfrak{A}$.

Stone's representation theorem has opened a new period in the development of the theory of Boolean algebras. However, it solves the representation problem for Boolean algebras from the point of view of finite Boolean operations only. In every Boolean algebra $\mathfrak{A}$, we can also define the notion of infinite join and meet which are Boolean analogues

([2]) Following Halmos, a subset of a topological space is said to be *clopen* provided it is both closed and open.

of the set-theoretical union and intersection of infinitely many sets. Viz. an element A is said to be the *join* of an indexed set $\{A_t\}_{t\epsilon T}$ of elements of $\mathfrak{A}$ provided

(j_1) $A_t \subset A$ for every $t \epsilon T$;

(j_2) if $A_t \subset B \epsilon \mathfrak{A}$ for every $t \epsilon T$, then $A \subset B$.

We then write

$$(2) \qquad A = \bigcup_{t\epsilon T} A_t.$$

Analogously an element A is said to be the *meet* of an indexed set $\{A_t\}_{t\epsilon T}$ of elements of $\mathfrak{A}$ provided

(m_1) $A \subset A_t$ for every $t \epsilon T$;

(m_2) if $B \subset A_t$ ($B \epsilon \mathfrak{A}$) for every $t \epsilon T$, then $B \subset A$.

We then write

$$(3) \qquad A = \bigcap_{t\epsilon T} A_t.$$

If $\overline{\overline{T}} \leqslant \mathfrak{m}$ ($\mathfrak{m}$ being a cardinal), then $\{A_t\}_{t\epsilon T}$ is called an $\mathfrak{m}$-*indexed set* (the same terminology will be applied for doubly indexed sets), and (2) and (3) are called an $\mathfrak{m}$-*join* and an $\mathfrak{m}$-*meet* respectively. The letter $\mathfrak{m}$ will always denote an infinite cardinal, and the letter σ—the cardinal of the set of all integers. The join and meet of an infinite set of elements of $\mathfrak{A}$ do not always exist. If they exist for every $\mathfrak{m}$-indexed set of elements of $\mathfrak{A}$, then $\mathfrak{A}$ is called $\mathfrak{m}$-*complete*. $\mathfrak{A}$ is said to be *complete* provided it is $\mathfrak{m}$-complete for every $\mathfrak{m}$.

We mention the simplest examples of $\mathfrak{m}$-complete Boolean algebras:

5° $\mathfrak{m}$-*fields of sets*, i. e. fields of sets $\mathfrak{F}$, which are closed with respect to the union and intersection of at most $\mathfrak{m}$ sets in $\mathfrak{F}$;

6° $\mathfrak{m}$-*quotient algebras*, i. e. algebras $\mathfrak{F}/\mathfrak{J}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets, and $\mathfrak{J}$ is an $\mathfrak{m}$-additive ideal.

The simplest example of a complete Boolean algebra is given by the algebra $\mathfrak{A}(\mathcal{X})$ of all subsets of a space $\mathcal{X}$. However, there are also complete Boolean algebras which are not isomorphic to any $\mathfrak{A}(\mathcal{X})$. For instance, any measure algebra, i. e. a σ-complete Boolean algebra with a finite (or, more generally, σ-finite) ([3]) strictly positive σ-measure (that is, a countably additive measure vanishing only on the zero element) is a complete Boolean algebra ([4]). In particular, the algebra of all Borel sets (of real numbers) modulo sets of the Lebesgue measure zero is complete. Similarly, the algebra of Borel subsets (of any topological space) modulo

([3]) A σ-measure on a Boolean algebra is said to be *σ-finite* if the unit element is the join of a sequence of elements of finite measure.

([4]) Wecken [30]. See also Sikorski [23], § 21.

sets of the first category is complete (5). The last example is typical: every complete Boolean algebra is isomorphic to a Boolean algebra of the kind just mentioned (6).

The Stone isomorphism h of a Boolean algebra $\mathfrak{A}$ onto the field of all clopen subsets of the Stone space X of $\mathfrak{A}$ does not transform infinite joins and meets into the corresponding set-theoretical unions and intersections. More precisely, if (2) holds, then $h(A)$ is not the set-theoretical union of all the sets $h(A_t)$ $(t \in T)$ except the case where the join (2) is not essentially infinite, i. e. $A = A_{t_1} \cup \ldots \cup A_{t_n}$ for a finite sequence $t_1, \ldots, t_n \in T$. The same remark is true for infinite meets. Both remarks easily follow from the compactness of the Stone space.

