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BORDISM OF ORIENTED 5-MANIFOLDS WITH T-STRUCTURE AND POLARIZATION

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0. Introduction. The notion of a T-structure on a manifold is a generalization of an action of a torus. Tori, possibly of different dimensions, act on open subsets of the manifold and these actions fit together on overlaps in such a way that the torus acting on one of the sets injects homomorphically into the torus acting on the second one. The T-structure was introduced in [G]. More general notions, namely an F-structure and a nilpotent Killing structure, were introduced in [CG1], [CG2] and [CFG].

Here we assume that the manifold is compact, orientable and differentiable and that the local actions of tori are smooth.

There is an extra notion of polarization which can be associated with a manifold with a T-structure (or an F-structure). For each open chart U_α there is a choice of a nontrivial linear subspace of the Lie algebra of the acting torus such that the derivative of the action sends it isomorphically onto a linear subspace of the tangent space of U_α at a given point.

The bordism groups of oriented 3- and 4-manifolds with T-structure and polarization were calculated in [HJ] and [Mi2]. These abelian groups have an uncountable set of generators. A cofinite subset of the $\mathbb{Z}$-basis is parametrized by the slope coefficient of polarizations near singular orbits in strata of codimension 2. The slope coefficient corresponds to an element of the interval $(0,1/2]$.

A similar situation occurs in dimension 5, where the slope coefficient is calculated near isolated singular orbits of codimension 4. Essential information is encoded in generators of type (11) in Section 5.

Let us denote the bordism group of compact oriented n-manifolds with T-structure and polarization by $\Omega_n^{\mathrm{pol},t}$. In this paper we obtain the following result.

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Main Theorem. *The group $\Omega_5^{\mathrm{pol},t}$ is generated by manifolds listed in Section* 5.

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0.1. Definition. We say that a manifold M admits a *T-structure* if there is a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of M by open sets such that for each $\alpha \in \Lambda$ there is a torus T^{k_α} which acts on U_α:

$$\theta_\alpha : T^{k_\alpha} \times U_\alpha \to U_\alpha$$

and if $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ then, up to changing the roles of α, β, $k_\alpha \le k_\beta$ and there exists a monomorphism $\xi_{\alpha\beta} : T^{k_\alpha} \to T^{k_\beta}$ such that the following diagram is commutative:

$$\begin{CD} T^{k_\alpha} \times U_{\alpha\beta} @>{\theta_\alpha}>> U_{\alpha\beta} \\ @V{\xi_{\alpha\beta}\times \mathrm{id}_{U_{\alpha\beta}}}VV @VV{\mathrm{id}_{U_{\alpha\beta}}}V \\ T^{k_\beta} \times U_{\alpha\beta} @>{\theta_\beta}>> U_{\alpha\beta} \end{CD}$$

Moreover, if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then $\xi_{\beta\gamma}\xi_{\alpha\beta} = \xi_{\alpha\gamma}$ $(k_\alpha \le k_\beta \le k_\gamma)$.

An *atlas* of the T-structure is the collection of sets and maps

$$\left\langle \{U_\alpha\}_{\alpha\in\Lambda}; \{\theta_\alpha\}_{\alpha\in\Lambda}; \{\xi_{\alpha\beta}\}_{(\alpha,\beta)\in\Lambda_0} \right\rangle .$$

Here Λ_0 is the set of those (α, β) for which $\xi_{\alpha\beta}$ is defined.

We say that the T-structure is *pure* if k_α does not depend on α.

We sometimes speak of a *local T^k action* while thinking of a pure T-structure of rank k.

0.2. Definition (cf. [CG1], Definition 1.6). We say that M admits a T-structure with *polarization* if there is an atlas

$$\left\langle \{U_\alpha\}_{\alpha\in\Lambda}; \{\theta_\alpha\}_{\alpha\in\Lambda}; \{\xi_{\alpha\beta}\}_{(\alpha,\beta)\in\Lambda_0} \right\rangle$$

of a T-structure on M and for each $\alpha \in \Lambda$ there is a linear subspace $K_\alpha < \mathrm{alg}(T^{k_\alpha})$, where alg denotes the Lie algebra, such that for each $x \in U_\alpha$ the derivative of the map

$$\theta_\alpha : T^{k_\alpha} \times \{x\} \to U_\alpha$$

sending K_α to the subspace $K_{\alpha,x}$ of $T_x(M)$ is a monomorphism.

Moreover, there is a compatibility condition: For each $(\alpha, \beta) \in \Lambda_0$ and $x \in U_{\alpha\beta}$, $K_{\alpha,x} < K_{\beta,x}$ or $K_{\beta,x} < K_{\alpha,x}$.

By an *atlas* of a manifold with T-structure and polarization we mean an atlas of the T-structure together with the collection $\{K_\alpha\}_{\alpha\in\Lambda}$ of linear subspaces.

We say that two T-structures or T-structures with polarization on the same manifold M are *compatible* if the corresponding atlases are subatlases of the same atlas of the structure on M.

0.3. Notation. Let $M^{(k)}$ denote the union of those U_α for which k_α is an integer k. It is an open submanifold of M. If $m \in M^{(k)}$ then the *orbit* of m (denoted by $[m]_k$) of the local T^k action is the orbit of m for an action θ_α such that $m \in U_\alpha$ and $k_\alpha = k$. According to Definition 0.1, $[m]_k$ does not depend on α. Let $X^{(k)}$ denote the *orbit space* of $M^{(k)}$ with the quotient topology which is determined by transversal slices. (See the slice theorem in [Br], II.4.4.) Denote the quotient map $M^{(k)} \to X^{(k)}$ by ϖ_k. Let $M_i^{(k)}$, where i is an index, denote connected components of $M^{(k)}$. Set $M^{(k,l)} = M^{(k)} \cap M^{(l)}$ and $M^{(k,l,r)} = M^{(k)} \cap M^{(l)} \cap M^{(r)}$. Observe that if $m \in M^{(k,l)}$ and $k \le l$ then the identity map on $M^{(k,l)}$ sends the orbit $[m]_k$ to $[m]_l$. Thus there is a map ϖ_l^k between the orbit spaces sending $[m]_k$ to $[m]_l$ such that $\varpi_l = \varpi_l^k \varpi_k$.

Thus M is covered by the open sets $M^{(l)}$ and each $M^{(l)}$ is a union of strata given by orbit types.

Let us denote by

- $\mathrm{CF}_k^{(l)}$ the k-dimensional stratum of $M^{(l)}$ consisting of the orbits with isotropy group $T^i \times G < T^l$ for G a finite abelian group;
- $\mathrm{Cr}_k^{(l)}$ the k-dimensional stratum of $M^{(l)}$ such that $\mathrm{Cr}_k^{(l)}/T^l \subset \partial(X^{(l)})$ with isotropy group $T^{(n-k)/2}$;
- $\mathrm{Ci}_k^{(l)}$ the k-dimensional stratum of $M^{(l)}$ such that $\mathrm{Ci}_k^{(l)}/T^l \subset \mathrm{int}(X^{(l)})$ with isotropy group T^i for some $0 < i < l$;
- $\mathrm{Fin}_k^{(l)}$ the k-dimensional stratum consisting of the orbits with finite isotropy group;
- $\mathrm{Pr}^{(l)}$ the stratum consisting of the principal orbits.

0.5. Definition. The *orbit space* of a T-structure with an atlas

$$\left\langle \{U_\alpha\}_{\alpha\in\Lambda}; \{\theta_\alpha\}_{\alpha\in\Lambda}; \{\xi_{\alpha,\beta}\}_{(\alpha,\beta)\in\Lambda_0} \right\rangle$$

is the collection of the orbit spaces $X^{(k)}$ together with the projections ϖ_l^k for $k \le l$ such that $M^{(k,l)} \ne \emptyset$.

