COLLOQUIUM MATHEMATICUM

VOL. 79 1999 NO. 2

VECTOR-VALUED ERGODIC THEOREMS FOR MULTIPARAMETER ADDITIVE PROCESSES

BY

RYOTARO SATO (OKAYAMA)

Abstract. Let X be a reflexive Banach space and (Ω, Σ, μ) be a σ-finite measure space. Let $d \geq 1$ be an integer and $T = \{T(u) : u = (u_1, \ldots, u_d),\ u_i \geq 0,\ 1 \leq i \leq d\}$ be a strongly measurable d-parameter semigroup of linear contractions on $L_1((\Omega, \Sigma, \mu); X)$. We assume that to each $T(u)$ there corresponds a positive linear contraction $P(u)$ defined on $L_1((\Omega, \Sigma, \mu); \mathbb{R})$ with the property that $\|T(u)f(\omega)\| \leq P(u)\|f(\cdot)\|(\omega)$ almost everywhere on Ω for all $f \in L_1((\Omega, \Sigma, \mu); X)$. We then prove stochastic and pointwise ergodic theorems for a d-parameter bounded additive process F in $L_1((\Omega, \Sigma, \mu); X)$ with respect to the semigroup T.

1. Introduction and the theorems. Let X be a reflexive Banach space and (Ω, Σ, μ) be a σ-finite measure space. For $1 \leq p \leq \infty$, let $L_p(\Omega; X) = L_p((\Omega, \Sigma, \mu); X)$ denote the usual Banach space of all X-valued strongly measurable functions f on Ω with the norm

$$\|f\|_p := \Big(\int \|f(\omega)\|^p \, d\mu\Big)^{1/p} < \infty \quad \text{if } 1 \leq p < \infty,$$
$$\|f\|_\infty := \operatorname{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

If $d \geq 1$ is an integer, we let $\mathbb{R}_d^+ = \{u = (u_1, \ldots, u_d) : u_i \geq 0,\ 1 \leq i \leq d\}$ and $\mathbb{P}_d = \{u = (u_1, \ldots, u_d) : u_i > 0,\ 1 \leq i \leq d\}$. Further $\mathcal{I}_d$ is the class of all bounded intervals in $\mathbb{R}_d^+$ and λ_d denotes the d-dimensional Lebesgue measure. In this paper we consider a strongly measurable d-parameter semigroup $T = \{T(u) : u \in \mathbb{R}_d^+\}$ of linear contractions on $L_1(\Omega; X)$. Thus T is strongly continuous on $\mathbb{P}_d$ (cf. Lemma VIII.7.9 in [1]). A linear operator U defined on $L_1(\Omega; X)$ is said to *have a majorant* P defined on $L_1(\Omega; \mathbb{R})$ if P is a positive linear operator on $L_1(\Omega; \mathbb{R})$ with the property that $\|Uf(\omega)\| \leq P\|f(\cdot)\|(\omega)$ a.e. on Ω for all $f \in L_1(\Omega; X)$. We assume in the theorems below that each $T(u)$, $u \in \mathbb{R}_d^+$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. As is known (cf. Theorem 4.1.1 in [7]), this holds automatically when $X = \mathbb{R}$ or $\mathbb{C}$ (= the complex numbers). But in general this is not the case, which can be seen by a simple counter-example (see [8]).

1991 *Mathematics Subject Classification*: Primary 47A35.

By a (d-dimensional) *process* F in $L_1(\Omega; X)$ we mean a set function $F : \mathcal{I}_d \to L_1(\Omega; X)$. It is called *bounded* if

$$K(F) := \sup\{\|F(I)\|_1/\lambda_d(I) : I \in \mathcal{I}_d,\ \lambda_d(I) > 0\} < \infty,$$

and *additive* (with respect to T) if it satisfies the following conditions:

(i) $T(u)F(I) = F(u + I)$ for all $u \in \mathbb{R}_d^+$ and $I \in \mathcal{I}_d$,

(ii) if $I_1, \ldots, I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$ then $F(I) = \sum_{i=1}^k F(I_i)$.

In particular, if $F(I) = \int_I T(u)f\,du$ for all $I \in \mathcal{I}_d$, where f is a fixed function in $L_1(\Omega; X)$, then $F(I)$ defines a bounded additive process in $L_1(\Omega; X)$ with respect to T.