The Stone representation theorem fails if infinite Boolean operations are taken into consideration: For every infinite cardinal $\mathfrak{m}$, there exists an $\mathfrak{m}$-complete Boolean algebra which is not isomorphic to any $\mathfrak{m}$-field of sets. This fact is closely connected with the fact that not all identities true for infinite set-theoretical unions and intersections are true for their Boolean analogues: the infinite join and meet. As an example of such an identity we quote here the infinite distributive law:

$$(4) \qquad \bigcap_{t \in T} \bigcap_{s \in S} A_{t,s} = \bigcup_{f \in S^T} \bigcap_{t \in T} A_{t,f(t)}$$

where S^T denotes the set of all mappings from T into S. Identity (4) holds for set-theoretical operations but, in general, it does not hold for infinite Boolean operation: it is possible that all infinite joins and meets in (4) exist but the equality does not hold.

A Boolean algebra $\mathfrak{A}$ is said to be $\mathfrak{m}$-*distributive* provided (4) holds for any $\mathfrak{m}$-indexed set $\{A_{t,s}\}_{t \in T, s \in S}$ of elements of $\mathfrak{A}$ such that all the joins and meets

$$\bigcup_{s \in S} A_{t,s}, \qquad \bigcap_{t \in T} \bigcup_{s \in S} A_{t,s}, \qquad \bigcap_{t \in T} A_{t,f(t)}$$

exist. It is called *completely distributive* iff it is $\mathfrak{m}$-distributive for every $\mathfrak{m}$ (7).

Every $\mathfrak{m}$-field of sets, and consequently every Boolean algebra isomorphic to an $\mathfrak{m}$-field of sets, is $\mathfrak{m}$-distributive. On the other hand, it is easy to construct $\mathfrak{m}$-complete Boolean algebras (of the form $\mathfrak{F}/\mathfrak{I}$

(5) This result is due to Birkhoff and Ulam. See [2].

The completeness of algebras of Borel sets modulo sets of measure zero or of the first category is an example of a very interesting phenomenon of overcompleteness of quotient algebras. For a detailed investigation of overcompleteness, see Smith and Tarski [25]. See also Sikorski [19], where a fundamental non-solved problem on overcompleteness is formulated.

(6) For the proof, see the remark below Theorem 10.

(7) For a systematic investigation of infinite distributivity see Scott [15], Sikorski [22], Smith [24], Smith and Tarski [25]. See also Sikorski [23], §§ 19, 30.

where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets, and $\mathfrak{I}$ is an $\mathfrak{m}$-additive ideal) which are not $\mathfrak{m}$-distributive. Hence it follows that they are not isomorphic to any $\mathfrak{m}$-field of sets. The first examples of $\mathfrak{m}$-complete Boolean algebras which are not isomorphic to any $\mathfrak{m}$-field of sets were based on the idea just mentioned (8). For instance, in this way we can prove that if $\mathfrak{F}$ is a σ-field (of sets of real numbers) containing all Borel sets, and $\mathfrak{I}$ is an σ-additive ideal containing all one-point sets ($\mathfrak{I} \neq \mathfrak{F}$), then the σ-complete Boolean algebra $\mathfrak{F}/\mathfrak{I}$ is not isomorphic to any σ-field of sets (9). If $\mathfrak{F}$ is the field of all Borel sets and $\mathfrak{I}$ is the ideal of all sets of the first category (or: of the Lebesgue measure zero), then $\mathfrak{F}/\mathfrak{I}$ is an example of an $\mathfrak{m}$-complete Boolean algebra which is not isomorphic to any $\mathfrak{m}$-field of sets.

We know some criteria for a given $\mathfrak{m}$-complete Boolean algebra to be isomorphic to an $\mathfrak{m}$-field of sets. Atomicity is the simplest sufficient (but not necessary) condition for the existence of such an isomorphism. An element $a \in \mathfrak{A}$ is said to be an *atom* of the Boolean algebra $\mathfrak{A}$ if $a \neq \wedge$ and the condition $\wedge \neq A \subset a$ implies $A = a$. The notion of atom is the Boolean analogue of the notion of one-element set. A Boolean algebra $\mathfrak{A}$ is said to be *atomic* iff, for every element $A \neq \wedge$, there exists an atom $a \subset A$.

