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# *J. E. McMILLAN'S AREA THEOREM*

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**1. Introduction.** The purpose of this note is to present a simple proof of a result of J. E. McMillan [3] concerning distortion of harmonic measure by conformal mappings and to re-introduce an interesting conjecture which he made in the same paper. Much has been discovered about the properties of harmonic measure since McMillan's contributions (see e.g. [2] and [5]) and the conjecture seems now to be more accessible.

A similar result of McMillan is used in his original proof of the celebrated twist point theorem [4] and the same considerations used here can be used to simplify the proof of that result as well. The proof given here may also admit generalization to higher-dimensional situations.

In Section 2 we give the simple proof of McMillan's harmonic measure result. Section 3 presents the result on boundary distortion for which the theorem in Section 2 is the key tool.

**2. A property of sets of zero harmonic measure.** Let $\mathbb{D}$ denote the unit disk in the complex plane and let $f:\mathbb{D}\to\Omega$ be a conformal map. Let $A$ denote the set of all $f(e^{i\theta})$ when $f$ has the nontangential limit $f(e^{i\theta})$ at $e^{i\theta}$.

Let $\Omega^*=\Omega\cup A$. If $S_1,S_2\subset\Omega^*$ then $\mathrm{dist}_\Omega(S_1,S_2)=\inf(\mathrm{diam}\,\gamma)$ where the infimum is over Jordan arcs $\gamma$ which lie in $\Omega$ and join $S_1$ and $S_2$.

Note that distinct values of $\theta$ give distinct points in $\Omega^*$ with respect to this distance.

THEOREM 2.1 (McMillan). *Let $E\subset A\subset\partial\Omega$ and suppose that for each $a\in E$ there exists a sequence $\{c_n\}$ of crosscuts of $\Omega$ each of which separates $a$ from $f(0)$ such that*

1. $\mathrm{diam}\,c_n\to 0$, *and*
2. $\displaystyle\sup_n \frac{\mathrm{diam}\,c_n}{\mathrm{dist}_\Omega(c_n,E)}<\infty$.

*Then* $\omega(E,f(0),\Omega)=0$.

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Proof. Write $E = \bigcup E_j$ where $j = 1, 2, \ldots$ and

$$E_j = \left\{ a \in E : \sup_n \frac{\operatorname{diam} c_n}{\operatorname{dist}_\Omega(c_n, E)} < j \text{ for some set of crosscuts} \right\}.$$

Fixing an integer $M$, we will show that for any compact subset $F \subset E_M$,

$$\omega(F, f(0), \Omega) = 0.$$

That will prove the theorem.

Fix a compact $F \subset E_M$. For each $a \in F$ let $c(a)$ denote a crosscut of $\Omega$ separating $a$ from $f(0)$ such that

$$\operatorname{diam} c(a) < \frac{1}{M} \operatorname{dist}_\Omega(a, f(0)) \quad \text{and} \quad \operatorname{diam} c(a) < M \operatorname{dist}_\Omega(c(a), E).$$

Let $D(c)$ denote the simply connected component of $\Omega \setminus c$ not containing $f(0)$. Considering $F$ as a compact subset of $\Omega^*$ we find a finite collection $\mathcal{C}_0 \subset \{c(a)\}$ whose union separates $F$ from $f(0)$ in $\Omega^*$. We delete any crosscuts from this collection satisfying

$$D(c) \subset \bigcup_{c' \neq c} D(c')$$

and name the collection of remaining crosscuts $\mathcal{C}_1$. The union of arcs in $\mathcal{C}_1$ still separates $F$ from $f(0)$ in $\Omega^*$. "Unnecessary" arcs have been deleted.

For $n \geq 2$ the collection $\mathcal{C}_n$ is formed likewise but starting with crosscuts which satisfy

$$\operatorname{dist}_\Omega(c, E) < \frac{1}{2M} \min_{c' \in \mathcal{C}_{n-1}} (\operatorname{diam} c').$$

It follows by condition 2 in the theorem that for any $c_n \in \mathcal{C}_n$ and any $c_{n+1} \in \mathcal{C}_{n+1}$,

$$\operatorname{dist}_\Omega (c_n, c_{n+1}) \geq \frac{1}{2M} \operatorname{diam} c_n. \tag{1}$$

Let $\omega(\mathcal{C}_n, f(0))$ denote the harmonic measure of the union of arcs in $\mathcal{C}_n$ from $f(0)$ in the connected component of $\Omega \setminus \bigcup_{c \in \mathcal{C}_n} D(c)$ containing $f(0)$. For each $n$ we have

$$\omega(F, f(0), \Omega) \leq \omega(\mathcal{C}_{n+1}, f(0)) \leq \omega(\mathcal{C}_n, f(0)) \cdot \sup_{z \in \bigcup_{c \in \mathcal{C}_n} c} \omega(\mathcal{C}_{n+1}, z) \tag{2}$$

by the strong Markov property of harmonic measure.