In the following, $q\text{-}\lim_{\alpha\to\infty}$ and $q\text{-}\limsup_{\alpha\to\infty}$ will mean that these limits are taken as α tends to infinity along a countable dense subset Q of the positive real numbers. Here we may assume that Q contains the positive rational numbers. A net (f_α) of strongly measurable X-valued functions on Ω is said to *converge stochastically* to a strongly measurable X-valued function f_∞ on Ω if for any $\varepsilon > 0$ and $A \in \Sigma$ with $\mu(A) < \infty$ we have

$$\lim_\alpha \mu(A \cap \{\omega : \|f_\alpha(\omega) - f_\infty(\omega)\| > \varepsilon\}) = 0.$$

It is now time to state the theorems.

THEOREM 1. *Let X be a reflexive Banach space and $T = \{T(u) : u \in \mathbb{R}_d^+\}$ a semigroup of linear contractions on $L_1(\Omega; X)$, strongly continuous on $\mathbb{P}_d$, such that each $T(u)$ with $u \in \mathbb{R}_d^+$ has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. Then for any d-dimensional bounded additive process F in $L_1(\Omega; X)$ with respect to T, the averages $\alpha^{-d}F([0, \alpha]^d)$ converge stochastically to a function F_∞ in $L_1(\Omega; X)$ invariant under T as α tends to infinity.*

In particular, if the operators $P_i = P(e^i)$, e^i being the ith unit vector in $\mathbb{R}_d^+$, satisfy the additional hypothesis

$$\|P_i\|_\infty \le 1 \quad (1 \le i \le d), \tag{1}$$

then

$$q\text{-}\lim_{\alpha\to\infty} \alpha^{-d}F([0, \alpha]^d)(\omega) = F_\infty(\omega) \quad \text{a.e. on } \Omega. \tag{2}$$

THEOREM 2. *Let X, $T = \{T(u) : u \in \mathbb{R}_d^+\}$, and F be the same as in Theorem 1. If the positive operators $P_i = P(e^i)$, $1 \le i \le d$, commute then the averages*

$$\Big(\prod_{i=1}^d \alpha_i\Big)^{-1} F([0, \alpha_1] \times \ldots \times [0, \alpha_d]) \tag{3}$$

converge stochastically to a function F_∞ *in* $L_1(\Omega; X)$ *invariant under* T *as* α_i *tends to infinity independently for each* $1 \le i \le d$. *If in addition the averages*

$$A_n(P_1, \ldots, P_d)f := A_n(P_1) \ldots A_n(P_d)f \quad (n \ge 1), \tag{4}$$

where

$$A_n(P_i) := n^{-1} \sum_{k=0}^{n-1} P_i^k \quad (1 \le i \le d), \tag{5}$$

converge a.e. for all $f \in L_1(\Omega; \mathbb{R})$, *then* (2) *holds.*

Theorems 1 and 2 may be considered to be vector-valued continuous refinements of Krengel's stochastic ergodic theorem (cf. Theorems 3.4.9 and 6.3.10 in [7]) and Dunford and Schwartz's pointwise ergodic theorem (cf. Theorem 6.3.7 in [7]). See also [5]. Concerning Theorem 2, some sufficient conditions for the a.e. convergence of $A_n(P_1, \ldots, P_d)f$ for all $f \in L_1(\Omega; \mathbb{R})$, where $P_1, \ldots, P_d$ are commuting positive linear contractions on $L_1(\Omega; \mathbb{R})$, have been examined in [4]. For example, one of such conditions is that the Brunel operator U corresponding to $P_1, \ldots, P_d$ satisfies the pointwise ergodic theorem.