In the theory of measures on Boolean algebras (10) a property weaker than the $\mathfrak{m}$-distributivity, but also of a distributive character, plays an important role. This property is called the *weak* $\mathfrak{m}$-*distributivity* (11). To explain this notion, let us introduce the following notation: if $\{A_{t,s}\}_{t \in T, s \in S}$ is any $\mathfrak{m}$-indexed set, then for every finite set $F \subset S$ the symbol $A_{t,F}$ denotes the finite join

$$A_{t,F} = \bigcup_{s \in F} A_{t,s}.$$

Let $\mathbf{S}$ denote the class of all finite subsets of S and, consequently let $\mathbf{S}^T$ denote the class of all mappings F from T into $\mathbf{S}$. A Boolean algebra $\mathfrak{A}$ is said to be *weakly* $\mathfrak{m}$-*distributive* provided the identity

$$(5) \qquad \bigcap_{t \in T} \bigcup_{s \in S} A_{t,s} = \bigcup_{F \in \mathbf{S}^T} \bigcap_{t \in T} A_{t,F(t)}$$

holds for every $\mathfrak{m}$-indexed set $\{A_{t,s}\}_{t \in T, s \in S}$ of elements of $\mathfrak{A}$ such that all the joins and meets

$$\bigcup_{s \in S} A_{t,s}, \qquad \bigcap_{t \in T} \bigcup_{s \in S} A_{t,s}, \qquad \bigcap_{t \in T} A_{t,F(t)}$$

exist.

(8) Tarski [2], Marczewski [1]. See also Sikorski [18].

(9) Stronger theorems of this kind were proved by Sikorski [18]. See also Sikorski [23], §§ 24, 26, 27, 28.

(10) Recently Kelley [5] has proved that the existence of a finite strictly positive σ-measure on a σ-complete Boolean algebra $\mathfrak{A}$ is equivalent to the existence of a finite strictly positive measure (finitely additive only!) and the weak σ-distributivity of $\mathfrak{A}$.

(11) For investigation of this notion, see e. g. Sikorski [22] and [23], § 30.

The most important case is $\mathfrak{m} = \sigma$. Every measure algebra is weakly σ-distributive [12]. In particular, the algebra of Borel sets modulo sets of the Lebesgue measure zero is weakly σ-distributive. However, the algebra of all Borel sets (of real numbers) modulo sets of the first category is not weakly σ-distributive. It follows from the continuum hypothesis that the algebra of all sets of real numbers modulo any proper σ-additive ideal containing all one-point sets is not weakly σ-distributive [13].

As we have stated above, there are $\mathfrak{m}$-complete Boolean algebras which are not isomorphic to any $\mathfrak{m}$-field of sets. The simplest examples are given in the form of $\mathfrak{m}$-*quotient algebras*, i. e. algebras $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets and the ideal $\mathfrak{I}$ is $\mathfrak{m}$-additive. The question arises whether every $\mathfrak{m}$-complete Boolean algebra is isomorphic to an $\mathfrak{m}$-quotient algebra. The answer is affirmative for $\mathfrak{m} = \sigma$. The answer is negative for $\mathfrak{m} \geqslant 2^\sigma$. For instance the algebra of all Borel sets (of real numbers) modulo sets of the first category or modulo sets of the Lebesgue measure zero is $\mathfrak{m}$-complete but is not isomorphic to any $\mathfrak{m}$-quotient algebra for $\mathfrak{m} \geqslant 2^\sigma$ (see Sikorski [18]).

The above consideration suggests to distinguish the following properties of Boolean algebras:

- A — atomicity;
- CD — complete distributivity;
- CSR — complete set-representability;
- $\mathfrak{m}$SR — $\mathfrak{m}$-set-representability;
- $\mathfrak{m}$D — $\mathfrak{m}$-distributivity;
- W$\mathfrak{m}$D — weak $\mathfrak{m}$-distributivity;
- $\mathfrak{m}$QR — $\mathfrak{m}$-quotient representability.

The atomicity and the distributivity properties being just defined, we are going to define more precisely the remaining properties.

A Boolean algebra $\mathfrak{A}$ is said to be *completely set-representable* if there exist a space X and an isomorphism h of $\mathfrak{A}$ into $\mathfrak{A}(X)$ which preserves all infinite joins and meets in $\mathfrak{A}$ (i. e. if (2) holds in $\mathfrak{A}$ then $h(A)$ is the set-theoretical union of all $h(A_t)$, $t \in T$; and the same holds for meets (3)).

