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FLAT SEMILATTICES

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Introduction. Let A and B be $\{\vee, 0\}$-semilattices. We denote by $A\otimes B$ the *tensor product* of A and B, defined as the free $\{\vee, 0\}$-semilattice generated by the set

$$(A-\{0\})\times(B-\{0\})$$

subject to the relations

$$\langle a, b_0\rangle \vee \langle a, b_1\rangle = \langle a, b_0 \vee b_1\rangle,$$

for $a \in A-\{0\}$, $b_0, b_1 \in B-\{0\}$, and symmetrically,

$$\langle a_0, b\rangle \vee \langle a_1, b\rangle = \langle a_0 \vee a_1, b\rangle,$$

for $a_0, a_1 \in A-\{0\}$, $b \in B-\{0\}$.

$A\otimes B$ is a universal object with respect to a natural notion of *bimorphism* (see [2], [5], and [6]). This definition is similar to the classical definition of the tensor product of modules over a commutative ring. Thus, for instance, *flatness* is defined similarly: The $\{\vee, 0\}$-semilattice S is *flat* if for every embedding $f : A \hookrightarrow B$, the canonical map $f \otimes \mathrm{id}_S : A \otimes S \to B \otimes S$ is an embedding.

Our main result is the following:

THEOREM. *Let S be a $\{\vee, 0\}$-semilattice. Then S is flat if and only if S is distributive.*

1. Background

1.1. *Basic concepts.* We adopt the notation and terminology of [6]. In particular, for every $\{\vee, 0\}$-semilattice A, we use the notation $A^- = A-\{0\}$. Note that A^- is a subsemilattice of A.

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A semilattice S is *distributive* if whenever $a \leq b_0 \vee b_1$ in S, then there exist $a_0 \leq b_0$ and $a_1 \leq b_1$ such that $a = a_0 \vee a_1$, or equivalently, iff the lattice $\operatorname{Id} S$ of all ideals of S, ordered under inclusion, is a distributive lattice; see [4].

1.2. *The set representation.* In [6], we used the following representation of the tensor product.

First, we introduce the notation:

$$\perp_{A,B} = (A \times \{0\}) \cup (\{0\} \times B).$$

Second, we introduce a partial binary operation on $A \times B$: let $\langle a_0, b_0 \rangle$, $\langle a_1, b_1 \rangle \in A \times B$; the *lateral join* of $\langle a_0, b_0 \rangle$ and $\langle a_1, b_1 \rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case it is the join $\langle a_0 \vee a_1, b_0 \vee b_1 \rangle$.

Third, we define bi-ideals: a nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$ if it satisfies the following conditions:

(i) I is hereditary;
(ii) I contains $\perp_{A,B}$;
(iii) I is closed under lateral joins.

The *extended tensor product* of A and B, denoted by $A \mathbin{\overline{\otimes}} B$, is the lattice of all bi-ideals of $A \times B$. It is easy to see that $A \mathbin{\overline{\otimes}} B$ is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \mathbin{\overline{\otimes}} B$ by

$$a \otimes b = \perp_{A,B} \cup \{\langle x, y \rangle \in A \times B \mid \langle x, y \rangle \leq \langle a, b \rangle\}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal.

Now we can state the representation:

Proposition 1.1. *The tensor product $A \otimes B$ can be represented as the $\{\vee, 0\}$-subsemilattice of compact elements of $A \mathbin{\overline{\otimes}} B$.*

1.3. *The construction of $A \mathbin{\vec{\otimes}} B$.* The proof of the Theorem uses the following representation of the tensor product (see J. Anderson and N. Kimura [1]).

Let A and B be $\{\vee, 0\}$-semilattices. Define

$$A \mathbin{\vec{\otimes}} B = \operatorname{Hom}(\langle A^-; \vee \rangle, \langle \operatorname{Id} B; \cap \rangle),$$

and for $\xi \in A \mathbin{\vec{\otimes}} B$, let

$$\varepsilon(\xi) = \{\langle a, b \rangle \in A^- \times B^- \mid b \in \xi(a)\} \cup \perp_{A,B}.$$

Proposition 1.2. *The map ε is an order preserving isomorphism between $A \mathbin{\overline{\otimes}} B$ and $A \mathbin{\vec{\otimes}} B$ and, for $H \in A \mathbin{\overline{\otimes}} B$, $\varepsilon^{-1}(H)$ is given by*

$$\varepsilon^{-1}(H)(a) = \{b \in B \mid \langle a, b \rangle \in H\},$$

for $a \in A^-$.