Here it may be appropriate to explain the role of the extra assumptions made about T in Theorems 1 and 2 (existence of a contraction majorant $P(u)$ and commutativity of operators P_i, $1 \le i \le d$). When $X = \mathbb{R}$ or $\mathbb{C}$, the existence of such a $P(u)$ is known; and it seems to the author that almost all known proofs of scalar-valued (stochastic and pointwise) ergodic theorems depend upon this fact. But, when $X \ne \mathbb{R}$ and $\mathbb{C}$, the existence of such a $P(u)$ does not follow, as remarked above. On the other hand, the continuous one-parameter version of Chacon's vector-valued ergodic theorem (see e.g. §4.2 of [7]) has been proved by Hasegawa, Sato and Tsurumi [6]; the key to the proof was Chacon's maximal ergodic lemma. Thus, in this case, such a $P(u)$ was not used at all. Incidentally, the reflexivity of X was only used there to deduce that the mean ergodic theorem holds for T, when T was considered to be a contraction semigroup on $L_p(\Omega; X)$ with $1 < p < \infty$. In this paper we also assume the reflexivity of X for this purpose.

Now, let $d \ge 2$. It is natural to ask if the continuous d-parameter version of Chacon's vector-valued ergodic theorem holds. This is an open problem. And, if we assume the existence of such a $P(u)$ which satisfies in addition $\|P(u)\|_\infty \le 1$ for each $u \in \mathbb{R}_d^+$, then an affirmative answer follows. In this connection we refer the reader to [5] and [8]. These are the reasons to assume the existence of such a $P(u)$ in Theorem 1. In Theorem 2 the commutativity of P_i is assumed. It is an open question whether Theorem 2 holds without the commutativity assumption.

2. Preliminaries. The next two theorems are slight modifications of Theorem 4 and Theorem 1(a) of [4]. Since these can be proved as in [4], we omit the details. The theorems will be used in order to prove those mentioned in the preceding section.

THEOREM A. *Let X be a reflexive Banach space. Let $T_1, \ldots, T_d$ be linear contractions on $L_1(\Omega; X)$, and $P_1, \ldots, P_d$ be positive linear contractions on $L_1(\Omega; \mathbb{R})$ such that $\|T_i f(\omega)\| \le P_i \|f(\cdot)\|(\omega)$ a.e. on Ω for all $f \in L_1(\Omega; X)$ and $1 \le i \le d$ and also such that $\|P_i\|_\infty \le 1$ for all $1 \le i \le d$. If either the operators $T_1, \ldots, T_d$ or the operators $P_1, \ldots, P_d$ commute, then for every $f \in L_1(\Omega; X)$ the averages $A_n(T_1, \ldots, T_d)f$ converge a.e. on Ω as n tends to infinity.*

THEOREM B. *Let X be a reflexive Banach space. Let $T_1, \ldots, T_d$ be commuting linear contractions on $L_1(\Omega; X)$, and $P_1, \ldots, P_d$ be commuting positive linear contractions on $L_1(\Omega; \mathbb{R})$ such that $\|T_i f(\omega)\| \le P_i \|f(\cdot)\|(\omega)$ a.e. on Ω for all $f \in L_1(\Omega; X)$ and $1 \le i \le d$. If the limit*

$$\lim_n A_n(P_1, \ldots, P_d) f(\omega)$$

exists a.e. on Ω for all $f \in L_1(\Omega; \mathbb{R})$, then the limit

$$\lim_n A_n(T_1, \ldots, T_d) f(\omega)$$

exists a.e. on Ω for all $f \in L_1(\Omega; X)$.

The next lemma is also a slight modification of Lemma 1 in [8]; we omit the details here.

LEMMA. *Let $T = \{T(u) : u \in \mathbb{R}_d^+\}$ be a semigroup of linear contractions on $L_1(\Omega; X)$, strongly continuous on $\mathbb{P}_d$, such that each $T(u)$ with $u \in \mathbb{R}_d^+$ has contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. Then there exists a positive linear contraction $\tau(u)$ on $L_1(\Omega; \mathbb{R})$ for each $u \in \mathbb{R}_d^+$, called the linear modulus of $T(u)$, such that*

(i) $\|T(u) f(\omega)\| \le \tau(u) \|f(\cdot)\|(\omega) \le P(u) \|f(\cdot)\|(\omega)$ *a.e. on Ω for all $f \in L_1(\Omega; X)$,*

(ii) $\tau(u) g = \operatorname{ess\,sup}\{\sum_{i=1}^k \|T(u) f_i(\cdot)\| : f_i \in L_1(\Omega; X),\ \sum_{i=1}^k \|f_i(\omega)\| \le g(\omega)$ *a.e. on $\Omega\}$ for all $g \in L_1^+(\Omega; \mathbb{R})$,*

