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ON THE EQUIVALENCE OF THE RICCI-PSEUDOSYMMETRY AND PSEUDOSYMMETRY

BY

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1. Introduction. Let (M,g) be a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g), respectively.

A semi-Riemannian manifold (M,g) is said to be *semisymmetric* [18] if

$$R \cdot R = 0 \tag{1}$$

on M. Further, a semi-Riemannian manifold (M,g) is said to be *pseudosymmetric* [6] if

$(*)_1$ the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent

at every point of M. This condition is equivalent to the equality

$$R \cdot R = L_R\, Q(g,R) \tag{2}$$

holding on $\mathcal{U}_R = \left\{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\right\}$, where L_R is a certain function on $\mathcal{U}_R$. The definitions of the tensors used here will be given in Section 2. There exist various examples of pseudosymmetric manifolds which are non-semisymmetric and a review of results on pseudosymmetric manifolds is given in [5] (see also [19]).

It is easy to see that if $(*)_1$ is satisfied on a semi-Riemannian manifold (M,g) then

$(*)_2$ the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent

at every point of M. The converse statement is not true ([5]). A semi-Riemannian manifold (M,g) is called *Ricci-pseudosymmetric* if at every point of M the condition $(*)_2$ is satisfied. If a manifold (M,g) is Ricci–

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pseudosymmetric then

$$R \cdot S = L_S\, Q(g, S) \tag{3}$$

on $\mathcal{U}_S = \left\{x \in M \mid S \neq \frac{\kappa}{n} g \text{ at } x\right\}$, where L_S is a certain function on $\mathcal{U}_S$. We note that $\mathcal{U}_S \subset \mathcal{U}_R$. It is easy to see that

($*$) the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent

at every point of a pseudosymmetric Einstein manifold. It is known that every hypersurface M, $\dim M \geq 4$, of a semi-Riemannian space of constant curvature satisfies ($*$) (cf. [12]). Recently, pseudosymmetric manifolds satisfying ($*$) were investigated in [8]. The condition ($*$) is equivalent to

$$R \cdot R - Q(S, R) = L_1\, Q(g, C) \tag{4}$$

on $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_1 is a certain function on $\mathcal{U}_C$. Warped products satisfying ($*$) were considered in [3]. For instance, in [3] it was shown that any warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = 3$, satisfies ($*$). In particular, every generalized Robertson–Walker spacetime satisfies ($*$).

Evidently, any semi-Riemannian semisymmetric manifold satisfies trivially at every point the following condition ([9]):

($**$) the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent.

This condition is equivalent to

$$R \cdot C = L_2\, Q(S, C) \tag{5}$$

on $\mathcal{U}_2 = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, where L_2 is a certain function on $\mathcal{U}_2$. There exist non-pseudosymmetric manifolds satisfying ($**$) (cf. [9]). Recently 4-dimensional manifolds satisfying ($**$) have been investigated in [10] and [11].

Semi-Riemannian manifolds satisfying $(*)_1$, $(*)_2$, ($*$) and ($**$) or other conditions of this kind, described in [5], are called *manifolds of pseudosymmetry type*.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *Ricci-semisymmetric* if

$$R \cdot S = 0 \tag{6}$$

on M. Evidently, every semisymmetric manifold is Ricci-semisymmetric. The converse is not true. However, if a Ricci-semisymmetric manifold satisfies certain additional assumptions then it is semisymmetric. For instance, every conformally flat Ricci-semisymmetric semi-Riemannian manifold is semisymmetric. This is an obvious consequence of the fact that every conformally flat Ricci-pseudosymmetric semi-Riemannian manifold is pseudosymmetric ([7], Lemma 2). It is a long-standing question whether (1) and (6) are equivalent for hypersurfaces of spaces of constant curvature; cf. Problem

P 808 of [17] by P. J. Ryan, and the references therein. The problem of the equivalence of Ricci-semisymmetry and semisymmetry was also studied in [1]. There one can find a review of partial solutions of this problem. The main result of [1] is the following:

THEOREM 1.1 ([1], Theorem 5.2). *Let (M, g), $n \geq 4$, be a Riemannian Ricci-semisymmetric manifold satisfying*

$$R \cdot R = Q(S, R). \tag{7}$$

If (M, g) has pseudosymmetric Weyl tensor then $R \cdot R = 0$ on $\mathcal{U}_S \subset M$.

We recall that every hypersurface M of $\mathbb{E}^{n+1}_s$, $n \geq 3$, satisfies (7) ([12], Corollary 3.1).

Extending the above problem we consider the problem of the equivalence of Ricci-pseudosymmetry and pseudosymmetry on semi-Riemannian manifolds. It is clear that if at a point x of a manifold (M, g) condition $(*)_1$ is satisfied then also $(*)_2$ holds at x. The converse is not true. For instance, every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 = n - 1 \geq 3$, of a manifold $(M_1, \overline{g})$ and a non-pseudosymmetric Einstein manifold $(M_2, \widetilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [5]). Further, in [4] (Theorem 4) it was shown that $(*)_1$ and $(*)_2$ are equivalent on the subset $\mathcal{U}_S$ of a 4-dimensional warped product $M_1 \times_F M_2$. In particular, (1) and (6) are equivalent on the subset $\mathcal{U}_S$ of a 4-dimensional warped product $M_1 \times_F M_2$. We also note that there exist non-semisymmetric Einsteinian 4-dimensional warped products $M_1 \times_F M_2$, e.g. the Schwarzschild spacetimes. Moreover, the Schwarzschild spacetimes are pseudosymmetric manifolds.

It was proved in [16] that (1) and (6) coincide for hypersurfaces of Riemannian space forms with non-zero constant sectional curvature. This result cannot be extended to the level of pseudosymmetry. Namely, the main result of [13] (Theorem 1) shows that the Cartan hypersurface in the sphere $S^{n+1}(c)$, $n = 6, 12$ or 24, is a non-pseudosymmetric Ricci-pseudosymmetric manifold with non-pseudosymmetric Weyl tensor.

The paper is organized as follows. In Section 2 we fix the notations and present auxiliary lemmas. Moreover, we describe some curvature properties of Ricci-pseudosymmetric manifolds satisfying $(**)$. In Section 3 we continue investigations of such manifolds assuming additionally condition $(*)$. Finally, in Section 4 we restrict our considerations to the special case when $L_S = \frac{\kappa}{n} L_2$. We prove that every Ricci-pseudosymmetric manifold satisfying $(*)$ and $(**)$, with $L_S = \frac{\kappa}{n} L_2$, must be pseudosymmetric (Theorem 4.1). We mention that a family of manifolds realizing the above conditions is described in [2]. Furthermore, we also show that the manifolds considered have additional, very interesting in our opinion, curvature properties. Some of them appeared in the earlier papers devoted to manifolds of pseudosymmetry type. Moreover, we note that a certain converse statement (Theorem 4.2) is also true.

Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞.

2. Preliminaries. Let (M,g) be an n-dimensional, $n\geq 3$, semi-Riemannian manifold. The *Ricci operator* $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X,Y \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M. Next, we define the endomorphisms $\mathcal{R}(X,Y)$, $\mathcal{C}(X,Y)$ and $X\wedge Y$ of $\Xi(M)$ by

$$\begin{aligned}
&\mathcal{R}(X,Y)Z = [\nabla_X,\nabla_Y]Z - \nabla_{[X,Y]}Z,\\
&\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X\wedge \mathcal{S}Y + \mathcal{S}X\wedge Y - \frac{\kappa}{n-1}X\wedge Y\right)Z,\\
&(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y,
\end{aligned}$$

where $X,Y,Z \in \Xi(M)$. Now the Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C and the (0,4)-tensor G of (M,g) are defined by

$$\begin{aligned}
R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\
C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4),\\
G(X_1,X_2,X_3,X_4) &= g((X_1\wedge X_2)X_3,X_4).
\end{aligned}$$

A tensor $\mathcal{B}$ of type $(1,3)$ on M is a *generalized curvature tensor* if

$$\begin{aligned}
&\mathop{\mathcal{S}}_{X_1,X_2,X_3} \mathcal{B}(X_1,X_2)X_3 = 0,\\
&\mathcal{B}(X_1,X_2) + \mathcal{B}(X_2,X_1) = 0,\\
&\qquad B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2),
\end{aligned}$$

where $B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)$. For a $(0,2)$-tensor field A on (M,g) we define the endomorphism $X\wedge_A Y$ of $\Xi(M)$ by $(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y$, where $X,Y,Z \in \Xi(M)$. In particular, $X\wedge_g Y = X\wedge Y$. For a $(0,k)$-tensor field T, $k\geq 1$, a $(0,2)$-tensor field A and a generalized curvature tensor $\mathcal{B}$ on (M,g) we define the tensors $B\cdot T$ and $Q(A,T)$ by

$$\begin{aligned}
(B\cdot T)(X_1,\ldots,X_k;X,Y) &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k) - \ldots\\
&\quad -T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k),\\
Q(A,T)(X_1,\ldots,X_k;X,Y) &= -T((X\wedge_A Y)X_1,X_2,\ldots,X_k) - \ldots\\
&\quad -T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k),
\end{aligned}$$

where $X,Y,Z,X_1,X_2,\ldots \in \Xi(M)$. Putting in the above formulas $\mathcal{B} = \mathcal{R}$ or

$\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$, we obtain the tensors $R \cdot R$, $R \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$, $Q(g, S)$ and $Q(S, C)$, respectively. We note that the Weyl conformal curvature tensor C can also be presented in the following form:

$$C = R - \frac{1}{n-2}U + \frac{\kappa}{(n-1)(n-2)}G, \tag{8}$$

where

$$\begin{aligned} U(X_1, X_2, X_3, X_4) = {} & g(X_1, X_4)S(X_2, X_3) - g(X_1, X_3)S(X_2, X_4) \\ & + g(X_2, X_3)S(X_1, X_4) - g(X_2, X_4)S(X_1, X_3). \end{aligned}$$

Let (M, g) be a semi-Riemannian manifold covered by a system of charts $\{W; x^k\}$. We denote by g_{ij}, R_{hijk}, S_{ij}, $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$ and

$$\begin{aligned} C_{hijk} = {} & R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) \\ & + \frac{\kappa}{(n-1)(n-2)}G_{hijk} \end{aligned} \tag{9}$$

the local components of the metric tensor g, the Riemann–Christoffel curvature tensor R, the Ricci tensor S, the tensor G and the Weyl tensor C, respectively. Further, we denote by $S^2_{ij} = S_{ir}S_j{}^r$ and $S_i{}^j = g^{jr}S_{ir}$ the local components of the tensor S^2 defined by $S^2(X, Y) = S(\mathcal{S}X, Y)$, and of the Ricci operator $\mathcal{S}$, respectively.

At the end of this section we present some results which will be used in the next sections.

LEMMA 2.1 ([8], Lemma 3.6). *If $\mathcal{B}$ is a generalized curvature tensor at a point x of a semi-Riemannian manifold* (M, g), $\dim M \geq 3$, *such that*

$$\underset{X,Y,Z}{\mathcal{S}} a(X)\mathcal{B}(Y, Z) = 0, \quad X, Y, Z \in T_x(M), \tag{10}$$

for a covector a at x, then $Q(a \otimes a, B) = 0$ at x.

Now we present the converse statement.

LEMMA 2.2. *Let* (M, g), $\dim M \geq 3$, *be a semi-Riemannian manifold. Let a be a non-zero covector and $\mathcal{B}$ a generalized curvature tensor at a point x of M satisfying $Q(a \otimes a, B) = 0$. Then* (10) *holds at x.*

P r o o f. In local coordinates the equality $Q(a \otimes a, B) = 0$ takes the form

$$\begin{aligned} & a_h a_l B_{mijk} - a_h a_m B_{lijk} + a_i a_l B_{hmjk} - a_i a_m B_{hljk} \\ & \quad + a_j a_l B_{himk} - a_j a_m B_{hilk} + a_k a_l B_{hijm} - a_k a_m B_{hijl} = 0. \end{aligned}$$

Alternating this identity in h, l, m, and making use of properties of $\mathcal{B}$, we obtain

$$2a_i(a_lB_{hmjk}+a_mB_{lhjk}+a_hB_{mljk})+a_j(a_lB_{hmik}+a_mB_{lhik}a_hB_{mlik}) \\ +a_k(a_lB_{hmji}+a_mB_{lhji}+a_hB_{mlji})=0.$$

Putting $P_{lhmjk}=a_lB_{hmjk}+a_mB_{lhjk}+a_hB_{mljk}$ and applying Lemma 2 of [14], we easily obtain (10).