Let $c \in \mathcal{C}_n$ and let $a$ be an endpoint of $c$. The boundary of a disk of radius $r(c) = \frac{1}{4M} \operatorname{diam} c$ centered at $a$ is connected to $a$ by $\partial\Omega \setminus E$. Denote a disk with center $z$ and radius $r$ by $\Delta_z(r)$ and let

$$U(c) = \bigcup_{z \in c} \Delta_z(r(c)).$$

By condition 2 in the theorem, Harnack's inequality and the Beurling projection theorem (see e.g. [1], p. 43), for any $z \in c$ we have

$$\omega(z, \partial\Omega \cap \Delta_a(r(c)), U) > \alpha > 0$$

where $\alpha$ is a constant depending only on $M$.

It follows from the maximum principle and equation (1) above that

$$\omega(\mathcal{C}_{n+1}, z) \le 1 - \alpha < 1$$

for any $z \in c$ and any $c \in \mathcal{C}_n$. Now equation (2) shows that $\omega(F, f(0), \Omega) = 0$ as desired.

**3. The area theorem.** In [3] Theorem 2.1 is used to prove an interesting result on the distortion at the boundary by a conformal mapping. To describe it we need a few more definitions.

Choose $r_0 < d(f(0), \mathcal{A})$ where $d$ denotes Euclidean distance. For each $a \in \mathcal{A}$ and $r < r_0$ let $\gamma(a, r) \subset \partial\Delta_a(r)$ be a crosscut of $\Omega$ separating $a$ from $f(0)$. Let $L(a, r)$ denote the Euclidean length of $\gamma(a, r)$.

Let $U(a, r) = \bigcup_{r' < r} \gamma(a, r')$. The sets $U(a, r)$ are Lebesgue measurable in the plane and the Fubini theorem applies. To see this, consider the circles centered at the same point $a$ but using the mappings $f(tz)$ with $t \to 1$ and notice that $L(a, r)$ is the pointwise limit of continuous functions.

Let

$$A(a, r) = \int_0^r L(a, \varrho)\, d\varrho$$

denote the Lebesgue measure of $U(a, r)$. In fact, McMillan shows that the sets $U(a, r)$ are each the union of an open set and an at most countable set of the crosscuts $\gamma(a, r)$ whose radii can tend only to zero.

McMillan used Theorem 2.1 to prove

THEOREM 3.1. *The set of* $a \in A$ *such that*

$$\limsup_{r \to 0} \frac{A(a, r)}{\pi r^2} < \frac{1}{2}$$

*has harmonic measure zero.*

Proof. The proof is McMillan's. It is only re-organized here.

Let

$$E_{m,k} = \left\{ a \in A : \frac{A(a, r)}{\pi r^2} < \frac{1}{2} - \frac{1}{m} \text{ for all } r < \frac{1}{k} \right\}.$$

It will be enough to show that any compact subset of $E_{m,k}$ has harmonic measure zero.

Fix a compact set $F \subset E_{m,k}$ and $1/k > \varepsilon > 0$. Given a point $a$ in $F$, we show how to construct a crosscut $c(a, \varepsilon)$ of $\Omega$ with diameter less than $C_1\varepsilon$ which separates $a$ from $f(0)$ and for which

$$\operatorname{diam} c < C_2(m, k) \operatorname{dist}_\Omega(c, F).$$

By Theorem 2.1 then, the set $F$ will be shown to have zero harmonic measure.

For any crosscut $c$ of $\Omega$, denote by $D(c)$ the component of $\Omega - c$ not containing $f(0)$.

There exists a $\delta > 0$ which only depends on $m$ and is such that for any $r < 1/k$ and $a \in E_{m,k}$ the set $\{r' \in (\delta r, r) : L(a,r) < (1/2 - 2/m)2\pi r'\}$ has positive Lebesgue measure. Cover the compact set $F$ with finitely many disks $\{\Delta_\nu\}$ with relative diameter less than $h\varepsilon$. The choice of $h > 0$ will soon be made more explicit. It will also only depend on $m$ and $\delta(m)$.