0.6. Assumption. We assume that the covering of M by the open sets U_α is locally finite, i.e. each compact set meets only a finite number of sets of the covering. If M is compact then the covering is finite. Moreover, we assume that the closures of U_α, $M^{(k)}$, $M^{(k,l)}$ and $M^{(k,l,r)}$ for all α, k, l and r are manifolds with piecewise smooth boundaries and that the corresponding pure tori actions can be extended to the closures. Denote the orbit spaces of the corresponding closures by $\mathrm{cl}(U_\alpha/T^2)$, $\mathrm{cl}(X^{(k)})$, $\mathrm{cl}(X^{(k;(k,l))})$

and $\mathrm{cl}(X^{(k;(k,l,r))})$. In the last two cases the orbit spaces are taken with respect to the T^k local action. Moreover, we assume that the image of $M^{(k,l)}$ for any k,l in $\mathrm{cl}(X^{(k)})$ and in $\mathrm{cl}(X^{(l)})$ is a collar neighbourhood of a part of the boundary.

It is sometimes more convenient to use a different definition of an atlas of a T-structure so that the subsets $M^{(k)}$, $M^{(k,l)}$ and $M^{(k,l,r)}$ etc. fit together in the following manner. $M^{(k)}$ is a manifold with corners and $M^{(k,l)} = M^{(k)} \cap M^{(l)}$ is a common face of codimension one. Similarly $M^{(k,l,r)}$ is a common face of codimension two of $M^{(k)}$, $M^{(l)}$ and $M^{(r)}$ etc. This convention was used in Yang's thesis [Y].

0.7. Definition. We say that two oriented n-manifolds M_1, M_2 with T-structure and polarization are *cobordant* if there exists an oriented $(n+1)$-manifold W with T-structure and polarization such that $\partial W = M_1 \cup -M_2$ as a manifold, and the T-structure and polarization on W restricted to ∂W give the corresponding T-structure and polarization on M_1 and M_2.

Let the bordism group of n-manifolds with T-structure and polarization be denoted by $\Omega_n^{\mathrm{pol},t}$.

If $M_1 = M_2$ and $W = M \times [0,1]$ then we say that the two T-structures with polarization on M are *concordant*.

To be more precise, here M_i $(i = 1,2)$ stands for any manifold equivariantly diffeomorphic to M_i, where the diffeomorphism preserves the polarization.

0.8. Assumption. We assume that for each i, outside a small neighbourhood of singular orbits in $M^{(i)}$ the polarization is maximal, i.e. it is given by the whole Lie algebra of T^i. Each manifold with T-structure and polarization admits a concordant structure satisfying this assumption.

0.9. Definition. Let k, n $(k < n)$ be any natural numbers. We define a simplicial complex $\mathrm{SC}_{\mathrm{pol}}(k,n)$ by the following construction. Let the vertices of $\mathrm{SC}_{\mathrm{pol}}(k,n)$ be indexed by linear subspaces of $\mathbb{R}^n$ of dimension less than or equal to k. A set $\{v_1, \ldots, v_l\}$ $(l \le k)$ of distinct vertices gives a simplex of dimension $l-1$ in $\mathrm{SC}_{\mathrm{pol}}(k,n)$ if up to permutation of indices the linear subspaces represent a flag. The complex $\mathrm{SC}_{\mathrm{pol}}(k,n)$ has dimension $k-1$.

If "W" is a condition on a linear subspace of $\mathbb{R}^n$ then by $\mathrm{SC}_{\mathrm{pol}}(k,n)(W)$ we denote the subcomplex of $\mathrm{SC}_{\mathrm{pol}}(k,n)$ spanned by the vertices corresponding to subspaces satisfying condition "W".

As an example we may take condition $W_{n,k}$:

A subspace $L < \mathbb{R}^n$ satisfies $W_{n,k}$ iff $L < \mathbb{R}^n \to \mathbb{R}^n/\mathbb{R}^k$ is an isomorphism of linear spaces, where $\mathbb{R}^k = \mathbb{R}^k \times \{0\} < \mathbb{R}^n$ is a fixed subspace.

We endow the complexes with a weak topology, i.e. a subset of a complex is open iff the intersection of this subset with each finite subcomplex is open.

0.10. FACT. $\mathrm{SC}_{\mathrm{pol}}(n-1,n)(W_{n,1})$ *is connected for* $n \geq 3$.

0.11. DEFINITION. Let k, n $(k < n)$ be any natural numbers. We define another simplicial complex $\mathrm{SC}_{k,n}$ by the following construction.

A vector from $\mathbb{Z}^n$ is *primitive* if the greatest common divisor of its entries is 1. Observe that primitive vectors correspond to subgroups $T^1 < T^n$.

Let the vertices of $\mathrm{SC}_{k,n}$ be indexed by primitive vectors from $\mathbb{Z}^n$ modulo multiplication by ± 1 (i.e. $(a_1, \ldots, a_n) \cong (-a_1, \ldots, -a_n)$). A sequence $v_1, \ldots, v_l$ $(l \leq k+1)$ of distinct vertices gives a simplex of dimension $l-1$ in $\mathrm{SC}_{k,n}$ if the sequence is primitive, i.e. can be completed to a base of $\mathbb{Z}^n$. The complex $\mathrm{SC}_{k,n}$ has dimension k. $\mathrm{SC}_{k,n}$ is the k-skeleton of $\mathrm{SC}_{k',n}$ if $k < k' < n$.

0.12. FACT. *The complex* $\mathrm{SC}_{k,n}$ *is* $(k-1)$*-connected.*

Proof. [DJ], Theorem 2.2, and [K], Theorem 2.6.

In this paper the we find generators of the bordism group by reducing step by step the complexity of the manifold and the atlas of the T-structure while staying within the same bordism class. The author hopes that the techniques developed in this paper will lead to stronger classification results.

1. The reduction of $M^{(5)} \cup M^{(4)} \cup M^{(3)}$

1.1. PROPOSITION. *A* 5*-manifold admitting a* T*-structure with polarization admits a concordant* T*-structure with polarization and with* $M^{(5)} = \emptyset$.

Proof. $M^{(5)}$ is a disjoint union of several copies of T^5 which are isolated orbits of the local T^5 action. We can assume that $M^{(5)} \cap (\bigcup_{i=1}^4 M^{(i)}) = \emptyset$. Up to concordance $M^{(5)}$ admits a pure polarization by the Lie algebra of a $T^1 < T^5$. We may replace the T^5 action by the T^1 action.

1.2. PROPOSITION. *A* 5*-manifold admitting a* T*-structure with polarization such that* $M^{(5)} = \emptyset$ *is concordant to a manifold* M' *such that* $M'^{(4)} = M'^{(5)} = \emptyset$.

Proof. If a connected component of $X^{(4)}$ is diffeomorphic to $(0,1)$ or S^1 there is a concordant manifold M' such that the component of $M^{(4)}$ is included in $M'^{(1)} \cup M'^{(2)} \cup M'^{(3)}$. The proof of this is given in [Mi3], provided Assumption 0.8 is satisfied.

In case $X_j^{(4)} = (0,1]$ or $X_j^{(4)} = [0,1]$ the proof is analogous to that in dimension 4, where $X_j^{(3)} = (0,1]$ or $X_j^{(3)} = [0,1]$. See [Mi2]. The proof is a consequence of the fact that $\mathrm{SC}_{\mathrm{pol}}(3,4)(W_{4,1})$ is connected (0.10).

1.3. LEMMA. *Let* M *be a manifold with pure* T*-structure of rank* 3 *and compatible* T*-structure such that* $M = M^{(1)} \cup M^{(2)}$. *Then there is a con-*

cordance $W = M \times [0,1]$ *such that* $W = W^{(1)} \cup W^{(2)}$ *and the* T-*structure is compatible with the pure* T-*structure of rank* 3 *on* W *which is the product of the* T-*structure on* M *with the interval* $[0,1]$. *Moreover*, *the following property holds*:

• *If an orbit* G_1 *of the local* T^i *action* $(i = 1, 2)$ *on* $M \times \{1\}$ *intersects an orbit* G_2 *of the local* T^3 *action on* $M \times \{1\}$ *then* $G_2 \subset (M \times \{1\})^{(i)}$.

Proof. See [Mi3].

1.4. *Change of atlas near singular strata.* According to [Mi3] we can extend a local T^1 or T^2 action to a local T^3 action in a tubular neighbourhood of

$$\mathrm{Fin}_1^{(1)} \cup \mathrm{Fin}_3^{(2)} \cup \mathrm{Ci}_1^{(2)} \cup \mathrm{CF}_1^{(2)}\,.$$

Similarly we can extend a local T^1 action to a T^2 action in a tubular neighbourhood of $\mathrm{Fin}_3^{(1)}$. The extensions can be chosen so that they define compatible local actions on overlaps. In particular, we obtain a manifold such that all singular strata of codimension 4 are included in $M^{(3)}$ and $M^{(1)} \cup M^{(2)}$ is free from orbits with finite isotropy groups.