From Theorem 4.1, Proposition 4.2 and Corollary 4.1 of [8] we get

LEMMA 2.3. *Let x be a point of a semi-Riemannian manifold (M,g), $\dim M\geq 4$, such that*

$$S=\mu g+\varrho a\otimes a,\qquad \underset{X,Y,Z}{\mathcal{S}}\, a(X)\mathcal{B}(Y,Z)=0$$

for some non-zero covector a, where $B=R-\gamma G$, $\mu,\varrho,\gamma\in\mathbb{R}$. Then at x we have

$$R\cdot R=\frac{\kappa}{n(n-1)}Q(g,R),\qquad R\cdot R=Q(S,R)-\frac{(n-2)\kappa}{n(n-1)}Q(g,C).$$

First we consider Ricci-pseudosymmetric manifolds satisfying $(**)$.

LEMMA 2.4. *Let (M,g), $\dim M\geq 4$, be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying condition (5). If $L_2\neq 0$ at $x\in\mathcal{U}_S\cap\mathcal{U}_2\subset M$ then*

$$S_h{}^rR_{rijk}+S_j{}^rR_{rikh}+S_k{}^rR_{rihj}=0, \tag{11}$$

$$S_h{}^rC_{rijk}+S_j{}^rC_{rikh}+S_k{}^rC_{rihj}=0, \tag{12}$$

$$C\cdot S=0, \tag{13}$$

$$S^2=\alpha S+\beta g \tag{14}$$

at x, where

$$\alpha=(n-2)L_S+\frac{\kappa}{n-1},\qquad \beta=\frac{\operatorname{tr}(S^2)}{n}-\frac{\kappa}{n}\left((n-2)L_S+\frac{\kappa}{n-1}\right).$$

Proof. In local coordinates, (3) takes the form

$$S_h{}^rR_{rijk}+S_i{}^rR_{rhjk}=L_S(g_{hj}S_{ik}-g_{hk}S_{ij}+g_{ij}S_{hk}-g_{ik}S_{hj}). \tag{15}$$

Summing cyclically this equation in h,j,k we obtain (11). Using (9) and (11) we easily obtain (12). On the other hand, the relation (5) in local coordinates takes the form

$$C_{rijk}R^r{}_{hlm}+C_{hrjk}R^r{}_{ilm}+C_{hirk}R^r{}_{jlm}+C_{hijr}R^r{}_{klm} \\ =L_2(S_{hl}C_{mijk}-S_{hm}C_{lijk}+S_{il}C_{hmjk}-S_{im}C_{hljk} \\ +S_{jl}C_{himk}-S_{jm}C_{hilk}+S_{kl}C_{hijm}-S_{km}C_{hijl}). \tag{16}$$

Contracting this equality with g^{hk} we get

$$L_2(S_l{}^r(C_{rijm}+C_{rjim})-S_m{}^r(C_{rijl}+C_{rjil}))=0,$$

whence, by the assumption that $L_2 \neq 0$, we obtain

$$S_l{}^r C_{rijm} + S_m{}^r C_{rilj} + S_l{}^r C_{rjim} + S_m{}^r C_{rjli} = 0.$$

Applying now (12) we have

$$(C \cdot S)_{ijlm} = S_i{}^r C_{rjlm} + S_j{}^r C_{rilm} = 0,$$

i.e. the equality (13). It is easy to see, in view of (9), that

$$C \cdot S = R \cdot S - \frac{1}{n-2} Q(g, S^2) + \frac{\kappa}{(n-1)(n-2)} Q(g, S).$$

Applying now (5) and (13) we get

$$Q\left(g, S^2 - \left((n-2)L_S + \frac{\kappa}{n-1}\right)S\right) = 0,$$

which, by Lemma 2.4(i) of [12], leads to

$$S^2 = \left((n-2)L_S + \frac{\kappa}{n-1}\right)S + \lambda g, \quad \lambda \in \mathbb{R}.$$

Hence we easily obtain (14). This completes the proof.

PROPOSITION 2.1. *Let* (M, g), $\dim M \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying condition* (5). *If* $L_2 \neq 0$ *at* $x \in \mathcal{U}_S \cap \mathcal{U}_2 \subset M$ *then*

$$(nL_S - \kappa L_2) S_l{}^r C_{rijk} = (\kappa L_S - \operatorname{tr}(S^2) L_2) C_{lijk} \tag{17}$$

at x. *Moreover, if* $L_S = \frac{\kappa}{n} L_2$ *at* x, *then*

$$\operatorname{tr}(S^2) = \frac{\kappa^2}{n}. \tag{18}$$

Proof. First we observe that (14) and (12) or (13) lead to

$$S^2_{hr} C^r{}_{ijk} + S^2_{jr} C^r{}_{ikh} + S^2_{kr} C^r{}_{ihj} = 0, \tag{19}$$

$$C \cdot S^2 = 0, \tag{20}$$

respectively. Transvecting (16) with $S_p{}^m$ we get

$$\begin{aligned}
S_p{}^s R_{slhr} C^r{}_{ijk} &+ S_p{}^s R_{slir} C_h{}^r{}_{jk} + S_p{}^s R_{sljr} C_{hi}{}^r{}_k + S_p{}^s R_{slkr} C_{hij}{}^r \\
&= L_2 (S_{hl} S_p{}^r C_{rijk} + S_{il} S_p{}^r C_{hrjk} + S_{jl} S_p{}^r C_{hirk} \\
&\quad + S_{kl} S_p{}^r C_{hijr} - S^2_{ph} C_{lijk} - S^2_{pi} C_{hljk} - S^2_{pj} C_{hilk} - S^2_{pk} C_{hijl})
\end{aligned}$$

and, after symmetrization in p, l, by (3),

$$\begin{aligned}
L_S (g_{ph} S_l{}^r C_{rijk} &+ g_{lh} S_p{}^r C_{rijk} - g_{pi} S_l{}^r C_{rhjk} - g_{li} S_p{}^r C_{rhjk} \\
&+ g_{pj} S_l{}^r C_{rkhi} + g_{lj} S_p{}^r C_{rkhi} - g_{pk} S_l{}^r C_{rjhi} \\
&- g_{lk} S_p{}^r C_{rjhi} - S_{lh} C_{pijk} - S_{ph} C_{lijk} \\
&- S_{li} C_{hpjk} - S_{pi} C_{hljk} - S_{lj} C_{hipk} \\
&- S_{pj} C_{hilk} - S_{lk} C_{hijp} - S_{pk} C_{hijl})
\end{aligned} \tag{21}$$

$$
\begin{aligned}
&= L_2(S_{hl}S_p{}^rC_{rijk} + S_{hp}S_l{}^rC_{rijk} + S_{il}S_p{}^rC_{hrjk}\\
&\quad + S_{ip}S_l{}^rC_{hrjk} + S_{jl}S_p{}^rC_{hirk} + S_{jp}S_l{}^rC_{hirk}\\
&\quad + S_{kl}S_p{}^rC_{hijr} + S_{kp}S_l{}^rC_{hijr} - S^2_{ph}C_{lijk} - S^2_{lh}C_{pijk}\\
&\quad - S^2_{pi}C_{hljk} - S^2_{li}C_{hpjk} - S^2_{pj}C_{hilk}\\
&\quad - S^2_{lj}C_{hipk} - S^2_{pk}C_{hijl} - S^2_{lk}C_{hijp}).
\end{aligned}
$$