If $a \in A$ and $b \in B$, then $\varepsilon(a \otimes b)$ is the map $\xi : A^- \to \operatorname{Id} B$ given by

$$\xi(x) = \begin{cases} (b] & \text{if } x \leq a, \\ \{0\} & \text{otherwise.} \end{cases}$$

If A is *finite*, then a homomorphism from $\langle A^-; \vee \rangle$ to $\langle \operatorname{Id} B; \cap \rangle$ is determined by its restriction to $\mathrm{J}(A)$, the set of all join-irreducible elements of A. For example, let A be a finite Boolean semilattice, say $A = \mathrm{P}(n)$ (n is a non-negative integer, $n = \{0, 1, \ldots, n-1\}$); then $A \mathbin{\overline{\otimes}} B \cong (\operatorname{Id} B)^n$, and the isomorphism from $A \mathbin{\overline{\otimes}} B$ onto $(\operatorname{Id} B)^n$ given by Proposition 1.2 is the unique complete $\{\vee, 0\}$-homomorphism sending every element of the form $\{i\} \otimes b$ ($i < n$ and $b \in B$) to $\langle (\delta_{ij} b] \mid j < n \rangle$ (where δ_{ij} is the Kronecker symbol). If $n = 3$, let $\beta : \mathrm{P}(3) \mathbin{\overline{\otimes}} S \to (\operatorname{Id} S)^3$ denote the natural isomorphism.

Next we compute $A \mathbin{\vec{\otimes}} B$, for $A = M_3$, the diamond, and $A = N_5$, the pentagon (see Figure 1). In the following two subsections, we let S be a $\{\vee, 0\}$-semilattice. Furthermore, we denote by $\widetilde{S}$ the ideal lattice of S, and identify every element s of S with its image, $(s]$, in $\widetilde{S}$.

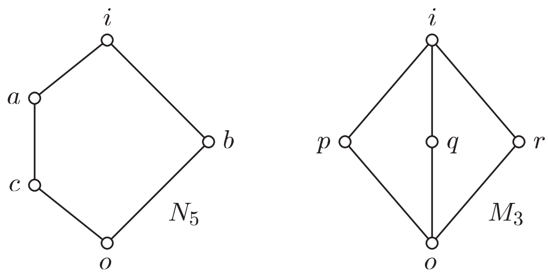

Fig. 1

1.4. *The lattices $M_3 \mathbin{\overline{\otimes}} S$ and $M_3[\widetilde{S}]$; the map i.* Let $M_3 = \{0, p, q, r, 1\}$, $\mathrm{J}(M_3) = \{p, q, r\}$ (see Figure 1). The nontrivial relations of $\mathrm{J}(M_3)$ are the following:

(1) $$p < q \vee r, \quad q < p \vee r, \quad r < p \vee q.$$

Accordingly, for every lattice L, we define

(2) $$M_3[L] = \{\langle x, y, z \rangle \in L^3 \mid x \wedge y = x \wedge z = y \wedge z\}$$

(this is the *Schmidt's construction*; see [9] and [10]). The isomorphism from $M_3 \mathbin{\overline{\otimes}} S$ onto $M_3[\widetilde{S}]$ given by Proposition 1.2 is the unique complete $\{\vee, 0\}$-homomorphism α such that, for all $x \in S$,

$$\alpha(p \otimes x) = \langle x, 0, 0 \rangle, \quad \alpha(q \otimes x) = \langle 0, x, 0 \rangle, \quad \alpha(r \otimes x) = \langle 0, 0, x \rangle.$$

We shall later make use of the unique $\{\vee, 0\}$-embedding

$$i : M_3 \hookrightarrow \mathrm{P}(3)$$

defined by

$$i(p) = \{1,2\}, \quad i(q) = \{0,2\}, \quad i(r) = \{0,1\}.$$

1.5. *The lattices $N_5 \mathbin{\overline{\otimes}} S$ and $N_5[\widetilde{S}]$; the map i'.* Let $N_5 = \{0,a,b,c,1\}$, $\mathrm{J}(N_5) = \{a,b,c\}$ with $a > c$ (see Figure 1). The nontrivial relations of $\mathrm{J}(N_5)$ are the following:

$$c < a \quad \text{and} \quad a < b \vee c. \tag{3}$$

Accordingly, for every lattice L, we define

$$N_5[L] = \{\langle x,y,z\rangle \in L^3 \mid y \wedge z \leq x \leq z\}.$$

The isomorphism from $N_5 \mathbin{\overline{\otimes}} S$ onto $N_5[\widetilde{S}]$ given by Proposition 1.2 is the unique complete $\{\vee,0\}$-homomorphism α' such that, for all $x \in S$,

$$\alpha'(a \otimes x) = \langle x,0,x\rangle, \quad \alpha'(b \otimes x) = \langle 0,x,0\rangle, \quad \alpha'(c \otimes x) = \langle 0,0,x\rangle.$$

We shall later make use of the unique $\{\vee,0\}$-embedding

$$i' : N_5 \hookrightarrow \mathrm{P}(3)$$

defined by

$$i'(a) = \{0,2\}, \quad i'(b) = \{1,2\}, \quad i'(c) = \{0\}.$$

1.6. *The complete homomorphisms $f \mathbin{\overline{\otimes}} g$.* The proof of the following lemma is straightforward:

LEMMA 1.3. *Let A, B, A', and B' be $\{\vee,0\}$-semilattices, let $f : A \to A'$ and $g : B \to B'$ be $\{\vee,0\}$-homomorphisms. Then the natural $\{\vee,0\}$-homomorphism $h = f \otimes g$ from $A \otimes B$ to $A' \otimes B'$ extends to a unique complete $\{\vee,0\}$-homomorphism $\overline{h} = f \mathbin{\overline{\otimes}} g$ from $A \mathbin{\overline{\otimes}} B$ to $A' \mathbin{\overline{\otimes}} B'$. Furthermore, if h is an embedding, then so is $\overline{h}$.*

We refer to Proposition 3.4 of [6] for an explicit description of the map $\overline{h}$.

2. Characterization of flat $\{\vee,0\}$-semilattices.

Our definition of flatness is similar to the usual one for modules over a commutative ring:

DEFINITION. A $\{\vee,0\}$-semilattice S is *flat* if for every embedding $f : A \hookrightarrow B$ of $\{\vee,0\}$-semilattices, the tensor map $f \otimes \mathrm{id}_S : A \otimes S \to B \otimes S$ is an embedding.

In this definition, id_S is the identity map on S.

In Lemmas 2.1–2.3, we let S be a $\{\vee,0\}$-semilattice and assume that both homomorphisms $f = i \otimes \mathrm{id}_S$ and $f' = i' \otimes \mathrm{id}_S$ are embeddings. As in the previous section, we use the notation $\widetilde{S} = \mathrm{Id}\, S$, and identify every element s of S with the corresponding principal ideal $(s]$.

We define the maps $g : M_3[\widetilde{S}] \to \widetilde{S}^3$ and $g' : N_5[\widetilde{S}] \to \widetilde{S}^3$ by

$$g(\langle x, y, z\rangle) = \langle y \vee z, x \vee z, x \vee y\rangle, \quad \text{for all } \langle x, y, z\rangle \in M_3[\widetilde{S}],$$
$$g'(\langle x, y, z\rangle) = \langle z, y, x \vee y\rangle, \quad \text{for all } \langle x, y, z\rangle \in N_5[\widetilde{S}].$$

Note that g and g' are complete $\{\vee, 0\}$-homomorphisms. The proof of the following lemma is a straightforward calculation.

LEMMA 2.1. *The following two diagrams commute*:

$$\begin{array}{ccc} M_3 \mathbin{\overline{\otimes}} S & \xrightarrow{f} & \mathrm{P}(3) \mathbin{\overline{\otimes}} S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ M_3[\widetilde{S}] & \underset{g}{\to} & \widetilde{S}^3 \end{array} \qquad \begin{array}{ccc} N_5 \mathbin{\overline{\otimes}} S & \xrightarrow{f'} & \mathrm{P}(3) \mathbin{\overline{\otimes}} S \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\beta} \\ N_5[\widetilde{S}] & \underset{g'}{\to} & \widetilde{S}^3 \end{array}$$