We can also change the T^3 action in a tubular neighbourhood of isolated orbits of $\mathrm{Fin}_3^{(3)}$ to a T^1 action by choosing for each such orbit a subgroup $T^1 < T^3$ intersecting trivially a finite isotropy subgroup. Thus we can assume that $\mathrm{Fin}_3^{(3)} = \emptyset$.

1.5. *Change of atlas on* $M^{(3)}$. Let us apply Lemma 1.3 to $(M^{(1)} \cup M^{(2)}) \cap M^{(3)}$. We find that each orbit of a local T^3 action is T^1 or T^2 saturated.

$X^{(3)}$ is a 2-manifold. Let $X_0^{(3)}$ denote a submanifold of $X^{(3)}$ such that all orbits intersect only those charts of the atlas that have a polarization corresponding to the Lie algebra of T^3 (see 0.8).

$\mathrm{SC}_{\mathrm{pol}}(2,3)$ is connected. The subcomplex generated by the Lie algebras of all subgroups $T^1 < T^3$ and $T^2 < T^3$ is also connected.

We can change the atlas over $X_0^{(3)} - A$, where A is a tubular neighbourhood of a finite set in $\varpi_3(\mathrm{Pr}^{(3)}) \cap X_0^{(3)}$, obtaining a manifold M'. After the change $\varpi_3^{-1}(X_0^{(3)} - A) \subset (M'^{(1)} \cup M'^{(2)}) - M'^{(3)}$, where ϖ_3 is the projection before the change.

1.6. *Reduction of* $\mathrm{Pr}^{(3)}$. (A) Polarizations over a connected component of ∂A define an element c of H_1 of the building at infinity associated with $\mathrm{SL}_3(\mathbb{Z})$, provided we have identified the orbits of the T^3 action on the component of $\varpi_3^{-1}(A)$ with a standard torus T^3. According to [AR] the element is a combination of basic elements $c_1, \ldots, c_s$ which are represented by boundaries of flats (in the symmetric space $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$) in the same $\mathrm{SL}_3(\mathbb{Z})$-orbit as the flat corresponding to the base of $\mathbb{Z}^3$:

$$(1,0,0);(0,1,0);(0,0,1).$$

There is a bordism between M and a manifold M', product outside $M - \varpi_3^{-1}(A)$, such that if N is the trace of the bordism then $P = N - (M - \varpi_3^{-1}(A)) \times [0,1]$ has the properties:

• There is a free T^3 action on each connected component of P.

• The closure of $\partial P - (M \cup M'^{(3)})$ has polarizations of dimensions 1 or 2 given by the Lie algebras of subgroups $T^1 < T^3$ or $T^2 < T^3$.

• The polarization on a connected component of the closure of $\partial P - (M \cup M'^{(3)})$ gives a labelling of the surface $\mathrm{cl}(\partial P - (M \cup M'^{(3)}))/T^3$ by subgroups $T^1 < T^3$ or $T^2 < T^3$. The surface represents a singular chain realizing homology between c and $c_1, \ldots, c_s$.

The singular chain can be constructed from the labelled surface as follows. We may assume that the surface is a union of two kinds of surfaces glued along 1-submanifolds of their boundaries. The first kind corresponds to labels by subgroups $T^1 < T^3$, the second to labels by subgroups $T^2 < T^3$. Given a sufficiently fine triangulation of $\mathrm{cl}(\partial P - (M \cup M'^{(3)}))/T^3$, with edges transversal to the 1-submanifolds, we can send vertices of the triangulation to vertices of the complex at infinity of $\mathrm{SL}_3(\mathbb{R})/\,\mathrm{SO}(3)$ corresponding to the labels. Then we extend the map simplicially.

Observe that the proof in [AR] based on the convex body theorem of Minkowski gives a method of constructing a concordance, i.e. $N = M \times [0,1]$. The surface $\mathrm{cl}(\partial P - (M \cup M'^{(3)}))/T^3$ is then a sphere with holes.

(B) The manifold M' obtained, cobordant to M, has the properties:

• $M' = M'^{(1)} \cup M'^{(2)} \cup M'^{(3)}$.

• $M'^{(3)}$ is a disjoint union of two manifolds:

(1) a tubular neighbourhood of singular strata in $M'^{(3)}$,

(2) a union of $D^2 \times T^3$ with the properties:

• ∂D^2 is a union of a cycle of six open arcs such that only the neighbouring ones (in the cyclic order) intersect. There is a base of $H_1(T^3)$ such that the polarization above a small neighbourhood of ∂D^2 corresponds to the boundary of the flat corresponding to the 0-1 base of $\mathbb{Z}^3$ in such a way that each arc corresponds to a 1- or 2-dimensional subgroup of T^3.

• The interior of $D^2 \times T^3$ has a polarization by the Lie algebra of T^3 (see 0.8).

Take the splitting $T^3 = T^1 \times T^2$, where T^1 is represented by the integer vector $(1,1,1)$. Replace each copy of $D^2 \times T^3$ by $\partial(D^2) \times D^2 \times T^2$, where the second factor is the standard filling of $T^1 \cong (1,1,1)$. This surgery is realized by a bordism of M' with a disjoint union of a manifold M'' and $S^3 \times T^2$ (see 1.9(2)). The polarizations over $\partial(D^2)$ in $D^2 \times T^3$ define polarizations on $\partial(D^2) \times T^1 \times T^2$, which can be extended on $\partial(D^2) \times D^2 \times T^2$.

The resulting manifold M'' has properties similar to those of M' except that $M'^{(3)}$ is a manifold of type (1) above.

1.7. *The generator* $S^5\#_{S^1}$. Let T^3 act on $S^5 \subset \mathbb{C}^3$ in the standard way:

$$(t_1, t_2, t_3)(z_1, z_2, z_3) = (t_1z_1, t_2z_2, t_3z_3).$$

Then the T^3 equivariant adjacent connected sum along two orbits with isotropy groups T^2 gives a 5-manifold, denoted here by $S^5\#_{S^1}$, with T-structure of rank 3 with one orbit having isotropy group T^2. Any polarization in a small neighbourhood of this orbit can be extended to a polarization on the whole manifold. (See 0.8 and 0.10.) In this way we obtain manifolds which are equivariantly diffeomorphic. They may be different as manifolds with T-structure and polarization.

1.8. *Reduction of* $\mathrm{Cr}_1^{(3)}$. A small neighbourhood of each orbit from $\mathrm{Cr}_1^{(3)}$ has a 1-dimensional polarization. There are generators of type $S^5\#_{S^1}$ such that the T^3 equivariant adjacent connected sum with M along the orbits with isotropy groups T^2 gives a 5-manifold without the stratum $\mathrm{Cr}_1^{(3)}$. In this way we show that M is cobordant to a manifold M' with $\mathrm{Cr}_1^{(3)} = \emptyset$ and a union of generators $S^5\#_{S^1}$.

1.9. *Reduction of* $\mathrm{Cr}_3^{(3)}$

(1) *Connected components of* $\mathrm{Cr}_3^{(3)}/T^3$ *which are intervals can be made closed by means of an equivariant bordism.*

Consider a connected component of $\mathrm{Cr}_3^{(3)}/T^3$ which is an interval with endpoints x, y. An orientation of the orbit space along the interval and an orientation of the principal bundle determine an orientation of M. Let K_x, K_y be subspaces of the Lie algebra of T^3 defining polarizations in the orbits x and y respectively. Here we choose an isomorphism between the standard torus T^3 and the tori acting on orbits x and y so that the linear spaces K_x, K_y and the Lie algebra of the isotropy subgroup are subspaces of the same space $\mathbb{R}^3$. K_x, K_y are subspaces of the Lie algebras of proper subtori of T^3 (i.e. the acting tori T^1 or T^2).