Contracting (21) with g^{hp} and using (12), (13), (19) and (20), we obtain

$$L_S(nS_l{}^rC_{rijk} - \kappa C_{lijk}) = L_2(\kappa S_l{}^rC_{rijk} - \operatorname{tr}(S^2)C_{lijk}),$$

which immediately leads to (17). Finally, if $L_S = \frac{\kappa}{n}L_2$, then (17), in view of $C \neq 0$ and $L_2 \neq 0$ at x, yields $\operatorname{tr}(S^2) = \kappa^2/n$. This completes the proof.

3. Ricci-pseudosymmetric manifolds satisfying $(*)$ **and** $(**)$**.** In the sequel we restrict our considerations to the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_2$.

LEMMA 3.1. *Let* (M, g), $\dim M \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying conditions* (4) *and* (5). *If* $L_2 \neq 0$ *at* $x \in \mathcal{U}$ *then*

$$(n-1)L_S^2 = \frac{2\kappa}{n}L_S + \frac{\operatorname{tr}(S^2)}{n} - \frac{\kappa^2}{n(n-1)} \tag{22}$$

at x.

Proof. Contracting (15) with g^{hk}, we find

$$A_{ij} = S^{rs}R_{rijs} = S^2_{ij} - nL_SS_{ij} + \kappa L_Sg_{ij}, \tag{23}$$

where $S^{rs} = g^{rt}S_t{}^s$. Applying the operation $R\,\cdot$ to this equality we obtain

$$(R\cdot S)_{rshk}R^r{}_{ij}{}^s + S^{rs}(R\cdot R)_{rijshk} = (R\cdot S^2)_{ijhk} - nL_S(R\cdot S)_{ijhk}. \tag{24}$$

In view of (3), (5) and $S^{rs}C_{rijs} = 0$, which follows immediately from (13), the left hand side of this identity is equal to

$$
\begin{aligned}
L_S(S_k{}^rR_{rjih} + S_k{}^rR_{rijh} - S_h{}^rR_{rjik} - S_h{}^rR_{rijk}) + S^2_{hr}R^r{}_{jik} + S^2_{hr}R^r{}_{ijk}\\
- S^2_{kr}R^r{}_{jih} + S^2_{kr}R^r{}_{ijh} + S_{ih}A_{jk} - S_{ik}A_{jh} + S_{jh}A_{ik} - S_{jk}A_{ih}\\
+ L_1(S_k{}^rC_{rjih} + S_k{}^rC_{rijh} - S_h{}^rC_{rjik} - S_h{}^rC_{rijk}).
\end{aligned}
$$

Using twice (12) and next (13) we can easily see that the expression in the last brackets vanishes. Moreover, in view of (11), we have $S_k{}^rR_{rjih} - S_h{}^rR_{rjik} = -S_i{}^rR_{rjhk}$. Analogously, using

$$S^2_{hr}R^r{}_{ijk} + S^2_{jr}R^r{}_{ikh} + S^2_{kr}R^r{}_{ihj} = 0,$$

which follows immediately from (14) and (11), we get

$$S^2_{hr}R^r{}_{jik} - S^2_{kr}R^r{}_{jih} = -S^2_{ir}R^r{}_{jkh} = S^2_{ir}R^r{}_{jhk}.$$

Taking into account all these identities one can easily see that the left hand side of (24) can be written as follows:

$$-L_S(R\cdot S)_{ijhk} + (R\cdot S^2)_{ijhk} + S_{ih}A_{jk} - S_{ik}A_{jh} + S_{jh}A_{ik} - S_{jk}A_{ih}.$$

Substituting this expression into (24) we obtain

$$(n-1)L_S(R\cdot S)_{ijhk} = S_{ik}A_{jh} - S_{ih}A_{jk} + S_{jk}A_{ih} - S_{jh}A_{ik}.$$

Using now (23) we get

$$(n-1)L_S(R\cdot S) = -Q(S,S^2) + \kappa L_S Q(g,S).$$

But, by (14), we have

$$Q(S,S^2) = Q(S,\alpha S+\beta g) = \beta Q(S,g) = -\beta Q(g,S).$$

Substituting this into the previous equality and using (3), we find

$$(n-1)L_S^2 Q(g,S) = (\beta + \kappa L_S)Q(g,S),$$

which, by the assumption that $x \in \mathcal{U}_S$, implies

$$\beta = L_S((n-1)L_S - \kappa). \tag{25}$$

Using the definition of β we immediately have (22). This completes the proof.

Now, taking into account Proposition 2.1 we have the following

Remark 3.1. Under the assumptions of Lemma 3.1, if $L_S = \frac{\kappa}{n}L_2$ at x then

$$L_S = \frac{\kappa}{n(n-1)}. \tag{26}$$

Moreover, if $\kappa \neq 0$ at x then $L_2 = 1/(n-1)$. In fact, substituting (18) into (22) we easily get (26). The equality $L_2 = 1/(n-1)$ now follows immediately from our assumptions.

Proposition 3.1. *Let* (M,g), $\dim M \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying conditions* (4) *and* (5). *If* $L_2 \neq 0$ *at* $x \in \mathcal{U}$ *then*

$$\begin{aligned} nL_1\left(S_l{}^r C_{rijk} - \frac{\kappa}{n}C_{lijk}\right) + (\kappa - nL_S)S_j{}^r R_{rijk} + nL_S(S_{ik}S_{lj} - S_{ij}S_{lk}) \\ - \kappa L_S(g_{lj}S_{ik} - g_{lk}S_{ij}) + nL_S^2(g_{ij}S_{lk} - g_{ik}S_{lj}) - \kappa L_S^2 G_{lijk} = 0 \end{aligned} \tag{27}$$

at x.