(iii) $\tau(s+t) \le \tau(s)\tau(t)$ *for all $s, t \in \mathbb{R}_d^+$,*

(iv) *if $u \in \mathbb{P}_d$ then*

$$\tau(u) = \text{strong-}\lim_{\substack{t \to u \\ t \ge u}} \tau(t). \tag{6}$$

In particular, if the semigroup T is strongly continuous on $\mathbb{R}_d^+$ then we have (6) for all $u \in \mathbb{R}_d^+$.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. By an easy argument we may assume that $d \geq 2$ (see e.g. [8]). Putting $T_i = T(e^i)$, $1 \leq i \leq d$, we then apply Theorem 6.3.4 of [7] to infer that there exists a constant $C_d > 0$ and a positive linear contraction U in $L_1(\Omega;\mathbb{R})$ of the form

$$U = \sum_{n_1,\ldots,n_d \geq 0} a(n_1,\ldots,n_d) P_1^{n_1} \ldots P_d^{n_d},$$

where

$$a(n_1,\ldots,n_d) > 0 \quad \text{and} \quad \sum_{n_1,\ldots,n_d \geq 0} a(n_1,\ldots,n_d) = 1,$$

so that for all $f \in L_1(\Omega;X)$,

$$\|A_n(T_1,\ldots,T_d)f(\omega)\| \leq C_d \cdot A_{d(n)}(U)\|f(\cdot)\|(\omega) \quad \text{a.e. on } \Omega. \tag{7}$$

Here $d(n)$ is a non-decreasing function, depending only on $d \geq 2$, from the positive integers to themselves. U will be called below the *Brunel operator* corresponding to the (not necessarily commuting) operators $P_1,\ldots,P_d$. We next use Krengel's stochastic ergodic theorem (cf. Theorem 3.4.9 in [7]) for U and see that $A_n(U)\|F([0,1]^d)(\cdot)\|(\omega)$ converges stochastically to a function $g \in L_1(\Omega;\mathbb{R})$ with $Ug = g \geq 0$.

Write

$$\Omega(g) = \{\omega : g(\omega) > 0\}.$$

By (7) we find that

$$A_n(T_1,\ldots,T_d)F([0,1]^d) \to 0 \quad \text{stochastically on } \Omega \setminus \Omega(g). \tag{8}$$

Since $Ug = g \geq 0$ and X is a reflexive Banach space, it follows from Eberlein's mean ergodic theorem (cf. Theorem 2.1.5 in [7]) that for any $f \in L_1(\Omega(g);X)$ the averages

$$A_n(T_1,\ldots,T_d)f \quad (n \geq 1)$$

converge in the L_1-norm to a function in $L_1(\Omega(g);X)$ invariant under $T_1,\ldots,T_d$ as n tends to infinity. Since $\Omega(g)$ is an absorbing set for the commuting operators $T_1,\ldots,T_d$, it is now routine (cf. the proof of Theorem 6.3.10 in [7]) to check that the functions

$$1_{\Omega(g)} \cdot A_n(T_1,\ldots,T_d)f, \quad \text{where } f \in L_1(\Omega;X),$$

converge in the L_1-norm to a function invariant under the operators $T_1,\ldots,T_d$. Combining these results, we conclude that the averages

$$n^{-d}F([0,n]^d) = A_n(T_1,\ldots,T_d)F([0,1]^d)$$

converge stochastically to a function F_∞ in $L_1(\Omega;X)$ invariant under the operators $T_1,\ldots,T_d$ as n tends to infinity. Since F is a bounded process, it follows that $\alpha^{-d}F([0,\alpha]^d)$ converges stochastically to F_∞ as α tends to infinity.

Now putting $S_i = T(r\cdot e^i)$, $1\le i\le d$, for an $r>0$, we have

$$(nr)^{-d}F([0,nr]^d) = A_n(S_1,\dots,S_d)[r^{-d}F([0,r]^d)],$$

and thus the averages

$$A_n(S_1,\dots,S_d)[r^{-d}F([0,r]^d)]$$

converge stochastically to F_∞. Hence F_∞ is invariant under the operators $S_1,\dots,S_d$. This shows the invariance of F_∞ under the semigroup $T=\{T(u): u\in\mathbb{R}_d^+\}$, and the first half of Theorem 1 has been proved.