Let $T^3 = T^1 \times T^2$ act on $S^3 \times T^2$:

$$(t_1, t_2, t_3)(z_1, z_2; s_1, s_2) = (t_1z_1, z_2; t_2s_1, t_3s_2).$$

Here z_1, z_2 are the complex coordinates of $S^3 \subset \mathbb{C}^2$. The orbit space is homeomorphic to $D^2 \cong [0,1] \times [0,1]$. Choose the polarization K_x over $[0,1/3) \times [0,1]$ and K_y over $(2/3,1] \times [0,1]$. Since $\mathrm{SC}_{\mathrm{pol}}(2,3)(W_{3,1})$ is connected the polarization can be extended to the whole manifold so that it is constant over the subsets $\{z\} \times [0,1]$ for $z \in [0,1]$.

The manifold is null-cobordant since it bounds $D^4 \times T^2$ equivariantly. The orbit space of $D^4 \times T^2$ is $[0,1] \times C([0,1])$, where $C([0,1]) \cong [0,1] \times$

$[0,1]/([0,1] \times \{1\} \sim *)$ is the cone over $[0,1]$ and the polarization does not depend on the radius coordinate of $C([0,1])$.

We can choose a polarization on $S^3 \times T^2$ in such a way that the linear spaces are represented by subspaces of the Lie algebras of proper subtori of T^3 since the corresponding subcomplex of $\mathrm{SC}_{\mathrm{pol}}(2,3)(W_{3,1})$ is also connected. It is then possible to change the T^3 action to an action of T^1 or T^2 over a small neighbourhood of

$$[0,1] \times \{1\} \cup \{0\} \times [1/2,1] \cup \{1\} \times [1/2,1].$$

Now it is possible to perform two adjacent connected sums:

- along orbits x and $(0,1/2)$;
- along orbits y and $(1,1/2)$,

so that the side of the orbit space of $S^3 \times T^2$ (suitably oriented) corresponding to the action of T^1 or T^2 fits the corresponding side of $X^{(3)}$. We assume that the orientation of $S^3 \times T^2/T^3$ and of a principal orbit agrees with the orientations of M near x and y. After the operation we can slightly redefine the atlas so that 0.8 is satisfied and the connected component of $\mathrm{Cr}_3^{(3)}/T^3$ is a circle.

(2) *The stratum* $\mathrm{Cr}_3^{(3)}$ *can be killed by means of an equivariant bordism.*

We assume that each connected component of $\mathrm{Cr}_3^{(3)}/T^3$ is a circle.

The monodromy matrices along $\mathrm{Cr}_3^{(3)}/T^3$ are in the group

$$G = \begin{pmatrix} \pm 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} < \mathrm{SL}_3(\mathbb{Z}).$$

The abelianization of this group is generated by the two matrices

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both have a common eigenvector $(0,1,0)$.

Each loop in $\mathrm{Cr}_3^{(3)}/T^3$ is homologous in $H_1(BG)$ to a union of loops with monodromy matrices B or C.

There is a bordism N between M and M' such that the stratum $\mathrm{Cr}_4^{(3)}/T^3$ of the orbit space of the trace of the bordism represents a homology between $\mathrm{Cr}_3^{(3)}/T^3$ and generators of $H_1(BG)$. The bordism is product outside some neighbourhood of $\mathrm{Cr}_4^{(3)}$ included in $N^{(3)}$.

Let us change the atlas on N so that $\mathrm{Cr}_4^{(3)} \cap M'$ is included in $M'^{(1)}$, where the action of T^1 and polarization correspond to the eigenvector

$(0,1,0)$ of the monodromy matrix B or C. (More precisely, the corresponding eigenvector of some conjugate of B or C.)

$\mathrm{Cr}_4^{(3)}/T^3$ is 2-dimensional thus it is possible to extend the polarization by 1- or 2-dimensional linear spaces on $\mathrm{Cr}_4^{(3)} \cap (M \cup M')$ over $\mathrm{Cr}_4^{(3)}/T^3 - D$, where D is a tubular neighbourhood of a finite set of points.

For each connected component $D^2 \subset D$ let $D_1^2 \subset \mathrm{int}(N/T^3)$ be an imbedded open disk such that $\partial(\mathrm{cl}(D_1^2)) = \partial(\mathrm{cl}(D^2))$. The submanifold of N lying above $\mathrm{cl}(D_1^2)$ is equivariantly diffeomorphic to $S^3 \times T^2$, provided the imbedding of $\mathrm{cl}(D_1^2)$ is differentiable and transversal to $\partial(N)$. The manifold inherits a polarization from N.

We have thus proved that M is cobordant to a union of M' and a disjoint sum of manifolds $S^3 \times T^2$ with the action of $T^3 = T^1 \times T^2$:

$$(t_1, t_2, t_3)(z_1, z_2; s_1, s_2) = (t_1 z_1, z_2; t_2 s_1, t_3 s_2).$$

The polarizations on both manifolds satisfy 0.8.

The stratum $\mathrm{Cr}_3^{(3)}$ of M' is empty.

(3) *Using the procedure of* 1.6 *we can prove that* M' *is cobordant to a manifold* M'' *such that* $M''^{(3)} = \emptyset$.

2. Separation of $M^{(1)}$ from $M^{(2)}$. The procedures in Section 1 show that a 5-manifold with T-structure and polarization is cobordant to a disjoint sum of some generators and a manifold M satisfying the following assumptions:

- $M = M^{(1)} \cup M^{(2)}$.
- There are no codimension 4 strata.
- $M^{(1)} = \mathrm{Pr}^{(1)}$.
- $M^{(2)} = \mathrm{Pr}^{(2)} \cup \mathrm{Cr}_3^{(2)}$.

We may assume that the atlas on M is such that $M^{(1)} \cap M^{(2)}$ is diffeomorphic to $N \times (0,1)$ for some 4-manifold N with polarization.

We have $N = N^{(1)} = N^{(2)}$. The polarization is pure and is given by the Lie algebra of the acting torus T^1.

$N = N^{(2)}$ consists of the strata $\mathrm{Cr}_2^{(2)}$ and $\mathrm{Pr}^{(2)}$. The monodromy group, a subgroup of $\mathrm{GL}_2(\mathbb{Z})$, fixes the direction of polarization. The monodromy matrices along loops in $\mathrm{Cr}_2^{(2)}$ are $\pm I$.

A surgery on N which is both T^1 and T^2 equivariant resulting in a manifold N' gives a bordism between M and a manifold M' such that a tubular neighbourhood of N' in M' can be chosen to be $M'^{(1,2)}$.

2.1. *Surgery on* N

(1) N/T^2 *is nonorientable.*

If there is a nonorientable simple closed curve γ on the orbit space $N^{(2)}/T^2$ then by means of the equivariant surgical operation corresponding to a connected sum in a neighbourhood of two points on γ we can change the manifold in such a way that the monodromy along the resulting nonorientable simple closed curve is conjugate to

$$E = \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The boundary δ of the corresponding Möbius strip has monodromy I. Cutting $N^{(2)}/T^2$ along δ and gluing in two disks corresponds to an equivariant surgical operation on N.

Thus by means of the above operations we obtain a manifold $N' = N'^{(2)} = N'^{(1)}$ with $N'^{(2)}/T^2$ orientable and a disjoint union of T^2 bundles over $\mathbb{R}P^2$ with monodromy E.

E has eigenvectors $(1,0),(0,1) \in \mathbb{Z}^2$. The first one corresponds to a polarization. Thus the T^2 bundle over $\mathbb{R}P^2$ with monodromy E bounds a $T^1 \times D^2$ bundle over $\mathbb{R}P^2$ with monodromy E. The polarization on the filling can be chosen to be the Lie algebra of the group $T^1 < T^2$ corresponding to $(1,0) \in \mathbb{Z}^2$.

We may thus assume that:

(2) N/T^2 *is orientable.*

A connected component of N/T^2 can be expressed as a connected sum:

$$T^2 \,\#\, \ldots \,\#\, T^2 \,\#\, (S^2 - (D_1^2 \cup \ldots \cup D_k^2))$$

or

$$T^2 \,\#\, \ldots \,\#\, T^2$$

if $\mathrm{Cr}_2^{(2)} = \emptyset$. In the latter case we may perform a surgery after which $\mathrm{Cr}_2^{(2)}/T^2 = S^1$ with monodromy I. We may then assume that there is a global section.

Each S^1, which is the locus of the connected sum, has monodromy I. Cutting $N^{(2)}/T^2$ along such a circle and gluing in two disks corresponds to an equivariant surgical operation on N.