A Boolean algebra $\mathfrak{A}$ is said to be $\mathfrak{m}$-*set-representable* iff there exist a space X and an isomorphism h of $\mathfrak{A}$ into $\mathfrak{A}(X)$ which preserves all the $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}$. By this definition, an $\mathfrak{m}$-complete Boolean algebra is $\mathfrak{m}$-set-representable iff it is isomorphic with an $\mathfrak{m}$-field of sets.

A Boolean algebra $\mathfrak{A}$ is said to be $\mathfrak{m}$-*quotient-representable* provided there exist a space X, an $\mathfrak{m}$-additive ideal $\mathfrak{I}$ and an isomorphism h of $\mathfrak{A}$ into $\mathfrak{A}(X)/\mathfrak{I}$ such that h preserves all $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}$ (i. e. if (2) holds in $\mathfrak{A}$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then $h(A)$ is the join of all $h(A_t)$ in $\mathfrak{A}(X)/\mathfrak{I}$; and the same holds for meets). By this definition, an $\mathfrak{m}$-complete Boolean algebra is $\mathfrak{m}$-quotient-representable iff it is isomorphic to a quotient algebra $\mathfrak{F}/\mathfrak{I}$ where $\mathfrak{F}$ is an $\mathfrak{m}$-field of sets and the ideal $\mathfrak{I}$ is $\mathfrak{m}$-additive.

[12] This theorem was explicitly formulated and proved by Horn and Tarski [4], but it had been applied earlier by Banach and Kuratowski [1].

[13] Banach and Kuratowski [1]. See also Sikorski [23], § 30.

The following theorems explain the mutual connection between the properties quoted on p. 6:

THEOREM 1 [14]. *The following equivalences hold*:

$$\mathrm{A} \Longleftrightarrow \mathrm{CD} \Longleftrightarrow \mathrm{CSR}.$$

THEOREM 2. *The following implications are true* [15]:

$$\mathrm{CSR} \Rightarrow \mathfrak{m}\mathrm{SR} \Rightarrow \mathfrak{m}\mathrm{D} \Rightarrow \mathrm{W}\mathfrak{m}\mathrm{D} \Rightarrow \mathfrak{m}\mathrm{QR}.$$

None of the converse implications is true for $\mathfrak{m} = \sigma$. A counter example for $\sigma\mathrm{SR} \not\Rightarrow \mathrm{CSR}$ is given by any non-atomic σ-field of sets. A counter example [16] for $\sigma\mathrm{D} \not\Rightarrow \sigma\mathrm{SR}$ is given by the algebra $\mathfrak{A}(X)/\mathfrak{I}$ where X is any set of cardinal $> 2^\sigma$ and $\mathfrak{I}$ is the ideal of sets of cardinal $\leqslant 2^\sigma$. A counter example for $\mathrm{W}\sigma\mathrm{D} \not\Rightarrow \sigma\mathrm{D}$ is given by the algebra of all Borel sets (of real numbers) modulo sets of the Lebesgue measure zero. A counter example for $\sigma\mathrm{QR} \not\Rightarrow \mathrm{W}\sigma\mathrm{D}$ is given by the algebra of Borel sets (of real numbers) modulo sets of the first category.

In the case of an arbitrary infinite cardinal $\mathfrak{m}$, we can show, by the same way, that the implications $\mathfrak{m}\mathrm{SR} \Rightarrow \mathrm{CSR}$ and $\mathfrak{m}\mathrm{D} \Rightarrow \mathfrak{m}\mathrm{SR}$ [17] are not true. The algebra $\mathfrak{A}(X)/\mathfrak{I}$ where X is a set of cardinal $> \mathfrak{m}$ and $\mathfrak{I}$ is the ideal of all sets of cardinal $\leqslant \mathfrak{m}$ is a counter example for $\mathfrak{m}\mathrm{QR} \not\Rightarrow \mathfrak{m}\mathrm{D}$. Assuming the generalized continuum hypothesis we can prove that the algebra in question is also a counter example for $\mathfrak{m}\mathrm{QR} \not\Rightarrow \mathrm{W}\mathfrak{m}\mathrm{D}$. The implication $\mathrm{W}\mathfrak{m}\mathrm{D} \Rightarrow \mathfrak{m}\mathrm{D}$ has not yet been examined for $\mathfrak{m} > \sigma$.