To prove the second half, let $\mathcal{P}(I)$, where $I\in\mathcal{I}_d$, denote the class of all finite partitions of I into pairwise disjoint intervals in $\mathbb{R}_d^+$, and let

$$F^0(I) = \operatorname{ess\,sup}\Big\{\sum_{i=1}^{k}\|F(I_i)(\cdot)\| : \{I_1,\dots,I_k\}\in\mathcal{P}(I)\Big\}.$$

Then

(i) $F^0(I)\in L_1^+(\Omega;\mathbb{R})$.
(ii) $\tau(u)F^0(I)(\omega)\ge F^0(u+I)(\omega)$ a.e. on Ω for all $u\in\mathbb{R}_d^+$.
(iii) If $\{I_1,\dots,I_k\}\in\mathcal{P}(I)$ then $F^0(I)=\sum_{i=1}^k F^0(I_i)$.

Since the operators $T_1,\dots,T_d$ commute and $\|P_i\|_\infty\le 1$ for all $1\le i\le d$ by hypothesis, Theorem A can be applied to show that

$$\lim_n n^{-d}F([0,n]^d)(\omega)=\lim_n A_n(T_1,\dots,T_d)F([0,1]^d)(\omega)=F_\infty(\omega)\quad \text{a.e. on } \Omega.$$

On the other hand, for $n\le\alpha<n+1$ we have

$$\begin{aligned}&\|\alpha^{-d}F([0,\alpha]^d)(\omega)-n^{-d}F([0,n]^d)(\omega)\|\\ &\quad\le |1-(\alpha/n)^d|\cdot\|\alpha^{-d}F([0,\alpha]^d)(\omega)\| + n^{-d}\|F([0,\alpha]^d)(\omega)-F([0,n]^d)(\omega)\|\end{aligned}$$

and

$$n^{-d}\|F([0,\alpha]^d)(\omega)-F([0,n]^d)(\omega)\|\le n^{-d}(F^0([0,n+1]^d)(\omega)-F^0([0,n]^d)(\omega)),$$

so that in order to prove the second half it suffices to show that

$$\lim_n n^{-d}(F^0([0,n+1]^d)(\omega)-F^0([0,n]^d)(\omega))=0\quad\text{a.e. on }\Omega. \tag{9}$$

To do this, given an $\varepsilon>0$, choose $g\in L_1(\Omega;\mathbb{R})\cap L_\infty(\Omega;\mathbb{R})$ so that

$$0\le g\le F^0([0,1]^d)\quad\text{and}\quad \|F^0([0,1]^d)-g\|_1<\varepsilon.$$

We then put $G(0)=g$, $H(0)=F^0([0,1]^d)-g$ and for $0\ne\widetilde{u}=(\widetilde{u}_1,\dots,\widetilde{u}_d)\in\{0,1,2,\dots\}^d$,

$$G(\widetilde{u})=\max\{P_{i(1)}\dots P_{i(k)}g : (i(1),\dots,i(k))\in\mathcal{S}(\widetilde{u})\} \tag{10}$$

(where $\mathcal{S}(\widetilde{u}) := \{(i(1),\ldots,i(k)) : k = \sum_{m=1}^{d} \widetilde{u}_m,\ \widetilde{u}_m = |\{j : i(j) = m\}|,\ 1 \le m \le d\}$ and $|A|$ is the cardinal number of a finite set A), and

$$H(\widetilde{u}) = [F^0(\widetilde{u} + [0,1]^d) - G(\widetilde{u})]^+. \tag{11}$$