Proof. In local coordinates (4) takes the form

$$\begin{aligned} R_{rijk}R^r{}_{hlm} &+ R_{hrjk}R^r{}_{ilm} + R_{hirk}R^r{}_{jlm} + R_{hijr}R^r{}_{klm} \\ &= S_{hl}R_{mijk} - S_{hm}R_{lijk} + S_{il}R_{hmjk} - S_{im}R_{hljk} + S_{jl}R_{himk} - S_{jm}R_{hilk} \\ &\quad + S_{kl}R_{hijm} - S_{km}R_{hijl} + L_1(g_{hl}C_{mijk} - g_{hm}C_{lijk} + g_{il}C_{hmjk} \\ &\quad - g_{im}C_{hljk} + g_{jl}C_{himk} - g_{jm}C_{hilk} + g_{kl}C_{hijm} - g_{km}C_{hijl}). \end{aligned}$$

Transvecting this with ${S_p}^m$ we obtain

$$
\begin{aligned}
&{S_p}^s R_{slhr} {R^r}_{ijk} - {S_p}^s R_{slir} {R^r}_{hjk} + {S_p}^s R_{sljr} {R^r}_{khi} - {S_p}^s R_{slkr} {R^r}_{jhi} \\
&\quad = S_{hl} {S_p}^s R_{sijk} - S_{il} {S_p}^s R_{shjk} + S_{jl} {S_p}^s R_{skhi} - S_{kl} {S_p}^s R_{sjhi} \\
&\qquad - S^2_{ph} R_{lijk} - S^2_{pi} R_{hljk} - S^2_{pj} R_{hilk} - S^2_{pk} R_{hijl} \\
&\qquad + L_1 (g_{hl} {S_p}^s C_{sijk} - g_{il} {S_p}^s C_{shjk} + g_{jl} {S_p}^s C_{skhi} - g_{kl} {S_p}^s C_{sjhi} \\
&\qquad - S_{ph} C_{lijk} - S_{pi} C_{hljk} - S_{pj} C_{hilk} - S_{pk} C_{hijl}).
\end{aligned}
$$

Symmetrizing the above equality in p, l and using (3), we have

$$
\begin{aligned}
(28)\qquad & L_S (g_{ph} S_{lr} {R^r}_{ijk} + g_{lh} S_{pr} {R^r}_{ijk} - g_{pi} S_{lr} {R^r}_{hjk} - g_{li} S_{pr} {R^r}_{hjk} \\
&\quad + g_{pj} S_{lr} {R^r}_{khi} + g_{lj} S_{pr} {R^r}_{khi} - g_{pk} S_{lr} {R^r}_{jhi} - g_{lk} S_{pr} {R^r}_{jhi} \\
&\quad - S_{lh} R_{pijk} - S_{ph} R_{lijk} + S_{li} R_{phjk} + S_{pi} R_{lhjk} - S_{lj} R_{pkhi} \\
&\quad - S_{pj} R_{lkhi} + S_{lk} R_{pjhi} + S_{pk} R_{ljhi}) \\
&= S_{hl} {S_p}^s R_{sijk} + S_{hp} {S_l}^s R_{sijk} - S_{il} {S_p}^s R_{shjk} - S_{ip} {S_l}^s R_{shjk} \\
&\quad + S_{jl} {S_p}^s R_{skhi} + S_{jp} {S_l}^s R_{skhi} - S_{kl} {S_p}^s R_{sjhi} - S_{kp} {S_l}^s R_{sjhi} \\
&\quad - S^2_{ph} R_{lijk} - S^2_{lh} R_{pijk} - S^2_{pi} R_{hljk} - S^2_{li} R_{hpjk} \\
&\quad - S^2_{pj} R_{hilk} - S^2_{lj} R_{hipk} - S^2_{pk} R_{hijl} - S^2_{lk} R_{hijp} \\
&\quad + L_1 (g_{hl} {S_p}^s C_{sijk} + g_{hp} {S_l}^s C_{sijk} - g_{il} {S_p}^s C_{shjk} \\
&\quad - g_{ip} {S_l}^s C_{shjk} + g_{jl} {S_p}^s C_{skhi} + g_{jp} {S_l}^s C_{skhi} \\
&\quad - g_{kl} {S_p}^s C_{sjhi} - g_{kp} {S_l}^s C_{sjhi} - S_{hp} C_{lijk} - S_{hl} C_{pijk} \\
&\quad - S_{ip} C_{hljk} - S_{il} C_{hpjk} - S_{jp} C_{hilk} - S_{jl} C_{hipk} \\
&\quad - S_{kp} C_{hijl} - S_{kl} C_{hijp}).
\end{aligned}
$$

Now we observe that the tensor A with components A_{ij} given by (23), in view of (14) and (25), can be written in the form

$$
A = \left(\frac{\kappa}{n-1} - 2L_S \right) S + (n-1) L_S^2 g. \tag{29}
$$

Moreover, (14) and (3) imply

$$
R \cdot S^2 = \alpha L_S Q(g, S). \tag{30}
$$

On the other hand, transvecting (15) with ${S_l}^i$ we get

$$
{S_h}^r {S_l}^s R_{rsjk} + S^2_{lr} {R^r}_{hjk} = L_S (S_{lj} S_{hk} - S_{lk} S_{hj} + g_{hj} S^2_{lk} - g_{ik} S^2_{lj}). \tag{31}
$$

Moreover, we note that the following identity is satisfied:

$$
\begin{aligned}
-{S_i}^r {S_l}^s R_{srjk} + {S_j}^r {S_l}^s R_{skri} - {S_k}^r {S_l}^s R_{sjri} &= {S_l}^s ({S_i}^r R_{rsjk} + {S_j}^r R_{risk} + {S_k}^r R_{rijs}) \\
&= {S_l}^s ({S_i}^r R_{rsjk} + {S_s}^r R_{rijk}) = {S_i}^r {S_l}^s R_{rsjk} + S^2_{lr} {R^r}_{ijk}.
\end{aligned}
$$

Contracting (28) with g^{ph} and using (11), (15), (29), (12), (25), the above equality and (31) and (30), we obtain, after standard but somewhat lengthy calculations, the relation (27). This completes the proof.