After the operation each connected component of N/T^2 (we use the same letter N for the resulting manifold) is homeomorphic to T^2 or $S^2 - (D_1^2 \cup \ldots \cup D_k^2)$.

Each T^2 bundle over T^2 with the property that there is a basic loop with trivial monodromy can be changed by means of the surgery corresponding to cutting along the loop in the base and gluing in two disks. The resulting T^2 bundle over S^2 is trivial and is the boundary of $T^2 \times D^3$.

Each connected component with orbit space of the local T^2 action equal to $S^2 - (D_1^2 \cup \ldots \cup D_k^2)$ can be changed by a similar surgery to a disjoint union of manifolds with orbit spaces equal to $S^1 \times [0,1]$. If the monodromy along S^1 is I then the manifold is the boundary of a manifold with orbit space $D^2 \times [0,1]$. If the monodromy along S^1 is $-I$ then there is a surgery which cuts the intervals ($[0,1]$ from $S^1 \times [0,1]$) into subintervals. The manifolds obtained have the property that the isotropy groups at the ends intersect in one point of T^2. Further surgery joins two ends of each of the intervals. The manifold obtained is a union of T^2 bundles over T^2.

We have shown that the manifold N is cobordant to a union of T^2 bundles over T^2 with monodromy matrices along basic loops of the base being

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

Both signs ± 1 can occur. Observe that the diffeomorphism of the base torus corresponding to the matrix $-I \in \mathrm{SL}_2(\mathbb{Z})$ changes the sign of the ± 1 entry. Moreover, the product of such a bundle with an interval gives a manifold with boundary which is a union of two such bundles with different orientations. The change of orientation can be realized by replacing one basic loop of the base by the opposite loop. Thus one of the bundles has entry 1 and the second -1 according to some basic loops on the base.

We have just proved that the double of such a bundle is a boundary. Thus we may assume that after the bordism there is only one bundle. Otherwise the number of such bundles is even and the manifold N is a T^2 and T^1 equivariant boundary.

2.2. *The manifold N is a T^1 and T^2 equivariant boundary.* The manifold N' obtained is a T^2 bundle over T^2 with monodromy matrices along basic loops as above. It is the boundary of $M'^{(1)}$, which is an S^1 bundle since the local T^1 action is free.

The S^1 bundle is characterized by:

- a monodromy representation: $\pi_1(X'^{(1)}) \to \mathbb{Z}_2$;
- an obstruction to existence of a global section, i.e. an element of $H_2(X'^{(1)}, \widetilde{\mathbb{Z}})$, where $\widetilde{\mathbb{Z}}$ is the locally constant coefficient system with stalks $\mathbb{Z}$ and monodromy as above.

According to the method of Knesser [GM], which is local, the cycle representing the obstruction can be chosen to be a submanifold of $X'^{(1)}$ with an integer ± 1 assigned to each point in a continuous manner. The submanifold is orientable.

The T^2 bundle over T^2 with monodromy matrices along basic loops being

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$$

is the total space of an S^1 bundle over the 3-dimensional base. The fibre corresponds to the eigendirection $(1,0)$. The bundle can be expressed as a Nil bundle over S^1, where Nil is an S^1 bundle over T^2 with Euler number ± 1.

The obstruction cycle in the 3-dimensional base of each bundle is represented by an S^1 (a lift of the base of the Nil bundle) with coefficient ± 1. The circle is the boundary of the obstruction surface K in $X'^{(1)}$.

Let w_1 denote a codimension 1 submanifold in $X'^{(1)}$ representing the obstruction to orientability. Then w_1 intersects the circle transversally. The intersection of w_1 with the obstruction cycle K to existence of a global section is a 1-submanifold $w_1 \cap K$ with boundary being a point on the circle. Since the boundary of a compact 1-manifold consists of an even number of points we come to a contradiction.

We have proved that complete separation of $M^{(1)}$ from $M^{(2)}$ by means of a bordism is possible.

3. Manifold with pure rank 1 T-structure. We assume that the local T^1 action is free. Thus $M = M^{(1)}$ is the total space of an S^1 bundle over $X^{(1)}$ and is characterized by a monodromy representation and an obstruction to existence of a global section, which is represented by an orientable surface $K \subset X^{(1)}$ with coefficient ± 1 (according to the method of Knesser [GM]). We can assume that w_1, the obstruction to orientability of $X^{(1)}$, intersects K transversally.

Take a loop γ in $w_1 \cap K$. Then γ is orientable and it is possible to perform a surgery on $X^{(1)}$, killing the homotopy class of the loop. It corresponds to an equivariant surgery on $M^{(1)}$. In this way we can make $w_1 \cap K = \emptyset$. By a similar surgery we can decompose K into a disjoint union of S^2 such that the normal bundle in $X^{(1)}$ has Euler number 0 or ± 1. In case the Euler number is 0 the sphere can be killed by means of a spherical surgery.

If the Euler number is ± 1 the boundary of the normal D^2 bundle in $X^{(1)}$ is an S^3 and we can make a "blow-down", i.e. we cut off a copy of $\pm \mathbb{C}P^2$ from $X^{(1)}$. A $\pm \mathbb{C}P^1 \subset \pm \mathbb{C}P^2$ is the obstruction cycle to existence of a global section.

After the operations $M^{(1)}$ has a global section. $X^{(1)}$ as a manifold representing an element of $\Omega_4^O \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is cobordant to $\mathbb{R}P^4$, $\mathbb{C}P^2$, $\mathbb{R}P^4 \cup \mathbb{C}P^2$ or is null-cobordant [GM]. This means that there is a 5-manifold Y such that $\partial(Y) = X^{(1)} \cup Z$, where Z is one of the four manifolds. We can assume that $X^{(1)}$ and Y are connected. Let w_1 be a codimension 1 submanifold of Y, which is the obstruction to orientability, extending that on $X^{(1)}$. Given a base point on $X^{(1)} - w_1$ and a local orientation we have a rule of changing the local orientation along a curve in Y transversal to w_1. We change the local orientation while crossing w_1. If we choose an orientation of a fibre of the S^1 bundle over the base point and assume a similar rule for changing the orientation of the fibre we obtain a procedure for constructing an S^1 bundle over Y extending $M^{(1)}$, with global orientation of the total space. Thus nonorientable bordism of bases corresponds to orientable bordism of total spaces for S^1 bundles with section.

We have shown that $M^{(1)}$ is cobordant as an S^1 bundle to a combination of:

- an S^1 bundle over $\pm\mathbb{C}P^2$ with obstruction to global section $\pm\mathbb{C}P^1$ with coefficient ± 1;
- $S^1 \times \mathbb{C}P^2$;
- an S^1 bundle over $\mathbb{R}P^4$ with orientable total space and global section.

4. Manifold with pure rank 2 T-structure. We assume that there are only two strata: $\mathrm{Cr}_3^{(2)}$ and $\mathrm{Pr}^{(2)}$. The manifold can be constructed from a T^2 bundle over $X^{(2)}$ with structure group $\mathrm{Aff}(T^2)$ with fibres over $\partial(X^{(2)})$ collapsed in the direction of isotropy subgroups.

4.1. *The obstruction to global section.* The obstruction to global section is represented by a 1-chain in $X^{(2)}$ with coefficients from the local system with stalks $\mathbb{Z}^2$ and monodromy $\mathrm{GL}_2(\mathbb{Z})$. Choose a sufficiently fine triangulation of $X^{(2)}$. For each 3-simplex $\triangle^3$ let us make a surgery on $M^{(2)}$:

- Remove a subset of $M^{(2)}$ lying over a smooth 3-ball B^3 such that $\mathrm{cl}(B^3) \subset \mathrm{int}(\triangle^3)$.
- Collapse the orbits over the resulting boundary in the direction of a subgroup $T^1 < T^2$.
- Choose a polarization in the neighbourhood of the new component of $\mathrm{Cr}_3^{(2)}$.