Observe that the above counter examples for $\sigma D \not\Rightarrow \sigma\mathrm{SR}$ and $\mathfrak{m}\mathrm{D} \not\Rightarrow \mathfrak{m}\mathrm{SR}$ were based on the following

THEOREM 3 [18]. *The following implication is true*:

$$2^{\mathfrak{m}}\mathrm{QR} \Rightarrow \mathfrak{m}\mathrm{D}.$$

[14] This theorem is due to A. Lindenbaum and A. Tarski. See Tarski [28], Horn and Tarski [4], and also Sikorski [23], § 25.

[15] The implication $\mathrm{W}\mathfrak{m}\mathrm{D} \Rightarrow \mathfrak{m}\mathrm{QR}$ follows from Theorems 6 and 7 below. See also Sikorski [22] and [23], §§ 29, 30.

[16] Sikorski [18].

[17] For a more general result, see Sikorski [21].

[18] See e. g. Sikorski [23], § 21.

A non-empty subset of an $\mathfrak{m}$-complete Boolean algebra is called an $\mathfrak{m}$-*subalgebra* provided it is closed with respect to complementation and forming of $\mathfrak{m}$-joins and $\mathfrak{m}$-meets. A Boolean algebra $\mathfrak{A}$ is said to be $\mathfrak{m}$-*generated* by a class $\mathfrak{C}$ of its elements if the smallest $\mathfrak{m}$-subalgebra containing the set $\mathfrak{C}$ coincides with the whole algebra $\mathfrak{A}$.

THEOREM 4 ([19]). *If an $\mathfrak{m}$-complete Boolean algebra $\mathfrak{A}$ is $\mathfrak{m}$-generated by at most $\mathfrak{m}$ elements, then the following equivalences are true for $\mathfrak{A}$:*

$$\mathrm{A} \Longleftrightarrow \mathfrak{m}\mathrm{SR} \Longleftrightarrow \mathfrak{m}\mathrm{D}.$$

THEOREM 5 ([20]). *An $\mathfrak{m}$-complete Boolean algebra $\mathfrak{A}$ has one of the properties*

$$\mathfrak{m}\mathrm{D}, \quad \mathrm{W}\mathfrak{m}\mathrm{D}, \quad \mathfrak{m}\mathrm{QR}$$

iff each of its $\mathfrak{m}$-subalgebras $\mathfrak{m}$-generated by at most $\mathfrak{m}$ elements has the property in question.

Now we are going to give some characterizations of Boolean algebras having one of the properties mentioned on p. 6. The first characterization is given in terms of two-valued homomorphisms. By a *two-valued homomorphism* on a Boolean algebra $\mathfrak{A}$ we understand any homomorphism of $\mathfrak{A}$ into a two-element Boolean algebra $\mathfrak{B}$ (see p. 1, 3°), i. e. any mapping h of $\mathfrak{A}$ into $\mathfrak{B}$ such that conditions (1) are satisfied. We say that a two-valued homomorphism h of $\mathfrak{A}$ into $\mathfrak{B}$ *preserves* a given join (2) in $\mathfrak{A}$ iff $h(A)$ is the join of all the $h(A_t)$ in $\mathfrak{B}$. The definition of preservation of meets (3) is analogous.

In the proof of the Stone representation theorem the following fact plays a fundamental part: for every element $A \neq \wedge$ of a Boolean algebra $\mathfrak{A}$, there exists a two-valued homomorphism h on $\mathfrak{A}$ such that $h(A) \neq \wedge$ (i. e. $h(A) = \vee$). The characterization given below is formulated in terms of similar but stronger properties:

THEOREM 6. *The following equivalences hold for every Boolean algebra $\mathfrak{A}$:*

CSR $\Longleftrightarrow$ *For every $A \neq \wedge$ there exists a two-valued homomorphism h on $\mathfrak{A}$ such that $h(A) \neq \wedge$ and h preserves all infinite joins and meets in $\mathfrak{A}$.*

$\mathfrak{m}$SR $\Longleftrightarrow$ *For every $A \neq \wedge$ there exists a two-valued homomorphism h on $\mathfrak{A}$ such that $h(A) \neq \wedge$ and h preserves all $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}$* ([21]).