From (1) we see that $\|G(\widetilde{u})\|_\infty \le \|g\|_\infty < \infty$ for all $\widetilde{u}$, and hence

$$\begin{aligned} n^{-d}(F^0([0,n+1]^d) &- F^0([0,n]^d)) \\ &\le n^{-d}\sum\{G(\widetilde{u}) + H(\widetilde{u}) : \\ &\qquad \widetilde{u} = (\widetilde{u}_1,\ldots,\widetilde{u}_d) \in \{0,1,\ldots,n\}^d \setminus \{0,1,\ldots,n-1\}^d\} \\ &\le n^{-d}[(n+1)^d - n^d]\cdot\|g\|_\infty \\ &\quad + n^{-d}\sum\{H(\widetilde{u}) : \widetilde{u} = (\widetilde{u}_1,\ldots,\widetilde{u}_d) \in \{0,1,\ldots,n\}^d\} \\ &= \mathrm{I}(n) + \mathrm{II}(n). \end{aligned}$$

Since $\lim_n \mathrm{I}(n) = 0$, it is enough to show that

$$\lim_n \mathrm{II}(n) = 0 \quad \text{a.e. on } \Omega. \tag{12}$$

Let $0 \ne \widetilde{u} = (\widetilde{u}_1,\ldots,\widetilde{u}_d) \in \{0,1,\ldots\}^d$ and $k = \sum_{i=1}^{d} \widetilde{u}_i$. For any sequence $(i(1),\ldots,i(k))$ in $\mathcal{S}(\widetilde{u})$, we have

$$\begin{aligned} P_{i(1)}\ldots P_{i(k)}(H(0) + G(0)) &= P_{i(1)}\ldots P_{i(k)}F^0([0,1]^d) \\ &\ge \tau(\widetilde{u})F^0([0,1]^d) \ge F^0(\widetilde{u} + [0,1]^d), \end{aligned}$$

whence

$$\begin{aligned} P_{i(1)}\ldots P_{i(k)}H(0) &\ge F^0(\widetilde{u} + [0,1]^d) - P_{i(1)}\ldots P_{i(k)}G(0) \\ &= F^0(\widetilde{u} + [0,1]^d) - P_{i(1)}\ldots P_{i(k)}g \\ &\ge F^0(\widetilde{u} + [0,1]^d) - G(\widetilde{u}) \quad \text{(by (10))}. \end{aligned}$$

Therefore we have

$$P_{i(1)}\ldots P_{i(k)}H(0) \ge [F^0(\widetilde{u} + [0,1]^d) - G(\widetilde{u})]^+ = H(\widetilde{u}). \tag{13}$$

Hence, if U denotes the Brunel operator corresponding to the operators $P_1,\ldots,P_d$, then (cf. the proof of Theorem 6.3.4 in [7])

$$\begin{aligned} n^{-d}\sum\{H(\widetilde{u}) : \widetilde{u} = (\widetilde{u}_1,\ldots,\widetilde{u}_d),\ 0 \le \widetilde{u}_i < n,\ 1 \le i \le d\} \\ \le C_d \cdot \sup_{m\ge1} A_m(U)H(0) \quad \text{a.e. on } \Omega. \end{aligned}$$

Since U satisfies $\|U\|_1 \le 1$ and $\|U\|_\infty \le 1$, we now apply Theorem 2.2.2 of [3] to infer that the function

$$H^*(0)(\omega) = \sup_{m\ge1} A_m(U)H(0)(\omega)$$

satisfies

$$\mu(\{\omega : H^*(0)(\omega) > \delta\}) \le \delta^{-1}\|H(0)\|_1 \quad (\delta > 0).$$

Therefore

$$\begin{aligned}\limsup_n \mathrm{II}(n) &= \limsup_n\ n^{-d}\sum\{H(\widetilde{u}_1,\dots,\widetilde{u}_d) : 0 \le \widetilde{u}_i < n,\ 1 \le i \le d\}\\ &\le C_d \cdot H^*(0) \quad \text{a.e. on } \Omega,\end{aligned}$$

and

$$\begin{aligned}\mu(\{\omega : \limsup_n \mathrm{II}(n)(\omega) > \delta\}) &\le \mu(\{\omega : C_d \cdot H^*(0)(\omega) > \delta\})\\ &\le \delta^{-1}C_d\|H(0)\|_1 < \delta^{-1}C_d \cdot \varepsilon \downarrow 0\end{aligned}$$

as $\varepsilon \downarrow 0$. This establishes (12), and the second half of Theorem 1 follows.