The surgery can be realized by an equivariant bordism between $M^{(2)}$ and a manifold $M'^{(2)}$. It preserves the 2-skeleton of the triangulation of $X^{(2)}$. $X'^{(2)}$ is a "thickening" of this 2-skeleton. As a 1-chain representing the obstruction to global section in $M'^{(2)}$ we can choose a union of intervals in $X'^{(2)}$ transversal to 2-simplexes of the remaining 2-skeleton of the triangulation of $X^{(2)}$. The tubular neighbourhood of each interval in $X'^{(2)}$ is $D^2 \times [0,1]$ with $D^2 \times \{0\}$ and $D^2 \times \{1\}$ corresponding to the stratum $\mathrm{Cr}_3^{(2)}$. Replacing each $D^2 \times [0,1]$ by $D^2 \times [0,1]$ without the obstruction interval inside corresponds to a surgery. The surgery is realized by a bordism and we obtain:

- a manifold with global section with orbit space $X'^{(2)}$;
- a manifold with orbit space being a union of $S^2 \times [0,1]$, each having an interval $\{s\} \times [0,1]$ representing the obstruction to global section.

4.2. *A manifold with global section and* $\mathrm{Cr}_3^{(2)} \neq \emptyset$. The monodromy group along a connected component of $\mathrm{Cr}_3^{(2)}/T^2$ preserves the vector, say $(1,0)$, corresponding to the isotropy subgroup, and also preserves the 1-dimensional direction of polarization.

A matrix of the form

$$\begin{pmatrix} 1 & \pm p \\ 0 & 1 \end{pmatrix}$$

for $p \neq 0$ has only one eigendirection, i.e. that of $(1,0)$. Thus it cannot occur as a monodromy matrix.

The matrix

$$\begin{pmatrix} -1 & p \\ 0 & 1 \end{pmatrix}$$

has two eigendirections: $(1,0)$ and $(p,2)$. The subgroup of $\mathrm{GL}_2(\mathbb{Z})$ stabilizing the directions is

$$G_p = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & p \\ 0 & 1 \end{pmatrix} \right\}.$$

The cobordism class of the pair:

- a connected component of $\mathrm{Cr}_3^{(2)}/T^2$;
- the monodromy representation $\pi_1(\mathrm{Cr}_3^{(2)}/T^2) \to G_p$ with the condition that orientability of a loop corresponds to the determinant of the monodromy matrix,

is trivial or is represented by $\mathbb{R}P^2$ with monodromy matrix

$$\begin{pmatrix} -1 & p \\ 0 & 1 \end{pmatrix}.$$

The groups G_p and $G_{p'}$ are conjugate in the stabilizer of $(1,0)$ ($\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ in the standard way) iff $p \equiv p'$ mod 2. Thus two connected components of $\mathrm{Cr}_3^{(2)}/T^2$ with monodromy groups G_p and $G_{p'}$ can be connected by means of a connected sum iff $p \equiv p'$ mod 2.

If the monodromy representation along a connected component of $\mathrm{Cr}_3^{(2)}/T^2$ is into $\mathrm{SL}_2(\mathbb{Z})$ then the monodromy group must be included in $\{\pm I\}$. Then any direction different from $(1,0)$ can be chosen as a polarization. The cobordism class of the pair is then trivial since $H_2(\mathbb{Z}_2) = 0$.

We conclude that by means of an equivariant bordism $\mathrm{Cr}_3^{(2)}/T^2$ can be made empty or can be changed to a union of two $\mathbb{R}P^2$, the generators corresponding to the groups G_p and $G_{p'}$.

The boundary of a tubular neighbourhood of the resulting stratum $\mathrm{Cr}_3^{(2)}/T^2$ in the second case is a disjoint union of two $\mathbb{R}P^2$ with the same monodromy representation as $\mathrm{Cr}_3^{(2)}/T^2$. Let us make a connected sum:

$\mathbb{R}P^2\#\mathbb{R}P^2$. The pair obtained, i.e. the Klein bottle with monodromy representation is the boundary of a 3-manifold with monodromy representation into $\mathrm{GL}_2(\mathbb{Z})$ such that for orientable loops the determinant of the monodromy matrix is equal to 1, and for nonorientable loops it is equal to -1. Thus $\mathbb{R}P^2\cup\mathbb{R}P^2$ with the given monodromy representation is the boundary of a 3-manifold with some monodromy representation. This defines a T^2 bundle over the 3-manifold, which is the boundary of the submanifold lying over $\mathbb{R}P^2\cup\mathbb{R}P^2$. If we glue in the tubular neighbourhood of the stratum $\mathrm{Cr}_3^{(2)}$ obtained, we obtain a generator of $\Omega_5^{\mathrm{pol},t}$ carrying all the information on $\mathrm{Cr}_3^{(2)}$ that this group can detect.

4.3. *A manifold with global section and* $\mathrm{Cr}_3^{(2)}=\emptyset$. Such a manifold is the total space of a T^2 bundle over $X^{(2)}$ with structure group $\mathrm{GL}_2(\mathbb{Z})$ and the condition:

- Orientability of a loop is determined by the determinant of the monodromy matrix.

In case $X^{(2)}$ is orientable the generators correspond to generators of $H_3(\mathrm{SL}_2(\mathbb{Z}))\cong\mathbb{Z}_{12}$, i.e. the lens spaces $L(4)$, $L(6)$ with monodromy representation given by isomorphism of fundamental groups with the subgroups $\mathbb{Z}_4,\mathbb{Z}_6$ of $\mathrm{SL}_2(\mathbb{Z})\cong\mathbb{Z}_4*_{\mathbb{Z}_2}\mathbb{Z}_6$.

Let us consider the general case, i.e. $X^{(2)}$ need not be orientable. Let H^2 be the upper half plane and let $\mathrm{GL}_2(\mathbb{Z})$ act on H^2 in the standard way by homographies. Then $H^2/\mathrm{GL}_2(\mathbb{Z})$ is an orbifold which can be identified with a subset of H^2 bounded by the lines

$$\mathrm{Re}(z)=0,\qquad \mathrm{Re}(z)=-1/2,\qquad |z|=1.$$

Let δ' denote the curve given by the equation $\mathrm{Re}(z)=-1/4$ included in this subset and let δ be a curve sufficiently close to δ' such that its lift to H^2 is smooth.

4.4. *Classifying map for the T^2 fibration*

4.5. Definition. We call a map from a manifold M to a surface H *stable* if it is generic in the space of C^∞ maps with C^∞ topology (see [GG]).

There are local coordinates $(u,x,z_1,\ldots,z_{n-2})$ in M and some coordinates in H such that near the singularity the map is of one of the forms

$$(u,x,z)\mapsto\Big(u;\pm x^2+\sum_{i=1}^{n-2}\pm z_i^2\Big),$$

$$(u,x,z)\mapsto\Big(u;\pm xu\pm x^3+\sum_{i=1}^{n-2}\pm z_i^2\Big).$$

The map is stable if the restriction of the map to each open subset of M is stable.

4.6. NOTATION. We adopt the following convention for denoting subsets or points of $H^2/\mathrm{GL}_2(\mathbb{Z})$ or $H^2/\mathrm{SL}_2(\mathbb{Z})$. A set or point in the quotient space is denoted in the same way as one of its lifts to H^2. For instance, i and $\exp(4\pi i/6)$ denote points in $H^2/\mathrm{SL}_2(\mathbb{Z})$ corresponding to i and $\exp(4\pi i/6)$ in H^2.

Let

$$J : H^2/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$$

be a modular function for $\mathrm{SL}_2(\mathbb{Z})$ with value $\infty, 0, 1$ at the cusp, i and $\exp(4\pi i/6)$ respectively.

4.7. PROPOSITION. *Assume that M is a manifold admitting a pure free local T^2 action. Then there is a T^2 invariant metric on M with the following properties*:

- *The map $\psi : X \to H^2/\mathrm{GL}_2(\mathbb{Z})$ sending an orbit to the conformal structure of its metric has the property*: *For any $x \in X$ there is an open neighbourhood U of x such that the map $\psi|_U$ can be lifted to H^2 and the lift is stable.*
- *The submanifold δ and its local lifts to H^2 are transversal to the strata of singular values of the mapping ψ or its local lifts*, *i.e. is disjoint from points given by singularities of cusp type and is transversal to the curve given by a singularity of fold type. Transversality is understood according to the smooth structure on H^2.*

Proof. Take a universal T^2 bundle over H^2, denoted by B, with an invariant metric such that the orbit map coincides with identification of the conformal structure of the metric of an orbit with a point in H^2. Take any invariant metric on M. Deform the map ψ to a stable (in the sense of 4.7) map ψ'. This can be done by means of a standard transversality argument (see [GG]). For any $x \in X$ there is an open neighbourhood U of x such that the map $\psi|_U$ has a lift to H^2, denoted by $\widetilde{\psi}|_U$, and the subset of M lying over U is a pull-back in the sense of [D] (or a pull-back in the bundle category) of the submanifold of B lying over the image of $\widetilde{\psi}|_U$.