$\mathfrak{m}$D $\Longleftrightarrow$ *For every $A \neq \wedge$ $(A \in \mathfrak{A})$ and for every $\mathfrak{m}$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ $(A \in \mathfrak{A}_0)$ $\mathfrak{m}$-generated by at most $\mathfrak{m}$ elements, there exists a two-valued homomorphism h on $\mathfrak{A}_0$ such that $h(A) \neq \wedge$ and h preserves all $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}_0$* (this equivalence is proved under the hypothesis that $\mathfrak{A}$ is $\mathfrak{m}$-complete).

([19]) See e. g. Sikorski [22] and [23], § 24.
([20]) See e. g. Sikorski [22] and [23], §§ 24, 29, 30.
([21]) Pauc [9], Horn and Tarski [4], Sikorski [18]. See also Sikorski [23], § 24.

W$\mathfrak{m}$D $\Longleftrightarrow$ *For every $A \neq \wedge$ $(A \in \mathfrak{A})$ and for every given set of at most $\mathfrak{m}$ $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}$:*

$$(*) \qquad \begin{aligned} A_s &= \textstyle\bigcup_{t \in T'_s} A_{t,s} \quad (\overline{\overline{T}}{}'_s \leqslant \mathfrak{m}, \quad s \in S', \ \overline{\overline{S}}{}' \leqslant \mathfrak{m}) \\ B_s &= \textstyle\bigcap_{t \in T''_s} B_{t,s} \quad (\overline{\overline{T}}{}''_s \leqslant \mathfrak{m}, \quad s \in S'', \ \overline{\overline{S}}{}'' \leqslant \mathfrak{m}), \end{aligned}$$

there exists an element $A_0 \in \mathfrak{A}$, $\wedge \neq A_0 \subset A$, such that for every two-valued homomorphism h on $\mathfrak{A}$ the condition $h(A_0) \neq \wedge$ implies that h preserves all the infinite joins and meets (*) ([22]).

$\mathfrak{m}$QR $\Longleftrightarrow$ *For every $A \neq \wedge$ $(A \in \mathfrak{A})$ and for every set* (*) *of at most $\mathfrak{m}$ $\mathfrak{m}$-joins and $\mathfrak{m}$-meets in $\mathfrak{A}$, there exists a two-valued homomorphism h such that $h(A) \neq \wedge$ and h preserves all the infinite joins and meets* (*) ([23]).

The next characterization of Boolean algebras having properties mentioned on p. 6 is given in terms of Stone spaces. To facilitate the formulation let us adopt the following terminology.

X will denote the Stone space of the Boolean algebra $\mathfrak{A}$ in question. A set $G \subset X$ $(F \subset X)$ is said to be $\mathfrak{m}$-*open* ($\mathfrak{m}$-*closed*) provided it is the union (the intersection) of at most $\mathfrak{m}$ clopen subsets of X. A subset of X is said to be $\mathfrak{m}$-*nowhere dense* provided it is a subset of an $\mathfrak{m}$-closed nowhere dense set. A subset of X is said to be of the $\mathfrak{m}$-*category* provided it is the union of at most $\mathfrak{m}$ sets $\mathfrak{m}$-nowhere dense in X.

THEOREM 7. *The following equivalences hold for every Boolean algebra $\mathfrak{A}$:*

CSR $\Longleftrightarrow$ *The union of all nowhere dense sets in X is nowhere dense.*

CSR $\Longleftrightarrow$ *The union of all nowhere dense sets in X is a boundary set.*

$\mathfrak{m}$SR $\Longleftrightarrow$ *The union of all $\mathfrak{m}$-nowhere dense sets in X is a boundary subset of X* ([24]).

W$\mathfrak{m}$D $\Longleftrightarrow$ *Every set of $\mathfrak{m}$-category is a nowhere dense subset of X* ([25]).

$\mathfrak{m}$QR $\Longleftrightarrow$ *Every set of $\mathfrak{m}$-category is a boundary subset of X* ([26]).

We do not know any simple characterization of this kind for $\mathfrak{m}$D.

Note that every set of σ-category is of the first category. Every set of the first category in a compact Hausdorff space is a boundary set. Hence the last equivalence implies the following fundamental

THEOREM 8. *Every Boolean algebra has the property σQR* ([27]).