LEMMA 3.2. *If* (M,g), $\dim M \geq 4$, *is a semi-Riemannian Ricci-pseudosymmetric manifold satisfying conditions* $(*)$ *and* $(**)$ *then we have on* $\mathcal{U}$:

$$(L_2-1)Q(S,R) - L_1Q(g,R) = \frac{1}{n-2}\left(L_2Q(S,U) + \left(\frac{L_2\kappa}{n-1} - L_1 - L_S\right)Q(g,U)\right), \tag{32}$$

$$\begin{aligned}(L_2-1)(\kappa R_{mijk} + S_i{}^r R_{rmjk}) - L_1(n-1)R_{mijk} &= \left(\frac{L_2}{n-2}+1\right)(S_{ik}S_{mj} - S_{mk}S_{ij})\\ &+ \frac{1}{n-2}\left(\frac{n\kappa L_2}{n-1} - (n-1)L_1 - L_S\right)(g_{mk}S_{ij} - g_{mj}S_{ik})\\ &+ \frac{n-1}{n-2}\left(\frac{L_2\kappa}{n-1} - L_1 - L_S\right)(g_{ij}S_{mk} - g_{ik}S_{mj})\\ &+ \frac{L_2}{n-2}(g_{mj}S^2_{ik} - g_{mk}S^2_{ij})\\ &+ \frac{\kappa}{n-2}\left(\frac{L_2\kappa}{n-1} - L_1 - L_S\right)(g_{mj}g_{ik} - g_{mk}g_{ij}).\end{aligned} \tag{33}$$

P r o o f. First we observe that (3) implies

$$R\cdot U = -L_SQ(S,G) = L_SQ(g,U).$$

Moreover, using (8), we obtain $R\cdot C = R\cdot R - \frac{1}{n-2}R\cdot U$. Substituting into this equality the previous one and (4) and (5) we get

$$L_2Q(S,C) = Q(S,R) + L_1Q(g,C) - \frac{L_S}{n-2}Q(g,U). \tag{34}$$

On the other hand, we have

$$\begin{aligned}Q(S,C) &= Q(S,R) - \frac{1}{n-2}Q(S,U) + \frac{\kappa}{(n-1)(n-2)}Q(S,G)\\ &= Q(S,R) - \frac{1}{n-2}Q(S,U) - \frac{\kappa}{(n-1)(n-2)}Q(g,U)\end{aligned}$$

and

$$Q(g,C) = Q(g,R) - \frac{1}{n-2}Q(g,U).$$

Combining the last three equalities we have (32). Using the definition of the tensor $Q(A,T)$, by a standard calculation, we obtain

$$\begin{aligned}g^{hl}Q(S,R)_{hijklm} &= \kappa R_{mijk} + S_i{}^r R_{rmjk} + S_{ik}S_{mj} - S_{ij}S_{km},\\ g^{hl}Q(g,R)_{hijklm} &= (n-1)R_{mijk} + g_{jm}S_{ik} - g_{km}S_{ij},\end{aligned}$$

$$g^{hl}Q(S,U)_{hijklm} = \kappa(g_{mk}S_{ij} - g_{mj}S_{ik}) + g_{mj}S^2_{ik} - g_{mk}S^2_{ij} + (n-1)(S_{ik}S_{mj} - S_{mk}S_{ij}),$$
$$g^{hl}Q(g,U)_{hijklm} = g_{mk}S_{ij} - g_{mj}S_{ik} + (n-1)(g_{ij}S_{mk} - g_{ik}S_{mj}) + \kappa(g_{jm}g_{ik} - g_{km}g_{ij}).$$

Contracting (32) with g^{hl} and using the above relations we get (33), which completes the proof.

4. On a certain subclass of pseudosymmetric manifolds. In this section we consider the special case of Ricci-pseudosymmetric manifolds satisfying conditions $(*)$ and $(**)$:

$$L_S = \frac{\kappa}{n}L_2 \text{ and } L_S \neq 0. \tag{S}$$

According to Remark 3.1 we have

$$L_S = \frac{\kappa}{n(n-1)}, \quad L_2 = \frac{1}{n-1}. \tag{35}$$

Moreover, in view of (14) and (18), we have

$$S^2 = \frac{2\kappa}{n}S - \frac{\kappa^2}{n^2}g. \tag{36}$$

LEMMA 4.1. *Let* (M,g), $\dim M \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying* (4) *and* (5). *If at a point* $x \in \mathcal{U}$ *the hypothesis* (S) *is satisfied then*

$$\left(S_l{}^r C_{rijk} - \frac{\kappa}{n}C_{lijk}\right)\left(L_1 + \frac{(n-2)\kappa}{n(n-1)}\right) = 0 \tag{37}$$

at x.

Proof. (27), in view of (35), takes the form

$$nL_1(S_l{}^r C_{rijk} - \frac{\kappa}{n}C_{lijk}) + \frac{n-2}{n-1}\kappa(S_l{}^r R_{rijk} - \frac{\kappa}{n}R_{lijk}) + \frac{\kappa}{n-1}(S_{ik}S_{lj} - S_{ij}S_{lk}) - \frac{\kappa^2}{n(n-1)}(g_{lj}S_{ik} - g_{lk}S_{ij}) + \frac{\kappa^2}{n(n-1)^2}(g_{ij}S_{lk} - g_{ik}S_{lj}) - \frac{\kappa^3}{n^2(n-1)^2}(g_{lk}g_{ij} - g_{lj}g_{ik}) = 0.$$

Now using (18) and (36) we easily obtain (37). Further, we define the tensor T by

$$T = S - \frac{\kappa}{n}g. \tag{38}$$

It is easy to see that (36) is equivalent to

$$T^2 = 0. \tag{39}$$

According to the above lemma we consider two a priori possible cases:

(i) $L_1 = -\dfrac{(n-2)\kappa}{n(n-1)}$,

(ii) $L_1 \neq -\dfrac{(n-2)\kappa}{n(n-1)}$.