For each $\nu$, a "step 0 arc for the disk $\Delta_\nu$" is constructed as follows: Choose a point $a_\nu \in \Delta_\nu \cap F$ and an $r' \in (\delta\varepsilon, \varepsilon)$ such that $L(a_\nu, r') < (1/2 - 2/m)2\pi r'$. Let $S$ be the open line segment whose endpoints are the same as $\gamma(a_\nu, r')$. This segment $S$ contains a crosscut $c_0$ of $\Omega$ that separates $a_\nu$ from $f(0)$ and that can be joined to $a_\nu$ by an open Jordan arc $\Lambda$ lying in $\Omega$ and satisfying

$$\Lambda \cap (S \cup \{w : |w - a_\nu| = r'\}) = \emptyset.$$

The segment $c_0$ is the desired building block for the construction. Notice that the Euclidean length of $c_0$ is less than $2\varepsilon$ since $r' \in (\delta\varepsilon, \varepsilon)$. Also note that the Euclidean distance between $c_0$ and $a_\nu$ is at least $\delta\varepsilon \sin(2/m)$. Let $h = (\delta/10)\sin(2/m)$.

We may assume that $a \in F \cap \Delta_1$ and we construct the step 0 arc for $\Delta_1$ and $a$. The $n$th stage in the construction will be a polygonal arc and the construction stops at the $n$th step $c_n$ if

$$\operatorname{dist}_\Omega(c_n, F) \geq h\varepsilon.$$

Otherwise there is some disk $\Delta_\nu$ such that

$$\operatorname{dist}_\Omega(c_n, \Delta_\nu \cap F) < h\varepsilon$$

and we have three alternatives according to whether the step 0 arc for $\Delta_\nu$ has $0, 1$, or more than 1 intersections with $c_n$. Label these possibilities as Case 0, Case I and Case II respectively. In each case the next step $c_{n+1}$ will satisfy

1. $D(c_n) \cup \Delta_\nu \subset D(c_{n+1})$.
2. $\operatorname{dist}_\Omega(c_{n+1}, \Delta_\nu) \geq h\varepsilon$.
3. Each edge of $c_{n+1}$ has Euclidean length at most $2\varepsilon$.
4. The angle on the $f(0)$ side of each corner is at most $\Psi < \pi$ where $\Psi$ is a fixed angle which only depends on $\delta$.

Conditions 1 and 2 imply that the construction halts in finitely many steps at a polygonal arc $c$ satisfying

$$\operatorname{dist}_\Omega(c, F) \geq h\varepsilon.$$

Conditions 3 and 4 imply that the constructed arc $c$ satisfies

$$\operatorname{diam} c < k\varepsilon$$

where $k > 0$ is a constant depending only on $\delta$. Then we have

$$\frac{\operatorname{diam} c}{\operatorname{dist}_{\Omega}(c, F)} < k$$

so that by Theorem 2.1 the proof of Theorem 3.1 will be completed when we construct $c_{n+1}$ satisfying conditions 1–4 in each of the cases 0, I, II.

Let $a^*$ denote a point of $\Delta_\nu \cap F$ such that $\operatorname{dist}_{\Omega}(c_n, a^*) < h\varepsilon$ and let $c_0^*$ denote the step 0 crosscut for $a^*$ and $\Delta_\nu$.

In Case 0 we have

$$\operatorname{dist}_{\Omega}(c_0^*, \Delta_\nu) > 9h\varepsilon, \qquad \operatorname{dist}_{\Omega}(c_n, \Delta_\nu) < h\varepsilon.$$

We see that $a^*$ can be joined to a point of $c_n$ by an open Jordan arc lying in $\Omega$ but not intersecting $c_0^*$. So $c_n \subset D(c_0^*)$. We take $c_{n+1} = c_0^*$ and easily verify 1–4.

For Cases I and II we may assume that $c_0^* \cap c_n$ contains no corner of $c_n$ by choosing a different $r'$ if necessary from the set of positive measure of possibilities in the construction of $c_0^*$.