There is a small neighbourhood of the orbit lying over x, T^2 equivariantly diffeomorphic to $T^2 \times D^3$.

Similarly, tubular neigbourhoods can be found near the image of the orbit in B. We can choose such neighbourhoods and identification with $T^2 \times D^3$ or $T^2 \times D^2$ respectively so that the map into B is of the form $\mathrm{id}|_{T^2} \times \phi$ for some function ϕ. We can define a T^2 invariant metric on the pull-back such that the map restricted to each set $T^2 \times \{d\}$ is an isometry.

By means of a partition of unity associated with the cover of M by such neighbourhoods, which are saturated open sets, we can patch together the metrics and obtain a T^2 invariant metric on M which agrees with the deformed map ψ'. In fact, let ω_i for $i \in I$ (I a set of indices) be a collection of metrics defined on open saturated neighbourhoods U_i of the point x. Let x lie in an orbit F. Moreover, assume that $\omega_i|_F = \omega$. Then $(\sum_i \phi_i \omega_i)|_F = \omega$, where $\{\phi_i\}$ is a set of functions nonzero at x taken from the partition of unity.

4.8. *Surgery along* $\psi^{-1}(\delta)$. Let a base point in X be situated in $\psi^{-1}(\delta)$. The monodromy representation of the T^2 bundle restricted to $\psi^{-1}(\delta)$ has its image in the group

$$\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

i.e. the stabilizer of $\delta \cap \{z : |z| = 1\}$. Observe that $\psi^{-1}(\delta)$ is two-sided in X since δ has a normal vector field in $H^2/\operatorname{GL}_2(\mathbb{Z})$ which lifts to a smooth vector field normal to the lift of δ in H^2.

The pair: $\psi^{-1}(\delta)$ and its monodromy representation is cobordant to an even number of $\mathbb{R}P^2$, each with monodromy matrix

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

and they are together null-cobordant. Thus by means of a bordism we can split X into two manifolds X_1, X_2 such that:

- The monodromy of X_1 is included in the stabilizer of i, isomorphic to D_4.
- The monodromy of X_2 is included in the stabilizer of $\exp(4\pi i/6)$, isomorphic to D_6.

4.9. *Generators of the "twisted" bordism group* $\widetilde{\Omega}_3(BD_m)$. Let m be an even integer. Let D_m denote the dihedral group having $2m$ elements. It acts on $\mathbb{R}^2 \cong \mathbb{C}$ as the symmetry group of a regular m-gon. The group D_m is generated by the maps $\mathbb{C} \to \mathbb{C}$:

$$z \mapsto \overline{z}, \qquad z \mapsto \exp(2\pi i/m)z.$$

Let $\mathrm{Re} < \mathbb{C}/D_m \cong \mathbb{R}^2/D_m$ denote the subset of the orbifold representing points of $\{(x, y) \in \mathbb{R}^2 : y = 0\}$. Its lift to $\mathbb{R}^2$ is a cone over the set $\mathbb{Z}_m$ included in the unit circle $S^1 < \mathbb{C}$.

4.10. Definition. Here by an element of the "twisted" bordism group $\widetilde{\Omega}_3(BD_m)$ we mean a pair:

- a smooth compact 3-manifold X;

• a monodromy representation $\pi_1(X) \to D_m$ with the property: orientability of a loop is determined by the sign of the determinant of the monodromy matrix. ($D_m < O(2)$.)

We assume that two manifolds X_1, X_2 represent the same element of $\widetilde{\Omega}_3(BD_m)$ iff $X_1 \cup -X_2 = \partial Y$ and the monodromy along loops of the 4-dimensional trace Y of the bordism satisfies a similar condition. Here $-X_2$ means that we have on X_2 a system of local orientations and the sign $-$ denotes that we interchange the two local orientations at each point.

Analogously to Proposition 4.7 we have:

4.11. Proposition. *There is a continuous map $\psi : X \to \mathbb{R}^2/D_m$ such that*:

• *For any $x \in X$ there is an open neighbourhood U of x such that the map $\psi|_U$ can be lifted to $\mathbb{R}^2$ and the lift is stable.*

• *The submanifold* Re *and its local lifts to* $\mathbb{R}^2$ *are transversal to the strata of singular values of the mapping ψ or its local lifts.*

• 0 *is a regular value of local lifts of ψ.*

Proof. Let BD_m be a classifying space of D_m. Let ED_m be the total space of a universal classifying bundle of D_m. $\mathbb{R}^2 \times ED_m$ is another model for such a space, where D_m acts diagonally. $\mathbb{R}^2 \times_{D_m} ED_m$ is thus another model for BD_m. We have a well defined continuous map

$$\psi' : X \to \mathbb{R}^2 \times_{D_m} ED_m \to \mathbb{R}^2/D_m$$

as the composition of the classifying map and the projection $\mathbb{R}^2 \times_{D_m} ED_m \to \mathbb{R}^2/D_m$. Using a transversality argument we can find a map ψ sufficiently C^0 close to ψ' which has the properties postulated above.

$\psi^{-1}(0)$ is a 1-submanifold of X and there is a small normal D^2 bundle having monodromy D_m with standard action on a fibre described above. Let w_1 be a submanifold of X representing the obstruction to orientability. We can assume that w_1 is transversal to $\psi^{-1}(0)$ and that w_1 intersects each connected component of $\psi^{-1}(0)$ in at most one point.

$\psi^{-1}(\mathrm{Re})$ is a $\mathbb{Z}_m$-manifold in X and we can assume that it is transversal to w_1.

4.12. *Lens spaces.* Choose a generator g of the group $\mathbb{Z}_m < D_m$. Let T_g be a D^2 bundle over S^1 with monodromy g. Denote its total space by $D^2 \times_g S^1$. The cone over the set $\mathbb{Z}_m$, a subset of a fibre, defines a $\mathbb{Z}_m$-manifold in the bundle. The boundary of the $\mathbb{Z}_m$-manifold is a "torus knot" on the boundary of the bundle, i.e. a loop representing a generator of $H_1(T^2)$. Let us glue to the D^2 bundle a copy of $D^2 \times S^1$ along the boundary torus such that in the resulting manifold the loop is contractible. The manifold obtained is a lens space $L(m)$. The loop is the boundary of a smooth disk D^2 in $D^2 \times S^1$ such that the union of the $\mathbb{Z}_m$-manifold and the D^2 is a well defined $\mathbb{Z}_m$-manifold in the lens space $L(m)$. The monodromy representation of $D^2 \times_g S^1$ can be extended to the lens space $L(m)$. There is a classifying map

$$\psi : L(m) \to \mathbb{R}^2/D_m$$

regular in the sense of 4.10 such that the $\mathbb{Z}_m$-manifold is equal to $\psi^{-1}(\mathrm{Re})$. The map can be constructed first on $D^2 \times_g S^1$ using the identification of a fibre and $D^2 < \mathbb{R}^2$ and then it can be extended over the rest of $L(m)$.

The lens spaces obtained for $m = 4$ and $m = 6$ represent the generators of $H_3(\mathrm{SL}_2(\mathbb{Z}))$ mentioned above. Note that in case $m = 4$ the square and in case $m = 6$ the third power is represented by a T^2 bundle with monodromy $\pm I$. Thus the T^2 bundle is the boundary of a $T^1 \times D^2$ bundle with the same monodromy. The polarization given by the direction $(1, 0)$ can be extended over the filling.

By means of a connected sum on the pair: (X, the $\mathbb{Z}_m$-manifold), along two points of $\psi^{-1}(0)$, we can change the manifold so that the monodromy along a connected component of $\psi^{-1}(0)$ is either a generator of $\mathbb{Z}_m$ or the conjugation $z \mapsto \overline{z}$.

By means of an adjacent connected sum with one of the lens spaces we can change the manifold in such a way that the monodromy along a connected component of $\psi^{-1}(0)$ is the conjugation.