Proof of Theorem 2. Since the commuting operators P_i satisfy $\|T(e^i)f(\omega)\| \le P_i\|f(\cdot)\|(\omega)$ a.e. on Ω for all $f \in L_1(\Omega; X)$, we may apply the proof of Theorem 6.3.10 in [7] to infer that the averages

$$\begin{aligned}&\Big(\prod_{i=1}^d n_i\Big)^{-1}\sum_{i_1=0}^{n_1-1}\dots\sum_{i_d=0}^{n_d-1} T_1^{i_1}\dots T_d^{i_d}F([0,1]^d)\\ &\qquad = \Big(\prod_{i=1}^d n_i\Big)^{-1}F([0,n_1]\times\dots\times[0,n_d])\end{aligned}$$

converge stochastically to a function $F_\infty \in L_1(\Omega; X)$ invariant under the operators $T_i = T(e^i)$, $1 \le i \le d$, as n_i tends to infinity independently for each $1 \le i \le d$. Since F is a bounded process, we then see that the averages

$$\Big(\prod_{i=1}^d \alpha_i\Big)^{-1}F([0,\alpha_1]\times\dots\times[0,\alpha_d])$$

converge stochastically to F_∞ as α_i tends to infinity independently for each $1 \le i \le d$. It is now immediate that F_∞ is invariant under the semigroup $T = \{T(u) : u \in \mathbb{R}_d^+\}$ (cf. the proof of Theorem 1).

To prove the second half of Theorem 2, we assume that for every $f \in L_1(\Omega; \mathbb{R})$,

$$\lim_n A_n(P_1,\dots,P_d)f(\omega) \text{ exists a.e. on } \Omega. \tag{14}$$

Then, by Theorem B,

$$\lim_n A_n(T_1,\dots,T_d)F([0,1]^d)(\omega) = \lim_n n^{-d}F([0,n]^d)(\omega)$$

exists a.e. on Ω. Hence, as in the proof of Theorem 1, it is enough to establish (9); and this follows from

$$\begin{aligned} n^{-d}(F^0([0,n+1]^d)(\omega) - F^0([0,n]^d)(\omega)) \\ &\le (1+1/n)^d A_{n+1}(P_1,\ldots,P_d)F^0([0,1]^d)(\omega) \\ &\quad - A_n(P_1,\ldots,P_d)F^0([0,1]^d)(\omega) \\ &\to 0 \quad \text{a.e. on } \Omega \quad (\text{by } (14)). \end{aligned}$$

The proof is complete.

4. Remarks. (a) *On continuity at the origin.* Let $T=\{T(u) : u\in\mathbb{P}_d\}$ be a strongly continuous semigroup of linear contractions on $L_1(\Omega;X)$, where X is a reflexive Banach space. In order that $\widetilde{T}(0) = \text{strong-lim}_{u>0,u\to 0}\, T(u)$ exists, it suffices that $\sup\{\|T(u)\|_p : u \in (0,1]^d\} < \infty$ for some $p > 1$.

To see this, we may assume $1 < p < \infty$ by the Marcinkiewicz interpolation theorem (see e.g. Theorem II.2.11 in [2], p. 148). Then, since X is a reflexive Banach space, it follows that $L_p(\Omega;X)$ is a reflexive Banach space. Let f be a function in $L_p(\Omega;X)$ and $\varepsilon_n > 0$ be such that $\varepsilon_n \downarrow 0$ as n tends to infinity. Putting $u_n = (\varepsilon_n,\ldots,\varepsilon_n) \in \mathbb{P}_d$ for each $n \ge 1$ and, if necessary, choosing a subsequence of (u_n), we may assume that for some $\widetilde{f} \in L_p(\Omega;X)$,

$$\widetilde{f} = \text{weak-}\lim_n T(u_n)f \quad \text{ in } L_p(\Omega;X).$$

Since $T = \{T(u) : u \in \mathbb{P}_d\}$ can be considered to be a strongly continuous semigroup of bounded linear operators in $L_p(\Omega;X)$, we see that for any $u \in \mathbb{P}_d$,

$$T(u)\widetilde{f} = \text{weak-}\lim_n T(u+u_n)f = \text{strong-}\lim_n T(u+u_n)f = T(u)f.$$

Further, by the Hahn–Banach theorem,

$$\widetilde{f} \in \Big[L_p\text{-norm closure of } \bigcup_{n=1}^{\infty} T(u_n)L_p(\Omega;X)\Big].$$