([22]) Sikorski [22] and [23], § 30.
([23]) Chang [3]. See also Sikorski [22] and [23], § 29.
([24]) Pierce [10]. See also Sikorski [23], § 24.
([25]) Kelley [5], Sikorski [22] and [23], § 30.
([26]) Pierce [10], Sikorski [22] and [23], § 29.
([27]) Under the hypothesis of σ-completeness, this theorem was proved independently by Loomis [6] and Sikorski [18]. Theorem 8 was proved in full generality by Sikorski [20].

The topological characterizations of CSR and $\mathfrak{m}$SR suggest also the distinction of the class of Boolean algebras whose Stone spaces have the following property: the union of all m-nowhere dense sets is nowhere dense. This class lies between CSR and $\mathfrak{m}$SR. For $m = \sigma$, we know an example of a Boolean algebra belonging to this class but not having the property CSR (for $m > \sigma$ such an example is not known). The example is given by the algebra of all subsets of an infinite space modulo the ideal of all finite subsets [28]. This algebra is not atomic, therefore it is not CSR. On the other hand, every σ-nowhere dense subset of the Stone space of this algebra is empty, and so is the union of all σ-nowhere dense sets.

The last property follows from the fact that there exists no essentially infinite σ-join in the Boolean algebra just defined. Observe that for $\mathfrak{m} > \sigma$ we do not know whether there exists an infinite Boolean algebra $\mathfrak{A}$ such that no $\mathfrak{m}$-join in $\mathfrak{A}$ is essentially infinite.

Theorems 6 and 7 show that there is a great similarity between the properties mentioned on p. 6. To underline that similarity we quote here an algebraic characterization of those properties.

In the next theorem $\{A_{t,s}\}$ denotes any indexed set of elements of the Boolean algebra $\mathfrak{A}$ in question, such that

(6) $$\bigcap_{t\in T}\bigcup_{s\in S}A_{t,s} = \vee,$$

and S^T and $\mathbf{S}^T$ have a meaning as in (4) and (5).

THEOREM 9. *The following equivalences hold for every Boolean algebra* $\mathfrak{A}$:

CSR $\Longleftrightarrow$ *For every indexed set* $\{A_{t,s}\}_{t\in T s\in S}$, *if* (6) *holds, then for every* $A \neq \wedge$ *there exists a mapping* $f\in S^T$ *such that*

$$A \cap \bigcap_{t\in T}A_{t,f(t)} \neq \wedge .$$

$\mathfrak{m}$D $\Longleftrightarrow$ *For every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T, s\in S}$, *if* (6) *holds, then for every* $A \neq \wedge$ *there exists a mapping* $f\in S^T$ *such that* [29]

$$A \cap \bigcap_{t\in T}A_{t,f(t)} \neq \wedge .$$

W$\mathfrak{m}$D $\Longleftrightarrow$ *For every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T, s\in S}$, *if* (6) *holds, then for every* $A \neq \wedge$ *there exists a mapping* $F\in \mathbf{S}^T$ *such that* [30]

$$A \cap \bigcap_{t\in T}A_{t,F(t)} \neq \wedge .$$

[28] For investigation of this algebra, see Sierpiński [16] and [17].
[29] Smith and Tarski [25]. See also Sikorski [23], § 19.
[30] Sikorski [22] and [23], § 30.

$\mathfrak{m}$QR $\Longleftrightarrow$ *For every* $\mathfrak{m}$*-indexed set* $\{A_{t,s}\}_{t\in T, s\in S}$, *if* (6) *holds, then for every* $A \neq \wedge$ *there exists a mapping* $f\in S^T$ *such that*

$$A \cap \bigcap_{t\in T'}A_{t,f(t)} \neq \wedge$$

for every finite set $T' \subset T$ [31].

The inequalities $\neq$ in the characterization of the first three properties should be read as follows: eithera the infinite meet on the left side does not exist, or it exists but is not equal to $\wedge$.

By an $\mathfrak{m}$*-filter* of a Boolean algebra $\mathfrak{A}$ we understand a non-empty class $\mathfrak{J}$ of elements of $\mathfrak{A}$ such that:

(f_1) if $A_t\in\mathfrak{J}$ for all $t\in T$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then there exists an element $A\in\mathfrak{J}$ such that $A \subset A_t$ foı every $t\in T$;

(f_2) if $A\in\mathfrak{J}$ and $A\subset B$, then $B\in\mathfrak{J}$.