CASE (i). In this case the equality (34) takes the form

$$\frac{1}{n-1}Q(S,C) = Q(S,R) - \frac{(n-2)\kappa}{n(n-1)}Q(g,C) - \frac{\kappa}{n(n-1)(n-2)}Q(g,U). \tag{40}$$

The identity $Q(S,G) = -Q(g,U)$ and (8) imply

$$Q(S,R) = Q(S,C) + \frac{1}{n-2}Q(S,U) + \frac{\kappa}{(n-1)(n-2)}Q(g,U).$$

Substituting this equality into (40) we obtain

$$\frac{n-2}{n-1}Q(T,C) + \frac{1}{n-2}Q(S+\frac{\kappa}{n}g,U) = 0. \tag{41}$$

Using the relations: $T_{ir}C^r{}_{mjk}+T_{mr}C^r{}_{ijk}=0$, $T_{mr}C^r{}_{ikj}+T_{kr}C^r{}_{ijm}+T_{jr}C^r{}_{imk} = 0$, and $T^{rs}C_{rijs} = 0$, which are obvious consequences of (13) and (12), by a standard calculation, we get $g^{hl}Q(T,C)_{hijklm} = -T_{mr}C^r{}_{ijk}$. Similarly, using (36) we find

$$g^{hl}Q\left(S+\frac{\kappa}{n}g,U\right)_{hijklm} = (n-1)(T_{ik}T_{mj} - T_{mk}T_{ij}).$$

Thus contraction of (41) with g^{hl} leads to

$$\frac{n-2}{n-1}T_{mr}C^r{}_{ijk} = \frac{n-1}{n-2}(T_{ik}T_{mj} - T_{mk}T_{ij}). \tag{42}$$

On the other hand, transvecting the equality

$$\frac{n-2}{n-1}Q(T,C)_{hijklm} + \frac{1}{n-2}Q\left(S+\frac{\kappa}{n}g,U\right)_{hijklm} = 0$$

with $T_p{}^l$ and using (39) and (42), in the same manner as above, we get

$$T_{ph}(T_{mj}T_{ik} - T_{mk}T_{ij}) + T_{pi}(T_{mk}T_{hj} - T_{mj}T_{hk}) + T_{pj}(T_{mh}T_{ik} - T_{mi}T_{hk}) + T_{pk}(T_{mi}T_{hj} - T_{mh}T_{ij}) = 0.$$

Putting $\overline{T}_{hijk} = T_{hk}T_{ij} - T_{hj}T_{ik}$ and using the fact that the tensor $\overline{T}$ with components T_{hijk} is a generalized curvature tensor, we can rewrite the above equality in the form

$$T_{ph}\overline{T}_{mijk} + T_{pi}\overline{T}_{hmjk} + T_{pj}\overline{T}_{himk} + T_{pk}\overline{T}_{hijm} = 0. \tag{43}$$

Since $T \neq 0$ at x we can choose a vector w at x such that $a_i = w^r T_{ri} \neq 0$. Transvecting now (43) with w^p we get

$$a_h \overline{T}_{mijk} + a_i \overline{T}_{hmjk} + a_j \overline{T}_{himk} + a_k \overline{T}_{hijm} = 0.$$

Applying now Lemma 4 of [15], in view of $a \neq 0$ at x, we have $\overline{T} = 0$, whence we immediately obtain $T = \varrho a \otimes a$, $\varrho \in \mathbb{R}$. Thus

$$S = \frac{\kappa}{n} + \varrho a \otimes a. \tag{44}$$

Applying (44) we have $Q\big(S + \frac{\kappa}{n} g, U\big) = \frac{2\kappa}{n} Q(g,U) + \varrho Q(a \otimes a, U)$. But $Q(g,U) = -Q(S,G) = -\varrho Q(a \otimes a, G)$. On the other hand, using (44) we easily obtain $Q(a\otimes a, U) = \frac{2\kappa}{n} Q(a \otimes a, G)$. Combining the last three equalities we have $Q\big(S + \frac{\kappa}{n} g, U\big) = 0$ and, in virtue of (41), $Q(T,C) = 0$, i.e. $Q(S,C) = \frac{\kappa}{n} Q(g,C)$. Thus (5) implies, in view of $L_2 = \frac{1}{n-1}$, $R\cdot C = \frac{\kappa}{n(n-1)} Q(g,C)$ and, in view of Ricci-pseudosymmetry of (M,g), $R\cdot R = \frac{\kappa}{n(n-1)} Q(g,R)$. Moreover, using (44) we have $Q(a \otimes a, C) = 0$ whence, by Lemma 2.2, we obtain

$$\underset{X,Y,Z}{\mathcal{S}} a(X)\mathcal{C}(Y,Z) = 0.$$

Case (ii). In this case (37) implies

$$T_{mr} C^r{}_{ijk} = 0. \tag{45}$$

The equality (33) leads to

$$\lambda R = \frac{n-1}{n-2} \overline{S} - \frac{1}{n-2}\left(\frac{\kappa}{n} - (n-1)L_1\right) U + \frac{\kappa}{n-2}\left(\frac{\kappa}{n^2(n-1)} - L_1\right) G,$$

where $\lambda = \frac{n-2}{n}\kappa + (n-1)L_1$. This implies $\lambda C = \frac{n-1}{n-2}\big(\overline{S} - \frac{\kappa}{n} U + \frac{\kappa^2}{n^2} G\big)$ and consequently $\frac{n-2}{n-1}\lambda C = \overline{T}$. Hence, in view of (39) and $\lambda \neq 0$, we have $C_{rijk} C^r{}_{pqt} = 0$, and next $C \cdot C = 0$. On the other hand, we can check that the following identity is satisfied on any semi-Riemannian manifold:

$$\begin{aligned}
(C\cdot C)_{hijklm} = (R \cdot C)_{hijklm} &+ \frac{1}{n-2} Q\left(\frac{\kappa}{n-1} g - S, C\right)_{hijklm} \\
&- \frac{1}{n-2}\big(g_{hl} S_m{}^r C_{rijk} - g_{hm} S_l{}^r C_{rijk} \\
&- g_{il} S_m{}^r C_{rhjk} + g_{im} S_l{}^r C_{rhjk} + g_{jl} S_m{}^r C_{rkhi} \\
&- g_{jm} S_l{}^r C_{rkhi} - g_{kl} S_m{}^r C_{rjhi} + g_{km} S_l{}^r C_{rjhi}\big).
\end{aligned}$$

Using now (5) and (45) we get $C \cdot C = -\frac{1}{(n-1)(n-2)} Q(T,C)$. Thus $Q(T,C) = 0$, which, as we saw in the previous case, leads to (2).

Thus we have proved the following

THEOREM 4.1. *Let* (M, g), $\dim M \geq 4$, *be a semi-Riemannian Ricci-pseudosymmetric manifold satisfying* $(*)$ *and* $(**)$ *and let* $x \in \mathcal{U}$. *Assume that* $L_S = \frac{\kappa}{n} L_2$ *at* x, $L_S \neq 0$, *and let* $\lambda = \frac{n-2}{n}\kappa + (n-1)L_1$. *Then* (2) *holds at* x *and*

(i) *if* $\lambda \neq 0$, *then* $\frac{n-2}{n-1}\lambda C = \overline{T}$ *at* x;
(ii) *if* $\lambda = 0$, *then the following identities hold at* x:

$$S = \frac{\kappa}{n} g + \varrho a \otimes a, \qquad \underset{X,Y,Z}{\mathcal{S}} a(X)\mathcal{C}(Y, Z) = 0$$

for some non-zero covector a *at* x *and* $\varrho \in \mathbb{R}$.