4.13. *Nonorientable generator.* As in case of lens spaces take a D^2 bundle over S^1 with monodromy $z \mapsto \overline{z}$. The classifying map ψ into $\mathbb{R}^2/D_m$ is well defined since we identify a fibre with $D^2 < \mathbb{R}^2$. The intersection of $\psi^{-1}(\mathrm{Re})$ with the boundary of the bundle is a union of two kinds of loops.

The first subset, which we can denote by $\psi^{-1}(\pm 1)$ (± 1 is preserved by monodromy), represents w_1, i.e. obstruction to orientability. It is the boundary of a band in the bundle corresponding to $\mathrm{Re} \cap D^2 < \mathbb{R}^2$ (not $\mathrm{Re} < D^2/D_m$). The band represents w_1 of the bundle.

The second subset consists of loops with trivial monodromy.

Let us make a surgery on the Klein bottle K which is the boundary of the bundle. We can kill the loops of the second subset and extend the classifying map over the trace of the resulting bordism in such a way that on the boundary surface K_1 which we obtain, $\psi^{-1}(\mathrm{Re}) = w_1 \cap K_1$. The tubular neighbourhood of $w_1 \cap K_1$ consists of two Möbius bands. The map ψ restricted to each one of them sends arcs transversal to the core onto the image in $\mathbb{R}^2/D_m$ of a small arc on the unit circle including 1 such that the local lifts of ψ are equivariant with respect to reflection along the core and conjugation on $\mathbb{R}^2$.

K_1 is a union of two copies of $\mathbb{R}P^2$ and an orientable surface disjoint from $\psi^{-1}(\mathrm{Re})$. Let us kill by surgery the orientable components of K_1. We obtain $K_2 = \mathbb{R}P^2 \cup \mathbb{R}P^2$ and the map ψ can be extended over the trace of the resulting bordism.

Let us glue together two copies of $\mathbb{R}P^2$, first along the cores, i.e. components of $w_1 \cap K_2$ in such a way that the resulting surface w_1 (in the 3-manifold with boundary obtained) is orientable and the gluing agrees with the map ψ. The gluing of two copies $\mathbb{R}P^2$ can be extended over the complement of the cores so that it agrees with the map ψ.

This defines a closed 3-manifold. The classifying map can be extended in an obvious way.

The manifold obtained represents a nonorientable generator of $\widetilde{\Omega}_3(BD_m)$.

Observe that in case $m = 4, 6$ we can choose an isomorphism between the abstract group D_m and the stabilizer in $\mathrm{GL}_2(\mathbb{Z})$ of i or $\exp(4\pi i/6)$ respectively such that the conjugation $z \mapsto \overline{z}$ is sent to the reflection fixing $\infty \in \partial H^2$ (∂H^2 is the ideal boundary of H^2). In case $m = 4$ the T^2 bundle has monodromy having $(1, 0)$ and $(0, 1)$ as eigendirections. Thus the T^2 bundle is the boundary of a $T^1 \times D^2$ bundle with the same monodromy. The polarization given by the direction $(1, 0)$ can be extended over the filling. In case $m = 6$ the T^2 bundle associated with the nonorientable generator has a T^1 subbundle corresponding to the eigendirection $(1, 0)$ of the monodromy group. The T^1 bundle has a global section. Thus the generator can be expressed as a union of generators obtained in Section 3.

4.14. *Further surgery.* By means of an adjacent connected sum with the nonorientable generator along $\psi^{-1}(0)$ we can change X so that $\psi^{-1}(0) = \emptyset$. Then $\psi^{-1}(\mathrm{Re})$ is a smooth orientable 2-submanifold of X.

By means of a surgery killing the homotopy classes of loops in X represented by $w_1 \cap \psi^{-1}(\mathrm{Re})$, we can make $w_1 \cap \psi^{-1}(\mathrm{Re}) = \emptyset$. Then since $\psi^{-1}(\mathrm{Re})$ is the boundary of an orientable 3-manifold, by further surgery (as in Section 3) we can make $\psi^{-1}(\mathrm{Re}) = \emptyset$. The surgeries can be performed in such a way that the classifying map can be defined on the traces of the bordism.

The manifold obtained has monodromy generated by the map $z \mapsto \overline{z}$, which is nontrivial on every nonorientable loop. Since $\Omega_3^O = \{1\}$ the manifold is a boundary.

5. Generators of $\Omega_5^{\mathrm{pol},t}$. In Sections 1, 3, 4 we have obtained the following types of generators:

(1) The generator $S^5 \#_{S^1}$ described in 1.7.

(2) The generator $S^3 \times T^2$ described in 1.9(2).

(3) The S^1 bundle over $\pm\mathbb{C}P^2$ with obstruction to global section $\pm\mathbb{C}P^1$ with coefficient ± 1 obtained in Section 3. The total space is an S^5, i.e.

the total space of the fibration $S^5 \to \mathbb{C}P^2$ defining $\mathbb{C}P^2$ with some orientations of the fibre and base. T^3 acts on $S^5 < \mathbb{C}^3$ and this action gives a noneffective T^3 action on the quotient space. Dividing T^3 by the kernel we obtain an effective action of T^2 on $\pm\mathbb{C}P^2$. The orbit space is a triangle with boundary labelled by a base of $\mathbb{Z}^3$, e.g. $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The T^1 action corresponds to a union of the base elements with appropriate signs, i.e. $(\pm1,\pm1,\pm1)$.

(4) $S^1 \times \mathbb{C}P^2$ obtained in Section 3. The double of this generator is null-cobordant. Since T^2 acts on $\mathbb{C}P^2$ the generator has a T^3 action with orbit space being a triangle with boundary labelled (according to some base of $\mathbb{Z}^3$) by 1-dimensional subgroups of T^3: $(1,0,0)$, $(0,1,0)$, $(1,1,0)$.

(5) The S^1 bundle over $\mathbb{R}P^4$ with orientable total space and global section obtained in Section 3. The double of this generator is null-cobordant.

(6) The generator obtained in 4.1. The orbit space has symmetry T^1, where the obstruction interval is included in the fixed point set. The generator thus has a T^3 action with orbit space a square with boundary labelled by four subgroups $T^1 < T^3$.

(7) The generator obtained in 4.2 corresponding to the stratum $\mathrm{Cr}_3^{(2)}$.

Further generators are the T^2 bundles over:

(8) The lens space $L(4)$ described in 4.12. The order of this generator divides 2.

(9) The lens space $L(6)$ described in 4.12. The order of this generator divides 3.

(10) The nonorientable generator of $\widetilde{\Omega}_3(BD_6)$ described in 4.13. This generator can be expressed as a union of generators of type (4) and (5).

It is convenient to add to the list the following type of generators:

(11) $S^4 \times T^1$ ($S^4 < \mathbb{C}^2 \times \mathbb{R}$) with T^3 action

$$(t_1,t_2,t_3)(z_1,z_2,x;s) = (t_1z_1,t_2z_2,x;t_3s).$$

The orbit space is D^2 with $\partial(D^2)$ labelled by subgroups: $(1,0,0)$, $(0,1,0)$. We can put any admissible polarization in the neighbourhood of the two orbits in $\mathrm{Cr}_1^{(3)}$ and extend the polarization on the whole manifold. Assume that the polarization in the neighbourhood of one orbit in $\mathrm{Cr}_1^{(3)}$ is "standard", i.e. represented by the Lie algebra of a chosen subgroup, e.g. $(0,0,1)$.

An adjacent connected sum along orbits in $\mathrm{Cr}_1^{(3)}$ (with the same polarization) with one of the generators of type (1), (3), (4), (6) gives an equivariantly diffeomorphic generator possibly with different polarization. This allows us to change the polarization in the neighbourhood of an orbit in $\mathrm{Cr}_1^{(3)}$ in these generators to a "standard" one. In this way the essential information about polarization in the neighbourhood of orbits in $\mathrm{Cr}_1^{(3)}$ can be encoded in a single type of generators, namely (11).

We have thus obtained the following

5.1. THEOREM. *The group $\Omega_5^{\mathrm{pol},t}$ is uncountable abelian with generators given above. The generators* (4), (5), $L(4)$ *and* $L(6)$ *are candidates for torsion elements in the bordism group.*

From the discussion above it is also possible to obtain some relations among the generators.

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