An $\mathfrak{m}$-filter $\mathfrak{J}$ is said to be *proper* iff $\wedge \notin \mathfrak{J}$.

The condition characterizing $\mathfrak{m}$QR in Theorem 9 can be formulated also in the following way: For every $\mathfrak{m}$-indexed set $\{A_{t,s}\}_{t\in T, s\in S}$, if (6) holds, then for every proper $\mathfrak{m}$-filter $\mathfrak{J}$ in $\mathfrak{A}$ there exists a mapping $f\in S^T$ such that

$$A \cap \bigcap_{t\in T'}A_{t,f(t)} \neq \wedge$$

for every $A\in\mathfrak{J}$ and for every finite set $T'\subset T$ [32]. Also conditions characterizing $\mathfrak{m}$QR in Theorems 6 and 7 can be modified in this way [33].

It follows from Theorem 1 that only atomic Boolean algebras can be isomorphically imbedded into a complete atomic algebras with the preservation of all infinite joins and meets. The situation is completely different if we do not require imbedding into atomic algebras.

THEOREM 10 [34]. *For every Boolean algebra* $\mathfrak{A}$ *there exist a complete Boolean algebra* $\mathfrak{A}'$ *and an isomorphism* h *of* $\mathfrak{A}$ *into* $\mathfrak{A}'$ *which preserves all infinite joins and meets in* $\mathfrak{A}$ (i. e. if (2) holds in $\mathfrak{A}$, then $h(A)$ is the join of all the $h(A_t)$ in $\mathfrak{A}'$; and the same holds for meets).

The algebra $\mathfrak{A}'$ can be defined as the quotient algebra $\mathfrak{F}/\mathfrak{J}$ where $\mathfrak{F}$ is the field of all Borel subsets of the Stone space X of $\mathfrak{A}$, and $\mathfrak{J}$ is the ideal of all subsets of the first category. The isomorphism h is defined by the formula

(7) $$h(A) = [h_0(A)]\in\mathfrak{F}/\mathfrak{J} \quad \text{for} \quad A\in\mathfrak{A},$$

[31] This equivalence, proved by Sikorski [22], is a modification of another condition given earlier by Smith [24]. See also [23], § 29.
[32] This characterization was formulated explicitly by Sikorski [5] (see also Sikorski [23], § 29) but it was based on an earlier result of Chang [3].
[33] Such a modification of characterization of $\mathfrak{m}$QR in Theorem 6 is due to Chang [3] who was the first to observe this fact.
[34] For the proof, see e. g. [23], § 35.

where h_0 is the Stone isomorphism of $\mathfrak{A}$ onto the field of all clopen subsets of X. The algebra $\mathfrak{F}/\mathfrak{J}$ is called the *minimal extension* of $\mathfrak{A}$ and has very special properties ([35]).

Observe that if $\mathfrak{A}$ is complete, then the isomorphism h defined by (7) maps $\mathfrak{A}$ onto $\mathfrak{F}/\mathfrak{J}$. This proves the remark on p. 4 that every complete Boolean algebra is isomorphic to a quotient algebra $\mathfrak{F}/\mathfrak{J}$, where $\mathfrak{F}$ is the field of all Borel subsets of a topological space and $\mathfrak{J}$ is the ideal of all subsets of the first category.

Without any hypothesis on the Boolean algebra $\mathfrak{A}$ in question, we can prove the following

THEOREM 11 ([36]). *For any given countable set of infinite joins and meets in a Boolean algebra* $\mathfrak{A}$,

$$(8)\qquad A_n = \bigcup_{t\in T'_n} A_{n,t}, \qquad B_n = \bigcup_{t\in T''_n} B_{n,t} \qquad (n = 1, 2, \ldots),$$

there exists a space X *and an isomorphism* h *of* $\mathfrak{A}$ *into* $\mathfrak{A}(X)$ *such that* h *preserves all the joins and meets* (8).

The proof of Theorem 11 is topological. It is based on the fact that, in a compact Hausdorff space, no open non-void set is of the first category.

Theorem 11 has important applications to Mathematical Logic. The Gödel theorem on completeness of the predicate calculus and the theorem on existence of models for any (finite or countable) consistent set of formulas are immediate consequences of Theorem 11 ([37]).

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([35]) The notion of minimal extension was introduced — in another way — by MacNeille [7]. The above definition was given by Sikorski [20]. For properties of minimal extensions, see Sikorski [23], § 35.

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