We now present the converse statement to (ii).

THEOREM 4.2. *Let* x *be a point of a semi-Riemannian manifold* (M, g), $\dim M \geq 4$, *such that* $x \in \mathcal{U}$ *and the following conditions are satisfied at* x:

$$S = \mu g + \varrho a \otimes a, \tag{46}$$

$$\underset{X,Y,Z}{\mathcal{S}} a(X)\mathcal{C}(Y, Z) = 0 \tag{47}$$

for some non-zero covector a *and* $\mu, \varrho \in \mathbb{R}$. *Then* (2), (4) *and* (5) *hold at* x. *Moreover*,

$$L_R = \frac{\kappa}{n(n-1)}, \quad L_1 = -\frac{(n-2)\kappa}{n(n-1)}, \quad L_2 = \frac{1}{n-1}, \quad \mu = \frac{\kappa}{n}, \quad \kappa \neq 0. \tag{48}$$

P r o o f. First we observe that the identity (47), which in local coordinates takes the form $a_l C_{hijk} + a_h C_{iljk} + a_i C_{lhjk} = 0$, implies $a_r C^r{}_{ijk} = 0$ and next $a_r a^r = 0$. Thus the relation (46) leads to $\mu = \kappa/n$. We assert that $\kappa \neq 0$ at x. Suppose that $\kappa = 0$. Then $S = \varrho a \otimes a$ and applying Lemma 2.1, in view of $\varrho \neq 0$, we get $Q(S, C) = 0$, a contradiction, because $x \in \mathcal{U}$. Using (8), (46) and $\mu = \kappa/n$ we have

$$C = R - \frac{\kappa}{n(n-1)} G + P, \tag{49}$$

where the $(0, 4)$-tensor P is defined by $P_{hijk} = g_{hk}a_i a_j + g_{ij}a_h a_k - g_{hj}a_i a_k - g_{ik}a_h a_j$. It is easy to see that $\mathcal{S}_{X,Y,Z}\, a(X)\mathcal{P}(Y, Z) = 0$. Taking into account this equality, (49) and (47) we obtain $\mathcal{S}_{X,Y,Z}\, a(X)\mathcal{B}(Y, Z) = 0$, where $B = R - \frac{\kappa}{n(n-1)} G$. Applying now Lemma 2.3 we see that (M, g) is pseudosymmetric and satisfies (4) with L_R and L_1 given by (48). We can check that on any semi-Riemannian manifold the equality $R \cdot R = \frac{\kappa}{n(n-1)} Q(g, R)$ implies

$$R \cdot C = \frac{\kappa}{n(n-1)} Q(g, C). \tag{50}$$

The relation (47), in view of Lemma 2.1, implies $Q(a \otimes a, C) = 0$. On the other hand, using (46) we have $Q(S, C) = \mu Q(g, C) + \varrho Q(a \otimes a, C)$. Thus

$Q(g,C) = \frac{1}{\mu}Q(S,C) = \frac{n}{\kappa}Q(S,C)$. Substituting this equality into (50) we get $R\cdot C = \frac{1}{n-1}Q(S,C)$. This completes the proof.

It is worth noticing that manifolds satisfying simultaneously conditions (5) and (47) have been considered in [9]. The main result of that paper says that the function L_2 of every such manifold must be equal to $1/(n-1)$ or $1/(n-2)$. Moreover, if $L_2 = 1/(n-1)$ then such a manifold is pseudosymmetric. On the other hand, as shown in Example 5.1 of [9], there exist manifolds with $L_2 = 1/(n-2)$ which are not pseudosymmetric.

REFERENCES

[1] K. Arslan, Y. Çelik, R. Deszcz and R. Ezentaş, *On the equivalence of Ricci-semisymmetry and semisymmetry*, Colloq. Math. 76 (1998), 279–294.

[2] F. Defever, R. Deszcz, M. Hotloś, M. Kucharski and Z. Şentürk, *On mani-folds of pseudosymmetry type*, in: Geometry and Topology of Submanifolds, IX, World Sci., River Edge, N.J., in print.

[3] F. Defever, R. Deszcz and M. Prvanović, *On warped product manifolds satisfying some curvature conditions of pseudosymmetry type*, Bull. Greek Math. Soc. 36 (1994), 43–67.

[4] R. Deszcz, *On four-dimensional Riemannian warped product manifolds satisfying certain pseudo-symmetry curvature conditions*, Colloq. Math. 62 (1991), 103–120.

[5] —, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A 44 (1992), 1–34.

[6] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 15 (1987), 311–322.

[7] —, —, *On certain curvature conditions on Riemannian manifolds*, Colloq. Math. 58 (1990), 259–268.

[8] R. Deszcz and M. Hotloś, *On a certain subclass of pseudosymmetric manifolds*, Publ. Math. Debrecen 53 (1998), 29–48.

[9] —, —, *On a certain extension of the class of semisymmetric manifolds*, Publ. Inst. Math. (Beograd) (N.S.) 63 (77) (1998), 115–130.

[10] R. Deszcz, M. Hotloś and Z. Şentürk, *On a certain application of Patterson's identity*, Dept. Math., Agricultural Univ. Wrocław, preprint No. 55, 1997.

[11] R. Deszcz and M. Kucharski, *On curvature properties of certain generalized Robertson–Walker spacetimes*, Tsukuba J. Math., in print.

[12] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, in: Geometry and Topology of Submanifolds, III, World Sci., River Edge, N.J., 1991, 131–147.

[13] R. Deszcz and Ş. Yaprak, *Curvature properties of Cartan hypersurfaces*, Colloq. Math. 67 (1994), 91–98.

[14] W. Roter, *On conformally symmetric Ricci-recurrent spaces*, ibid. 31 (1974), 87–96.

[15] —, *On generalized curvature tensors on some Riemannian manifolds*, ibid. 37 (1977), 233–240.

[16] P. J. Ryan, *Hypersurfaces with parallel Ricci tensor*, Osaka J. Math. 8 (1971), 251–259.

[17] —, *A class of complex hypersurfaces*, Colloq. Math. 26 (1972), 175–182.
[18] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$. I. The local version*, J. Differential Geom. 17 (1982), 531–582.
[19] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz*, in: Geometry and Topology of Submanifolds, VI, World Sci., River Edge, N.J., 1994, 199–209.

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