Therefore, *both g and g' are embeddings.*

LEMMA 2.2. *The lattice $\widetilde{S}$ does not contain a copy of M_3.*

P r o o f. Suppose, on the contrary, that $\widetilde{S}$ contains a copy of M_3, say $\{o, x, y, z, i\}$ with $o < x, y, z < i$. Then both elements $u = \langle x, y, z\rangle$ and $v = \langle i, i, i\rangle$ of L^3 belong to $M_3[\widetilde{S}]$, and $g(u) = g(v) = \langle i, i, i\rangle$. This contradicts the fact, justified by Lemma 2.1, that g is one-to-one. ∎

LEMMA 2.3. *The lattice $\widetilde{S}$ does not contain a copy of N_5.*

P r o o f. Suppose, on the contrary, that $\widetilde{S}$ contains a copy of N_5, say $\{o, x, y, z, i\}$ with $o < x < z < i$ and $o < y < i$. Then both elements $u = \langle x, y, z\rangle$ and $v = \langle z, y, z\rangle$ of L^3 belong to $N_5[\widetilde{S}]$, and $g'(u) = g'(v) = \langle z, y, i\rangle$. This contradicts the fact (again Lemma 2.1) that g' is one-to-one. ∎

Lemmas 2.2 and 2.3 together prove that $\widetilde{S}$ is distributive, and therefore S is a distributive semilattice. Now we are in a position to prove the main result of this paper in the following form:

THEOREM 1. *Let S be a $\{\vee, 0\}$-semilattice. Then the following are equivalent*:

(i) *S is flat.*
(ii) *Both homomorphisms $i \otimes \mathrm{id}_S$ and $i' \otimes \mathrm{id}_S$ are embeddings.*
(iii) *S is distributive.*

P r o o f. (i)⇒(ii). This is trivial.

(ii)⇒(iii). This was proved in Lemmas 2.2 and 2.3.

(iii)⇒(i). Let S be a distributive $\{\vee, 0\}$-semilattice; we prove that S is flat. Since the tensor product by a fixed factor preserves direct limits (see Proposition 2.6 of [6]), flatness is preserved under direct limits. By P. Pudlák [8], every distributive join-semilattice is the direct union of all its finite distributive subsemilattices; therefore, it suffices to prove that every finite distributive $\{\vee, 0\}$-semilattice S is flat. Since S is a distributive lattice, it admits a lattice embedding into a finite Boolean lattice B. We have seen in Section 1.3 that if $B = \mathrm{P}(n)$, then $A \otimes B = A^n$ (up to a natural isomorphism), for every $\{\vee, 0\}$-semilattice A. It follows that B is flat. Furthermore, the inclusion map $S \hookrightarrow B$ is a lattice embedding; in particular, with the terminology of [6], it is an *L-homomorphism*. Thus, the natural map from $A \otimes S$ to $A \otimes B$ is, by Proposition 3.4 of [6], a $\{\vee, 0\}$-semilattice embedding. This implies the flatness of S. ■

3. Discussion. It is well known that a module over a given principal ideal domain R is flat if and only if it is torsion-free, which is equivalent to the module being a direct limit of (finitely generated) free modules over R. So the analogue of the concept of torsion-free module for semilattices is the concept of distributive semilattice. This analogy can be pushed further, by using the following result, proved in [3]: *a join-semilattice is distributive iff it is a direct limit of finite Boolean semilattices.*

PROBLEM 1. Let $\mathbf{V}$ be a variety of lattices. Let us say that a $\{\vee, 0\}$-semilattice S is *in* $\mathbf{V}$ if $\operatorname{Id} S$ as a lattice is in $\mathbf{V}$. Is every $\{\vee, 0\}$-semilattice in $\mathbf{V}$ a direct limit (resp., direct union) of *finite* join-semilattices in $\mathbf{V}$?

If $\mathbf{V}$ is the variety of all lattices, we obtain the obvious result that every $\{\vee, 0\}$-semilattice is the direct union of its finite $\{\vee, 0\}$-subsemilattices. If $\mathbf{V}$ is the variety of all distributive lattices, there are two results (both quoted above): P. Pudlák's result and K. R. Goodearl and the second author's result.

PROBLEM 2. Let $\mathbf{V}$ be a variety of lattices. When is a $\{\vee, 0\}$-semilattice S flat with respect to $\{\vee, 0\}$-semilattice embeddings in $\mathbf{V}$? That is, when is it the case that for all $\{\vee, 0\}$-semilattices A and B in $\mathbf{V}$ and every semilattice embedding $f : A \hookrightarrow B$, the natural map $f \otimes \mathrm{id}_S$ is an embedding?

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