Thus an approximation argument shows that

$$\lim_{\substack{u\to 0\\ u>0}} \|T(u)f - \widetilde{f}\|_p = \lim_{\substack{u\to 0\\ u>0}} \|T(u)\widetilde{f} - \widetilde{f}\|_p = 0.$$

In particular, if $f \in L_p(\Omega;X)\cap L_1(\Omega;X)$, then choosing a suitable sequence (v_n) in $\mathbb{P}_d$ with $v_n \to 0 \in \mathbb{R}_d^+$ as n tends to infinity, and putting

$$f_n = T(v_n)f \quad (n \ge 1),$$

we get $\widetilde{f} = \lim_n f_n$ a.e. on Ω, and hence $\|\widetilde{f}\|_1 = \lim_n \|f_n\|_1$ by Fatou's lemma together with the fact that $\|f_n\|_1 = \|T(v_n)f\|_1 = \|T(v_n)\widetilde{f}\|_1 \le \|\widetilde{f}\|_1$. It follows from Lebesgue's convergence theorem that

$$\lim_n \|\widetilde{f} - f_n\|_1 = \lim_n \|\widetilde{f} - T(v_n)f\|_1 = 0,$$

whence $\lim_{u>0,u\to 0}\|\widetilde{f}-T(u)f\|_1=0$ by approximation. Since $L_p(\Omega;X)\cap L_1(\Omega;X)$ is dense in $L_1(\Omega;X)$, this completes the proof.

(b) *An improvement of Theorem 1.* The first part of Theorem 1 holds even if $T=\{T(u)\}$ is a strongly continuous $L_1(\Omega;X)$-contraction semigroup defined only on the interior $\mathbb{P}_d$ of $\mathbb{R}_d^+$.

In fact, if $F:\mathcal{I}_d\to L_1(\Omega;X)$ is a bounded additive process in $L_1(\Omega;X)$ with respect to the semigroup T, then by Lemma 4 in [8] we may assume without loss of generality that

$$\widetilde{T}(0)=\text{strong-}\lim_{\substack{u\to 0\\ u>0}} T(u)$$

exists. Then obviously the domain of T can be continuously extended to $\mathbb{R}_d^+$. Denote by $\widetilde{T}=\{\widetilde{T}(u):u\in\mathbb{R}_d^+\}$ its extended semigroup. Since $\widetilde{T}(u)$ has a contraction majorant $P(u)$ defined on $L_1(\Omega;\mathbb{R})$ for every $u\in\mathbb{P}_d$ by hypothesis, modifying the proof of Lemma 1 in [8] we see that there exists a family $\{\tau(u):u\in\mathbb{R}_d^+\}$ of positive linear contractions on $L_1(\Omega;\mathbb{R})$ such that

$$\|\widetilde{T}(u)f(\omega)\|\le\tau(u)\|f(\cdot)\|(\omega)\qquad\text{a.e. on }\Omega$$

for all $f\in L_1(\Omega;X)$ and $u\in\mathbb{R}_d^+$. From this, together with Theorem 1, the desired conclusion follows.

REFERENCES

[1] N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory*, Interscience, New York, 1958.

[2] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[3] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Markham, Chicago, 1970.

[4] S. Hasegawa and R. Sato, *On d-parameter pointwise ergodic theorems in L_1*, Proc. Amer. Math. Soc. 123 (1995), 3455–3465.

[5] —, —, *On a d-parameter ergodic theorem for continuous semigroups of operators satisfying norm conditions*, Comment. Math. Univ. Carolin. 38 (1997), 453–462.

[6] S. Hasegawa, R. Sato and S. Tsurumi, *Vector valued ergodic theorems for a one-parameter semigroup of linear operators*, Tôhoku Math. J. 30 (1978), 95–106.

[7] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[8] R. Sato, *Vector valued differentiation theorems for multiparameter additive processes in L_p spaces*, Positivity 2 (1998), 1–18.

Department of Mathematics
Faculty of Science
Okayama University
Okayama, 700-8530 Japan
E-mail: satoryot@math.okayama-u.ac.jp

Received 23 March 1998;
revised 26 June 1998