Also notice that, since the Euclidean length of $c_0^*$ is at most $2\varepsilon$ and

$$\operatorname{dist}_{\Omega}(c_0^*, \Delta_\nu) > 9h\varepsilon,$$

the largest angle that any line segment $l \subset \Omega$ which intersects $c_0^*$ and satisfies $\operatorname{dist}_{\Omega}(l, \Delta_\nu) < h\varepsilon$ can make with $c_0^*$ is

$$\Psi \equiv \pi - \arctan(7h/2) < \pi.$$

We are assuming inductively that all angles on the $f(0)$ side of $c_n$ are less than $\Psi$.

In Case I, let $w^* \in c_n$ be a point such that

$$\operatorname{dist}_{\Omega}(w^*, a^*) < h\varepsilon$$

and let $\alpha$ be the open Jordan subarc of $c_n$ joining $c_0^* \cap c$ to $w^*$. Because the Euclidean distance between $c_0^*$ and $a^*$ is larger than $9h\varepsilon$ we must have $w^* \in D(c_0^*)$ and therefore $\alpha \subset D(c_0^*)$. Recall that by definition $c_0^*$ separates $a^*$ from $f(0)$ and can be joined to $a^*$ by an open Jordan arc in $\Omega$ which intersects neither the line segment with the same endpoints as $\gamma(a^*, r')$ nor the circle of radius $r'$ centered at $a^*$. Since $\operatorname{dist}_{\Omega}(w^*, a^*) < h\varepsilon$, we can join $w^*$ and $a^*$ by an open Jordan arc $\lambda$ in $\Omega$ which does not exit the open half-plane $H$ containing $a^*$ and having $c_0^* \subset \partial H$.

Let $\beta$ denote the closed segment (or point) on $c_0^*$ joining the endpoints of $\alpha$ and $\lambda$ on $c_0^*$. As $\Omega$ is simply connected, the endpoints of $c_0^*$ (which are not in $\Omega$) are in the unbounded component of $\overline{\alpha} \cup \lambda \cup \beta$. This implies that $\alpha \subset H$, because by the induction assumption on the angles of $c_n$, if after entering $H$, $\alpha$ exits on one side of $c_0^*$, the only way for it to re-enter $H$ is to wind around $c_0^*$. Now we can take $c_{n+1}$ to be the union of the component of $c_n - c_0^*$ on the $f(0)$ side of $c_0^*$ together with its endpoint on $c_0^*$ and the component of $c_0^* - c_n$ on the $f(0)$ side of $c_n$. Conditions 1–4 are clear.

In Case II, there exists a component $\alpha$ of $c_n - c_0^*$ with both endpoints on $c_0^*$. Let $\beta(\alpha)$ denote the open line segment joining the endpoints of $\alpha$. Then $\alpha \cup \overline{\beta(\alpha)} \subset \Omega$ is a Jordan curve and as $\Omega$ is simply connected, its interior domain $\Delta(\alpha)$ must be contained in $\Omega$. Since each angle on the $f(0)$ side of $c_n$ is less than $\Psi < \pi$, we see that $\alpha$ is contained in an open half-plane whose boundary contains $c_0^*$. If there were another such arc $\alpha'$ it would have an endpoint in common with $\alpha$ and $\overline{\beta(\alpha')} \cap \overline{\beta(\alpha)} \neq \emptyset$. Further, one of the two components of $c - (\overline{\alpha} \cup \overline{\alpha'})$ would be in $\overline{\Delta(\alpha)} \cup \overline{\Delta(\alpha')} \subset \Omega$, but then an endpoint of $c_n$ would be contained in $\overline{\Delta(\alpha)} \cup \overline{\Delta(\alpha')} \subset \Omega$. This contradiction shows that $c_n \cap c_0^*$ consists of the two endpoints of $\alpha$.

Since points of $\Delta(\alpha)$ are on the $f(0)$ side of $c_n$, we must have $c_n - \alpha \subset \Omega - D(c_0^*)$. For otherwise $c - \overline{\alpha} \subset D(c_0^*)$ and it would be possible to join $f(0)$ to $\alpha$ without touching $c_0^*$ or $D(c_0^*)$ and so to reach the $a$ side of $c_n$ without touching $c_n$.

This means that

$$\operatorname{dist}_\Omega(c_n - \alpha, \Delta_\nu) > 9h\varepsilon$$

so $\alpha$ must intersect $\{w : |w - a^*| < h\varepsilon\}$. Now since the Euclidean length of $c_0^*$ is less than $2\varepsilon$ and the Euclidean distance from $c_0^*$ to $\Delta_\nu$ is more than $h$, we can set $c_{n+1} = (c_n - \alpha) \cup \beta(\alpha)$. The proof is complete.

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